Earthquake Engineering, Tenth World Conference © 1992 Balkema, Rotterdam. ISBN 90 5410 060 5

Modeling of layered soil-structure interaction by infinite elements

Francisco Medina
Dames & Moore, Oakland, Calif., USA

ABSTRACT: A finite/infinite element technique is presented to solve three-dimensional soil-structure interaction problems in layered media. A family of axisymmetric infinite elements is defined to model the three-dimensional linear elastic wave propagation problem in the far field. These elements are developed to transmit Rayleigh, shear and compressional waves in the frequency domain. The technique is used to find the harmonic dynamic response of rigid foundations embedded and on the surface of layered media. By using finite and infinite elements, the number of degrees of freedom used to model the near field is small. In addition, no extra degrees of freedom are needed to model the far field. Thus, the systems are characterized by relatively few degrees of freedom, yielding inexpensive numerical solutions.

1 INTRODUCTION

In the presence of dynamic excitations, structures and the surrounding soil interact. This interaction is the subject of considerable interest to researchers as well practitioners in the area of earthquake engineering. Due to the difficulties involved in the analysis of the interaction process, there are is not a unique procedure to deal with the problem. This still poses a challenge. It is possible to differentiate two main sources of difficulties: far-field modeling and proper definition of excitation forces. This work is only concerned with the former problem. The latter is an entirely different problem (in its own merit) and it will not be dealt with herein. Finding input excitations is not a limitation of the method outlined herein, but it represents the lack of knowledge in that field.

Several approaches have emerged as alternatives to treat the problem of modeling the far field. These approaches present different degrees of generality. Amongst these approaches it is possible to mention the generalized Winkler's medium approach,[1] the consistent boundary approach,[2,3] the integral equation approach,[4] the cloning algorithm,[5] the boundary integral method,[6] and the finite/infinite element technique.[7] None of these approaches is sufficiently general or widely accepted, although the last one presents some degree of generality and efficiency. Nevertheless, this technique has not been widely used, and a conclusive statement cannot be made before it is applied to practical engineering problems. In effect, there is not a general, uniquely accepted, procedure to treat nonhomogeneous, inelastic, anisotropic, soil-structure interaction problems in three dimensions, considering semi-infinite and/or layered soil conditions.

To illustrate the method, what follows briefly presents the finite/infinite element technique applied to problems of soil-foundation interaction. The foundations and surrounding media (near field, *nf*) are modeled with finite elements. The far field (*ff*) is modeled with infinite elements and considered linear. The infinite elements simultaneously transmit Rayleigh, shear and compressional elastic waves. The interface between finite and infinite elements is defined as far-field boundary (*ffb*) for convenience. The soil in the *nf* as well as in the *ff* may be nonhomogeneous, anisotropic, and/or viscoelastic. This work presents a more general application to those already presented in the literature.[8,9] Theoretical considerations on the infinite elements applied to elastic multi-wave propagation may be found elsewhere.[10]

2 FORMULATION OF THE SOIL-FOUNDATION INTERACTION PROBLEM

A foundation system may be discretized into finite and infinite elements, as shown in Figure 1. The element displacement component y^e for both finite and infinite elements is approximated by

$$y^e(\mathbf{x}) = \mathbf{N}_y^e(\mathbf{x})\mathbf{r}_y^e \tag{1}$$

where $\mathbf{N}_y^e(\mathbf{x})$ contains the assumed shape functions at point $\mathbf{x}$ within element e and $\mathbf{r}_y^e$ contains the nodal displacements for element e, corresponding to displacement component y ($y = u, v, w$). For finite elements $\mathbf{N}_y^e(\mathbf{x}) = \mathbf{N}^e(\mathbf{x})$, all y.

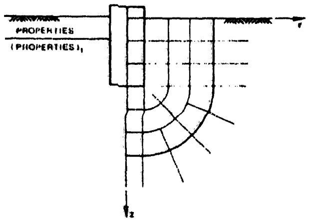


Figure 1. Soil-foundation system embedded in a semi-infinite layered medium and discretized with finite and infinite elements.

For simplicity, the total system is considered linear. Then, by assuming harmonic excitations of the type $e^{i\omega t}$ that may be represented by a load vector of Fourier amplitude $\mathbf{p}(\omega)$, the system displacement solution as a function of frequency may be obtained from

$$[\mathbf{K}(\omega) + i\omega\bar{\mathbf{C}}(\omega) - \omega^2\bar{\mathbf{M}}(\omega)]\mathbf{r}(\omega) \;=\; \mathbf{p}(\omega) \tag{2}$$

where the stiffness matrix $\mathbf{K}$, viscous damping matrix $\bar{\mathbf{C}}$, and mass matrix $\bar{\mathbf{M}}$ are defined in the usual way. For instance, the stiffness matrix for element e is

$$\mathbf{K}^e \;=\; \int_{V^e} [\mathbf{B}^e]^T \mathbf{D}^e \mathbf{B}^e dV \tag{3}$$

where $\mathbf{D}^e$ is the element constitutive relationship matrix, and $\mathbf{B}^e$ relates strains with nodal displacements within element e. No restrictions are imposed on the constitutive relationship and problem dimension. Hence, $\mathbf{D}$ may be as general as needed and the problem may be three-dimensional.

