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Eliminating apparent limitations in modal combination

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ABSTRACT: Spectral Modal Superposition is hampered in its use by an apparent inability to provide information needed in design, which in static or time-history analyses is ordinarily obtained from equilibrium or compatibility. It is shown that this inability can be completely removed. A formula for obtaining spectral estimates of linear combinations of two basic response variables, is presented. The estimates provided for the combinational responses have the same level of quality as the standard estimates of the basic variables. The formula introduces an important cross-estimator as the only additional quantity needed to remove the said inability, at post-processor stage. The formula is first applied to the problems of determining if a member is in single or double curvature. It is also used in establishing a design criterion for cases where the requirements are expressed through a nonlinear interaction relationship. Examples showing the significance of the latter criterion, are also presented.

1 INTRODUCTION

Spectral modal superposition, when used together with a design spectra that appropriately describes the local seismicity of the site, is the most suitable present day design procedure for a vast majority of structures. However, the use of this technique is met by the problem of an apparent inability to provide the information required by certain design provisions, namely, those involving combination of end forces, and very specifically, interaction equations.

Actually, such a recent code as the 1988 Recommended Lateral Force Requirements, of the Structural Engineers Association of California (1988), in the commentary to Section 1F.5.b., states: "Modal combinations present several important problems in the interpretation of results. First, all computed terms are positive. Second, the value associated with each term may correspond to a different point in time. Thus member and joint equilibrium cannot be checked; moments, shears, and deformations at points between the nodes in the model cannot be directly calculated. The designer needs to consider these conditions when using the terms, and should assign signs to the individual terms to assure that the results are conservative. An examination of individual modes may be useful in those assessments. Some code provisions require that the designer know if a member is in single or double curvature, and the predominant mode response could be used to determine this condition."

However, most of these problems are largely inexistent. That the "terms" - as moments, shears and deformations are generically referred to in the code - are all calculated as absolute values is certainly a characteristic of the spectral technique. But such a characteristic should not be understood as meaning that the terms are to be regarded as essentially positive quantities, rather, that as estimations of response variables reversible in nature, they should be taken both with plus and minus sign. Of course, the equations of statics and those of geometrical deformation do not apply to spectral estimates. But this does in no way imply that response quantities not ordinarily obtained explicitly from analysis programs have, of necessity, a lack of estimates of the same level of accuracy as those of the member end forces.

On the contrary, the modal spectral superposition technique is bound to yield an estimate for any response variable that can be expressed as a linear combination of the structure's displacement degrees of freedom, and through them, as a linear combination of the modal normal coordinates. Certainly, there is no reason for a preferential status in one type of response variable, for example the end moment of a certain member, with respect to any other type of response, as for instance the moment in a point in the span of the same member. The relevant question is to decide beforehand which information must be retrieved from the analysis in order to have after its completion the capability to calculate estimates of given combinational response variables that, of course, in a static analysis would be computed directly from equilibrium or compatibility considerations.

The second section of this paper deals with this point, developing a general procedure by which an estimate of any linear combination of two basic variables can be obtained from their standard estimates and an additional "cross-estimator", as the only supplementary information required from the main analysis process. The third section addresses the special problem of deciding if a flexural member is in single or double curvature.

Interaction equations that are often included in code provisions can be readily handled, except in cases when they involve a nonlinear relationship. When the interaction is associated with a linear expression, the requirement is equivalent to imposing a limit to a response variable that is a weighted average of end forces. It can be readily estimated through the procedures for combinational variables. A generalized method for nonlinear interactions, based in the property of convexity of the safe interaction region, typical of ductile members, is presented in the fourth section of the paper. The fifth section is devoted to an application example.

The problem here posed has in fact been addressed before, pointedly by Çakiroğlu (1987), and Gupta (1990). However, the presentation in the work of both these authors is somewhat obscured by the fact that analysis of nonlinear interaction under spectral superposition does not appear to be the main point of interest in either study. Rather, the first one is trying to find the most unfavorable direction of the seismic excitation, while the second one is mainly concerned with the effects of a multicomponent earthquake. The two methods are in some way graphical, and they both lead to a safety criterion based in that a certain ellipse is to remain within the safe region. However, it is not easy to establish that the two ellipses are in fact coincident. Actually, Gupta's method can be shown to be equivalent to the criterion here presented, and a discussion thereof is included as the sixth section of the present paper. The formulation of this graphical method is rederived, so as to remove the here unrequired references to multicomponent excitation, and to eliminate a distracting definition of a certain "equivalent modal response" that is really unnecessary.