The formulation as well as the application of the method is straightforward. Definition of proper infinite elements however requires special attention. This subject is presented in the next section.

3 INFINITE ELEMENTS FOR THREE-DIMENSIONAL LINEAR ELASTIC WAVE PROPAGATION

A family of axisymmetric infinite elements is defined to model problems having a *ff* with axisymmetric geometry. The characteristic matrices of these elements depend upon the n-th harmonic of the angle θ. Hence, these elements may be used to model problems having nonaxisymmetric excitations. The geometry of these elements, shown in Figure 2, consists of at most six nodes: two or three on the *ffb* and at most three located in the *ff*. The elements have been developed having degrees of freedom only on the *ffb* nodes. The nodes in the *ff* are degree-of-freedomless nodes. Having the degrees of freedom only on the *ffb* nodes presents the following advantages: (a) the total number of degrees of freedom does not increase beyond that already defined by the *nf* discretization, and (b) the *ff* dynamic stiffness matrix, which is complex, is narrow banded, amenable to fast numerical solutions. However, the *ff* is represented by few degrees of freedom, which results in a poor *ff* numerical solution, unless the shape functions used are good approximations to the problem being modeled. Having degrees of freedom on the nodes located in the *ff* may increase the accuracy of the model. However, this has not been studied.

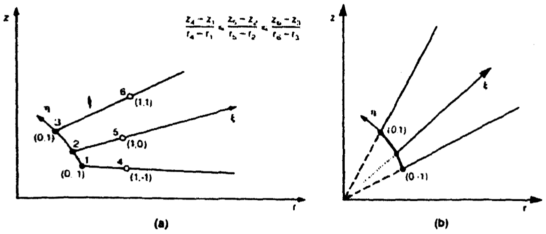


Figure 2. Parametric axisymmetric infinite elements. (a) Six-node element (only three nodes having degrees of freedom: on $\xi = 0$). (b) Three-node radial element.

The infinite elements developed to transmit elastic waves in (layered) semi-infinite media are characterized by having distinct mapping and shape functions.

3.1 *Mapping functions*

From the six-node axisymmetric element shown in Figure 2a a family of elements consisting of two to six nodes is defined. However, in order to model layered systems, the mapping function must at least refer to to all nodes on the sides extending to infinity ($\eta = -1$ and $\eta = 1$). Mathematically, the mapping function may be expressed as

$$M_j(\xi,\eta) \;=\; L_j(\xi)L_j(\eta) \tag{4}$$

($0 \le \xi < \infty$, $-1 \le \eta \le 1$), where L_j is the Lagrange polynomial for node j ($j = 1, 2, \cdots, 6$). In particular, for a five-node element (element with nodes 1, 2, 3, 4 and 6, see Figure 2a),

$$\mathbf{M}^T(\xi,\eta) \;=\; \begin{bmatrix} M_1(\xi,\eta) \\ M_2(\xi,\eta) \\ M_3(\xi,\eta) \\ M_4(\xi,\eta) \\ M_6(\xi,\eta) \end{bmatrix} \;=\; \begin{bmatrix} -\frac{1}{2}(1-\xi)\eta(1-\eta) \\ (1-\xi)(1-\eta^2) \\ \frac{1}{2}(1-\xi)\eta(1+\eta) \\ \frac{1}{2}\xi(1-\eta) \\ \frac{1}{2}\xi(1+\eta) \end{bmatrix} \tag{5}$$

On the other hand, if the element has only nodes on the *ffb*, the element is a radial element, as shown in Figure 2b. In this case, the mapping function is

$$M_j(\xi,\eta) \;=\; (1+\eta)L_j(\eta) \tag{6}$$

($j = 1, 2, 3$).

The location of nodes 4 to 6 is relevant as it shall be shown further below. It is readily seen that an element having node 4 and/or 6 (see Figure 2a) can model nonhomogeneous systems. In particular, a layered system may be modeled with elements having at least nodes 1, 3, 4 and 6. The layers may have any orientation with respect to the global coordinate axis. The element sides extending to infinity need not be parallel to each other, however, the angle formed by these two sides must be greater or equal to zero and less than π. On the other hand, radial elements may only model homogeneous semi-infinite media.

3.2 *Shape functions for multi-wave propagation*

The shape functions, different for each displacement component, only refer to the nodes on the *ffb*. Details of the shape function derivation may be found elsewhere.[9] For a typical displacement component y, the shape function is[10]

$$\mathbf{N}_y(r, z, k_R, k_S, k_P) \;=\; \mathbf{f}_y(r, z, k_R, k_S, k_P)\mathbf{F}_y^{-1} \tag{7}$$

($y = u, v, w$), where $\mathbf{F}_y$ contains the nodal values of $\mathbf{f}_y$, which in turn contains the wave components being transmitted. In the equation above, k_R, k_S and k_P are the Rayleigh, shear and compressional elastic wave propagation numbers, respectively. The infinite elements were developed to transmit Rayleigh, shear and compressional elastic waves into the *ff*.

The wave components described elsewhere[9,10] and used for the family of infinite elements presented above are undefined for the zero frequency. It is then necessary to define another set of shape functions for zero frequency. Although there are several infinite elements developed to solve elastostatic problems, an element consistent with the equation shown above was developed for the zero frequency case. For this element, equation (7) yields

$$\mathbf{N}_y(r, z) = \mathbf{f}(r, z)\mathbf{F}^{-1} \tag{8}$$

(all $y = u, v, w$), where

$$\mathbf{f}(r, z) = \begin{bmatrix} \dfrac{1}{R} & \dfrac{1}{R+r} & \dfrac{1}{R+|z|} \end{bmatrix} \tag{9}$$

such that $R = \sqrt{r^2 + z^2}$. This element, despite being more expensive to use than those already reported in the literature, leads to better results.

The frequency dependent elements described above, by definition transmit different elastic waves. Therefore, by modeling the *ff* with these elements, energy is transmitted away from the *nf* into the *ff*, in the form of the elastic waves present in the shape functions. These elements, however, are noncompatible and inconsistent with the finite element discretization along the finite direction. The noncompatible and inconsistent nature of these elements is not a severe drawback, since a more important factor is to simultaneously transmit various elastic waves into the *ff* (radiation condition). So far, infinite elements satisfying compatibility and radiation conditions for multi-wave propagation have not been defined for nonhomogeneous media. It is desirable to have compatible infinite elements, but those already developed can only transmit a single wave, which leads to larger errors than those produced by noncompatible infinite elements developed to treat multi-wave propagation problems.[10]

Furthermore, the wave components (and henceforth the shape functions) are only rough approximations to solutions of particular problems. Therefore, the numerical solutions obtained for the *ff* may not in general be accurate. This problem may be overcome in two different ways: (a) by including additional wave components (i.e., additional degrees of freedom) into the displacement field approximation, or (b) by systematically refining the discretization on the *ffb* along the finite direction. Nevertheless, the *ff* is represented and included in the global equilibrium equation by means of a dynamic stiffness matrix, which leads to good numerical solutions for the *nf*, as it shall be shown further below.

The constitutive relationship matrix used to compute the stiffness matrices, equation (3), may be that of anisotropic media. On the other hand, the shape functions are defined only for isotropic media. Although this combination may be used to roughly model anisotropies in the *ff*, no examples have been carried out.

3.3 *Shape functions for single-wave propagation*

For torsional wave propagation, simpler shape functions have been developed.[7] In this case, the shape function for node j and displacement v is

$$N_j(\xi, \eta, k_S) = \mathrm{e}^{-(1+ik_S R_0)\xi} L_j(\eta) \tag{10}$$

where R_0 is a characteristic distance, say $R_0 = R(\xi = 0, \eta = 0)$. For homogeneous media this element is compatible. The element is also consistent with the finite element approximation along the finite direction.

4 APPLICATION: DYNAMIC RESPONSE OF RIGID FOUNDATION

4.1 *Foundation dynamic response*

This problem reduces to that of finding the frequency-dependent foundation compliance functions, generally expressed in matrix form as

$$\mathbf{C}(\omega) = \begin{bmatrix} C_H^{rr} & C_{HM}^{r\theta} & 0 & 0 \\ C_{MH}^{\theta r} & C_M^{\theta\theta} & 0 & 0 \\ 0 & 0 & C_V^{zz} & 0 \\ 0 & 0 & 0 & C_T^{zz} \end{bmatrix} \tag{11}$$

such that $\Delta = \mathbf{C}(\omega)\mathbf{p}$, where Δ contains the harmonic foundation displacements due to harmonic loads. In equation (11), subscripts denote loading conditions, whereas superscripts denote coordinate axes on which the corresponding loads are applied. Subsequently, superscripts are dropped. The coupling terms must satisfy the condition $C_{HM} = C_{MH}$.

4.2 *Discretization*

To take advantage of the technique, the rigid foundations and the surrounding media in the *nf* are modeled with a reasonably small number of nine-node axisymmetric finite elements, and the *ff* is modeled with even fewer six-node axisymmetric infinite elements. The rigidity of the foundations is not a limitation of the technique, the foundations were assumed rigid only to make comparisons with available solutions. The response of the foundations is obtained for torsional (T), vertical (V), horizontal (H), and rocking (M) harmonic loadings. The *nf* and *ff* are assumed to have no material damping. Therefore, the *nf* characteristic matrices $\mathbf{K}$ and $\bar{\mathbf{M}}$ are frequency independent; they are then computed only once and later combined with the corresponding frequency dependent *ff* dynamic stiffness matrices. The *nf* characteristic matrices are computed using Gauss-Legendre quadrature. The *ff* dynamic stiffness is computed using reduced Gauss-Laguerre quadrature with two integration points in the finite direction and a larger but limited number of points in the infinite direction. This number varies with frequency, as indicated elsewhere.[10]

Numerical solutions are computed for discrete values of the dimensionless frequency

$$a_0 = ak_S = a\omega/V_S \tag{12}$$

where a is the foundation characteristic dimension and V_S is the shear wave velocity of the medium adjacent to the foundation. For a typical load J, the error (discrepancy) ε_J, is defined as

$$\varepsilon_J = \frac{\max\limits_{a_0} |C_J(a_0) - \hat{C}_J(a_0)|}{|C_J(0)|} \tag{13}$$

where $C_J(a_0)$ and $\hat{C}_J(a_0)$ respectively are the *exact* and approximate numerical complex values of the compliance diagonal term, corresponding to load J. It is understood by *exact* as the available solutions used for comparison. The global error in computing the foundation dynamic response is defined as

$$\varepsilon = \left(\frac{1}{4} \sum_{J=T,V,H,M} \varepsilon_J^2 \right)^{\frac{1}{2}} \tag{14}$$

A summary of errors for all the cases treated is shown in Table 1.

Table 1. Compliance function errors for rigid foundations loaded harmonically.

LOADING	RIGID HEMISPHERE		RIGID PLATE		RIGID PLATE ON LAYER OVER HALF-SPACE		RIGID PLATE ON LAYER OVER RIGID BASE		RIGID CYLINDER	
	DOF†	ERROR, %	DOF†	ERROR, %	DOF†	ERROR, %	DOF†	ERROR, %	DOF†	ERROR, %
TORSIONAL	24	2.5	67	4.1	87	--	103	11.7	86	0.8
VERTICAL	44	8.7	137	7.3	179	6.5	208	--	173	4.8
HORIZONTAL	80	12.2	225	6.5	289	10.0	331	--	284	11.4
ROCKING	80	4.3	225	8.7	289	12.8	331	--	284	10.9
GLOBAL ERROR		7.9		6.9		10.1		11.7		8.3

†Number of degrees of freedom.

5 NUMERICAL EXAMPLES

5.1 *Rigid hemispherical body*

The response of a rigid hemispherical body embedded in a homogeneous, isotropic semi-infinite medium was obtained for harmonic loadings. The *nf* is modeled by eight finite elements and the *ff* by four infinite elements, as shown in Figure 4a. The *ffb* is approximately located at one and one-half hemisphere radii (R_0) away from the hemisphere origin. The degree-of-freedomless nodes are located at a distance R^* from the hemisphere origin. Since the infinite element shape functions, equation (7), depend on the global coordinates, if R^* varies, the shape functions vary. Hence, the computed foundation response depends upon the location of the degree-of-freedomless nodes. Figure 3 shows the compliance errors for different values of the infinite element mapping ratio R^*/R_0. The global error is minimum at a ratio R^*/R_0 around 2.25. Hence, this ratio was used for all the cases treated. For a ratio of 2.25, the computed compliance functions obtained are compared with other available solutions[11,12] in Figures 4b-f. Despite the few degrees of freedom used for discretization, there is good agreement between both sets of solutions.

5.2 *Rigid circular plate*

The response of a rigid circular plate on the surface of a homogeneous, isotropic semi-infinite medium was obtained for harmonic loadings. The *nf* is modeled by nineteen finite elements and the *ff* by four infinite elements, as shown in Figure 5a. The *ffb* is located at two plate radii away from the plate origin. The computed compliance functions obtained are compared with the corresponding exact solutions[13,14] in Figures 5b-f. In this case the error shows similar behavior to that found previously when the ratio R^*/R_0 varies. The minimum error is also found in the neighborhood of 2.25. Table 2 shows the variations of the compliance errors with respect to R^*/R_0 for a few values of this ratio.

5.3 *Rigid circular plate on a single layer over a half-space*

The response of a rigid circular plate on the surface of a homogeneous, isotropic single layer resting over a homogeneous, isotropic semi-infinite medium was obtained for harmonic loadings. The thickness of the layer is 0.2 times the plate radius. The *nf* is modeled by twenty-four finite elements and the *ff* by five infinite elements, as shown in Figure 6a. The *ffb* is approximately located between 2 and 2.2 plate radii away from the plate origin. The computed compliance functions obtained are shown in Figures 6b-f, where the exact solutions[15] are also shown for comparison.

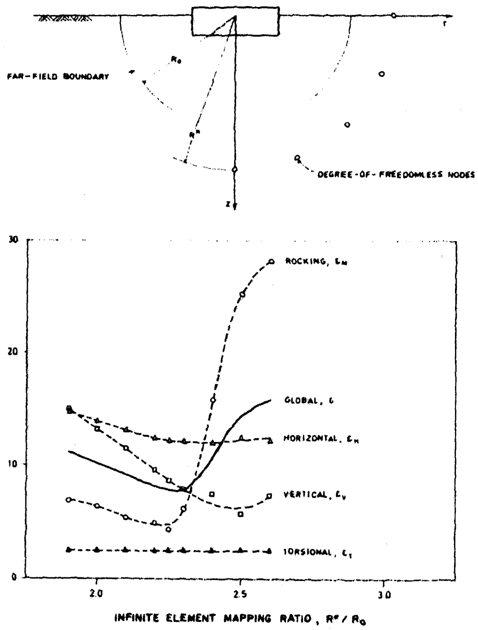


Figure 3. Influence of the infinite element mapping ratio on the response of a rigid foundation embedded in a semi-infinite medium.

5.4 *Rigid circular plate on a single layer over a rigid base*

The response of a rigid circular plate on the surface of a homogeneous, isotropic single layer resting over a rigid base was obtained for harmonic loadings. The thickness of the layer is two plate radii. The *nf* is modeled by twenty-eight finite elements and the *ff* by five infinite elements, as shown in Figure 7a. The *ffb* is approximately located at two plate radii away from the plate vertical axis.

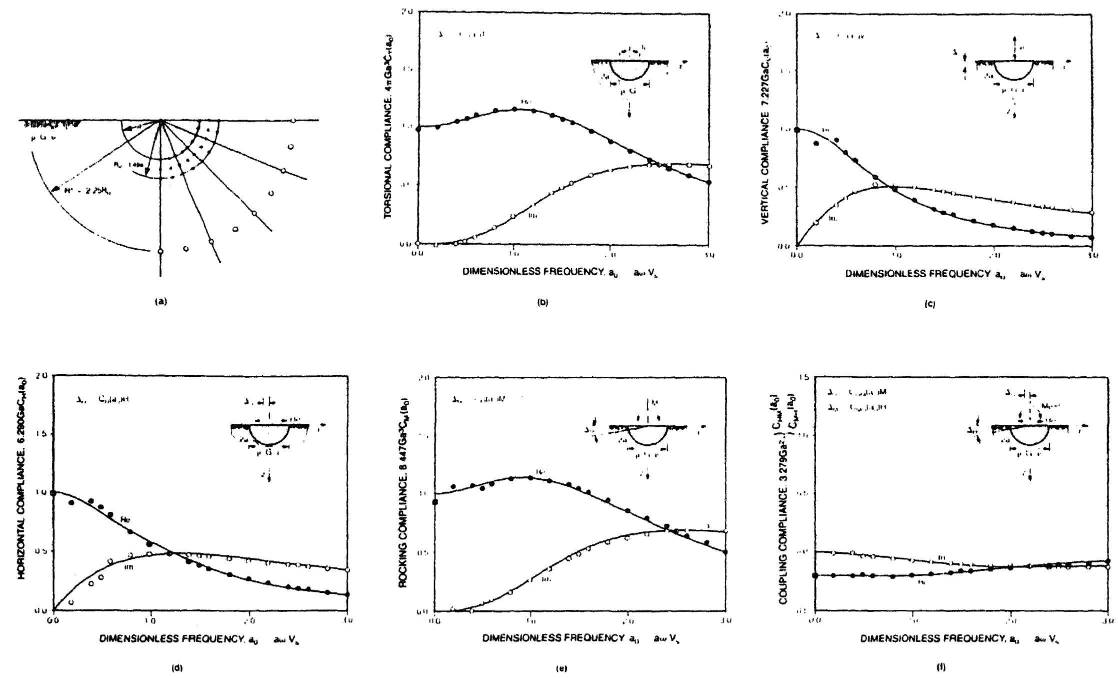


Figure 4. Rigid hemispherical body embedded in a homogeneous, isotropic semi-infinite medium ($\nu = 1/4$). (a) Element mesh. (b-f) Compliance functions.

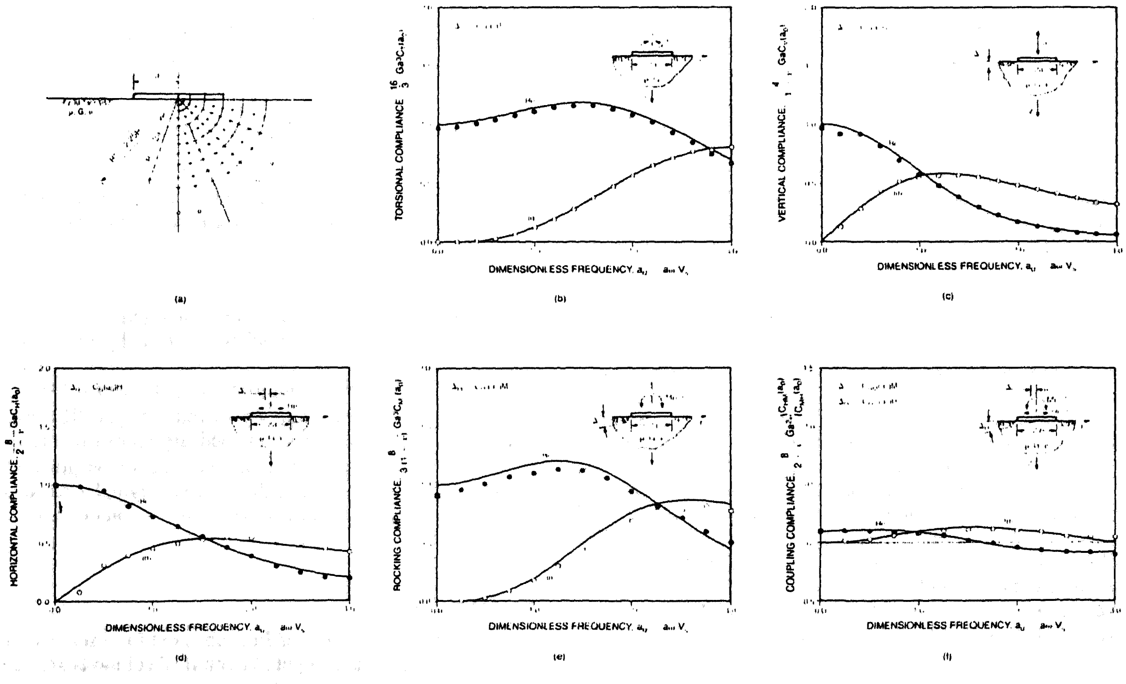
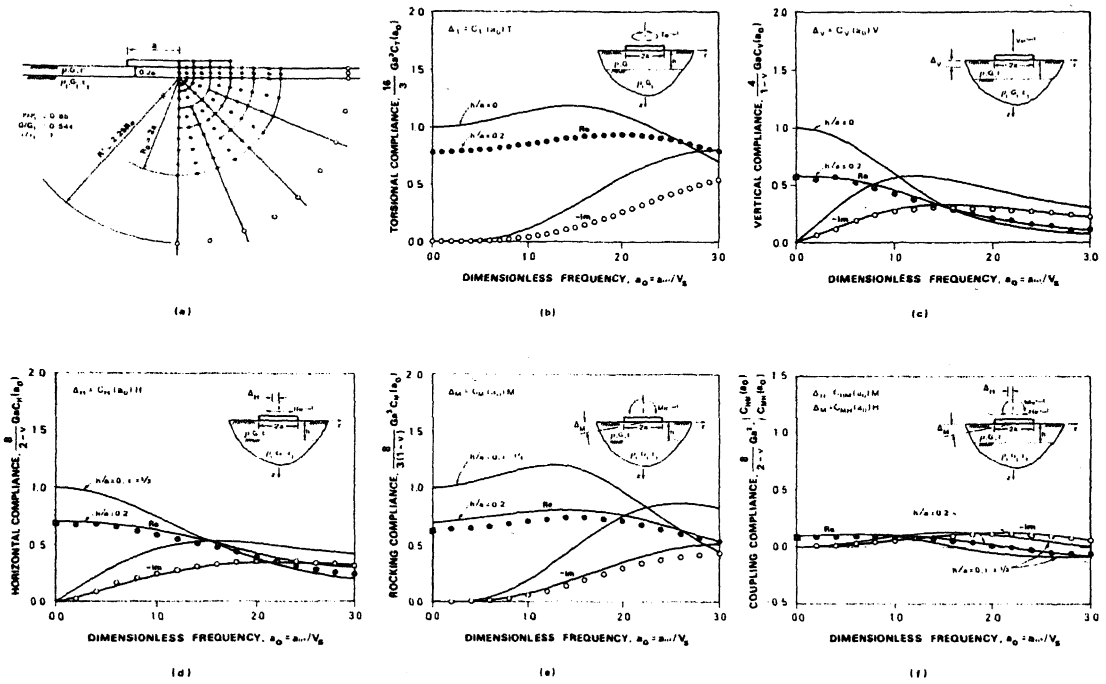


Figure 5. Rigid circular plate on a homogeneous, isotropic semi-infinite medium ($\nu = 1/3$). (a) Element mesh. (b-f) Compliance functions.

Table 2. Influence of the infinite element mapping ratio on the response of a rigid plate on a semi-infinite medium.

LOADING	ERROR, %			
	$R^*/R_0 = 2.00$	$R^*/R_0 = 2.20$	$R^*/R_0 = 2.25$	$R^*/R_0 = 2.30$
TORSIONAL	4.2	3.4	4.1	5.4
VERTICAL	8.8	7.5	7.3	7.2
HORIZONTAL	7.9	6.9	6.7	8.8
ROCKING	8.9	9.4	8.7	10.4
GLOBAL ERROR	7.7	7.1	6.9	8.2



(a) (b) (c)

(d) (e) (f)

Figure 6. Rigid circular plate on a homogeneous, isotropic single layer, resting over a homogeneous, isotropic semi-infinite medium ($\nu = 1/4$). (a) Element mesh. (b-f) Compliance functions. (The compliance functions for a rigid circular plate on a homogeneous, isotropic semi-infinite medium are shown for reference.)

The computed compliance functions obtained are shown in Figures 7b-f, where another solution,[2] available for the torsional case, is also shown for comparison. Solutions for zero frequency (the static case) are also found in the literature.[2,3,16] In Table 3 they are compared with the computed solutions obtained using the shape functions given by equations (8) and (9). In Figure 7c it may be observed that the foundation vertical response presents a marked amplification between the dimensionless frequencies $a_0 = 1.5$ and $a_0 = 1.7$. In addition, In Figures 7e and 7f it may be observed that the foundation rocking response is severely amplified in the neighborhood of $a_0 = 2.1$. These amplification peaks occur at two of the *nf* natural frequencies of vibration: $a_0 = 1.687$ and $a_0 = 2.124$.

5.5 *Rigid cylinder embedded in a half-space*

The response of a rigid cylinder embedded in a homogeneous, isotropic semi-infinite medium was obtained for harmonic loadings. The *nf* is modeled by twenty-five finite elements and the *ff* by six infinite elements, as shown in Figure 8a. The *ffb* is approximately located between 2 and $\sqrt{5}$ plate radii away from the cylinder origin. The computed compliance functions obtained are shown in Figures 8b-f, where other available solutions[12] are shown for comparison. For zero frequency, the torsional cylinder response global error is only 1.3%, compared to the solution considered exact.[17]

5.6 *General remarks*

In spite of the few degrees of freedom used for discretization it is observed that the computed numerical solutions are generally in good agreement with the solutions shown for comparison. Maximum discrepancies are below 13%, see Table 1.

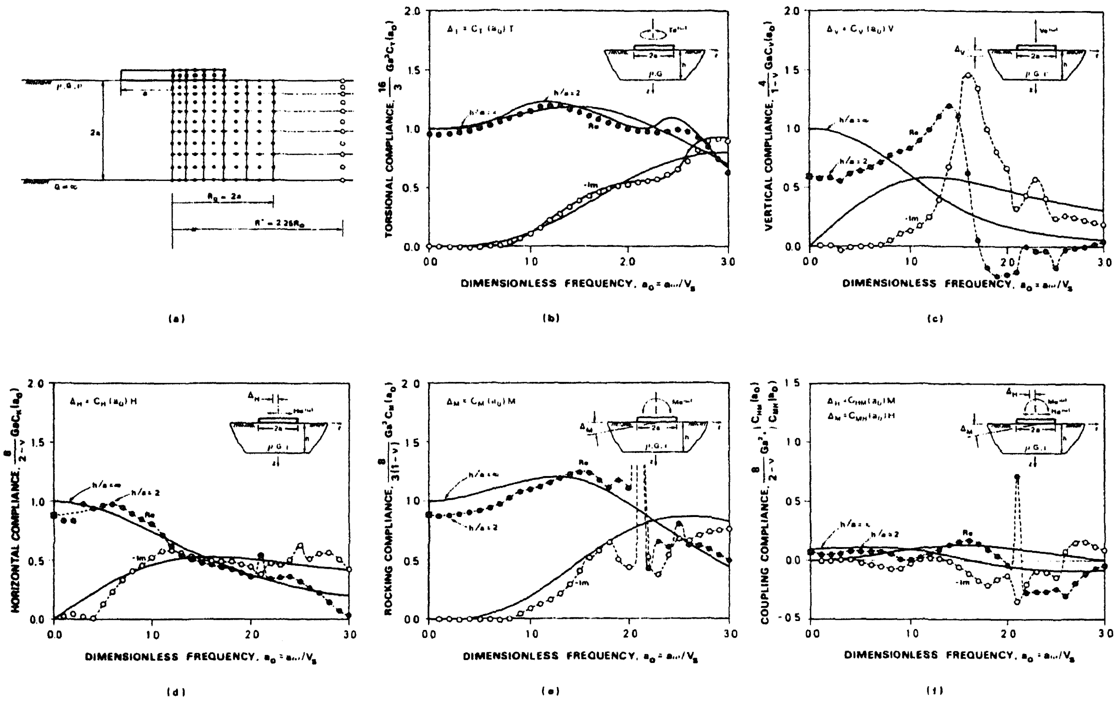


Figure 7. Rigid circular plate on a homogeneous, isotropic single layer ($\nu = 1/3$), resting over a rigid base. (a) Element mesh. (b-f) Compliance functions. (The compliance functions for a rigid circular plate on a homogeneous, isotropic semi-infinite medium are shown for reference.)

Table 3. Compliance functions at zero frequency for a rigid plate on a single layer over a rigid base.

LOADING	REFERENCE	NORMALIZED COMPLIANCE		ERROR, %
		AVAILABLE SOLUTION	PRESENT METHOD	
TORSIONAL	Waas[2]	0.982	0.951	3.2
VERTICAL	Poulos[16]	0.613	0.598	2.4
HORIZONTAL	Kausel[3]	0.800	0.883	10.4
ROCKING	Kausel[3]	0.923	0.883	4.3
GLOBAL ERROR				6.0

The numerical solutions may be improved by refining the discretization in the *nf* as well as in the *ff*. This approach is not attempted herein since the main point to emphasize is that reasonable numerical results may be obtained in the *nf* by modeling the system with only few degrees of freedom.

Due to the few degrees of freedom used to discretize the *ff* and due to the noncompatible shape functions used, the *ff* solution is poor. However, by transmitting outgoing elastic waves, the infinite elements approximately model the elastic *ff*, leading to satisfactory solutions in the *nf* for all the cases considered.

6 CONCLUSIONS

A direct simple finite/infinite element technique for the analysis of dynamic soil-structure interaction has been presented. The technique permits the analysis of unbounded problems considering homogeneous or layered media. The treatment of rigid bases may also be conducted. A family of axisymmetric infinite elements has been defined to solve elastic wave propagation problems in unbounded media. The infinite elements approximately represent and incorporate the elastic far field into the solution procedure. By transmitting outgoing Rayleigh, shear and compressional waves, the elements carry energy away into the far field. By modeling the far field with infinite elements, this technique permits a significant reduction in the size of the near field. Solutions were carried out with a small number of degrees of freedom, which provided inexpensive answers and satisfactory accuracy.

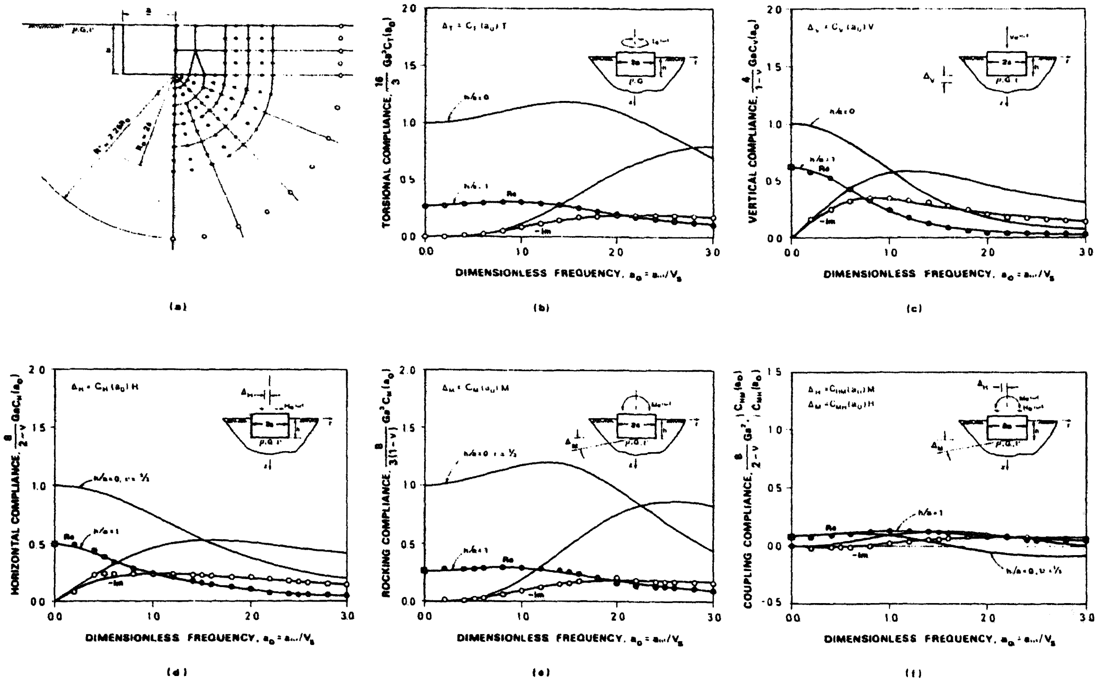


Figure 8. Rigid cylinder embedded in a homogeneous, isotropic semi-infinite medium ($\nu = 1/4$). (a) Element mesh. (b-f) Compliance functions. (The compliance functions for a rigid circular plate on a homogeneous, isotropic semi-infinite medium are shown for reference.)

7 ACKNOWLEDGEMENTS

Most of the work reported herein was partially supported by the Chilean National Fund for Science and Technology (FONDECYT), while the author was at the University of Chile.

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