2 LINEAR COMBINATIONS OF VARIABLES

As it has already been stated, whether basic in the analysis stage or not, any response variable can be expressed as a weighted sum of the modal normal coordinates having the form

$$r(t) = \Sigma r_i \zeta_i(t) \tag{1}$$

The weighing factors affecting the normal coordinates are the values that can be calculated for each response variable when the structure is deformed in the given modal shape. It will be assumed that the modal vector normalization has included the corresponding spectral displacement ordinate for each shape, so as to render a maximum of one for all the normal coordinates. The spectral estimate of the response variable can then be written as

$$R^2 = \Sigma\Sigma\rho_{ij} r_i r_j \tag{2}$$

where the modal coupling coefficients are those of the CQC method (Wilson et. al., 1981) or those proposed by Der Kiureghian (1981), or eventually, zeros and ones, as in the SRSS method.

In order to establish the minimum information about a set of two reference response variables that has to be retrieved from the analysis process in order to retain capability of later calculating an estimate of a response variable that is a linear combination of the former two, such as

$$t = \alpha r + \beta s \tag{3}$$

the expression given by the direct application of the superposition formula (2)

$$T^2 = \Sigma\Sigma\rho_{ij} t_i t_j \tag{4}$$

has to be appropriately expanded.

This expansion can be performed due to the fact that the linear equation (3) is valid within the context of a geometrically and statically compatible modal shape deformation, i.e.,

$$\begin{aligned} t_i &= \alpha r_i + \beta s_i \\ t_j &= \alpha r_j + \beta s_j \end{aligned} \tag{5}$$

which upon substitution into equation (4) leads to

$$T^2 = \Sigma\Sigma\rho_{ij}(\alpha r_i + \beta s_i)(\alpha r_j + \beta s_j) \tag{6}$$

Equation (6) can now be rewritten

$$\begin{aligned} T^2 = \Sigma\Sigma\rho_{ij}[&\alpha^2 r_i r_j + \\ &\alpha\beta(r_i s_j + r_j s_i) + \beta^2 s_i s_j] \end{aligned} \tag{7}$$

so as to allow the derivation of the fundamental formula

$$T^2 = \alpha^2 R^2 + 2\alpha\beta\overline{RS} + \beta^2 S^2 \tag{8}$$

which includes the standard estimators of the basic response variables

$$\begin{aligned} R^2 &= \Sigma\Sigma\rho_{ij} r_i r_j \\ S^2 &= \Sigma\Sigma\rho_{ij} s_i s_j \end{aligned} \tag{9}$$

and what will be called their cross-estimator, which due to the necessary symmetry of the modal coupling coefficients, can be written asymmetrically as

$$\overline{RS} = \Sigma\Sigma\rho_{ij} r_i s_j \tag{10}$$

It is thus concluded that equation (8) solves the problem posed, and that the cross-estimator of equation (10) is indeed the only additional piece of information required to calculate the estimator of any linear combination of the two reference variables. A generalization to three or more basic variables can be achieved through a clearly straightforward extension of the procedure.

As a direct example, consider the calculation of the bending moment at point X in the beam of Figure 1. The linear expression in this case is

$$m_x = -(1-\xi)m_a + \xi m_b \qquad (11)$$

so that on application of the formula of equation (8), the following result is obtained

$$M_x^2 = (1-\xi)^2 M_a^2 - 2(1-\xi)\xi\overline{M_a M_b} + \xi^2 M_b^2 \qquad (12)$$

Here, the notation derived from (9) and (10) is

$$M_a^2 = \sum\sum \rho_{ij} m_{ai} m_{aj}$$
$$M_b^2 = \sum\sum \rho_{ij} m_{bi} m_{bj} \qquad (13)$$

for the estimators of the end moments, and

$$\overline{M_a M_b} = \sum\sum \rho_{ij} m_{ai} m_{bj} \qquad (14)$$

for their cross-estimator.

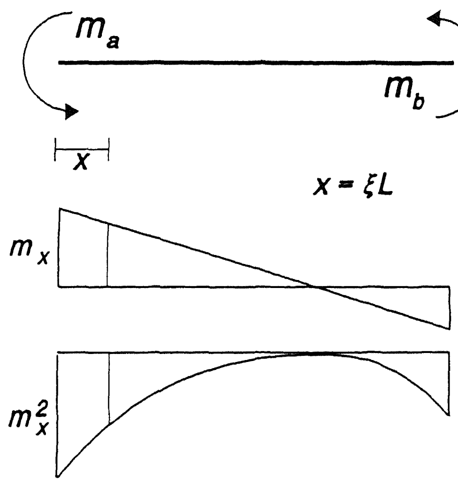

Figure 1. Flexural Member

3 CURVATURE PATTERN OF A MEMBER

Under static loading, or in a time history analysis, the curvature pattern of a flexural member due to lateral deformation can be established by comparing the signs of the two end moments. However, such a test cannot be expressed through a linear relationship. Actually, it is equivalent to obtaining from equation (11) the point X for which the bending moment is zero, and determining if it is located within the end points of the beam or not. The position of that point is given in terms of its dimensionless distance to end A by equation (15). Of course, due to the inherent nonlinearity of a ratio such as this expression, it is not possible to estimate it directly by modal spectral superposition.

$$\xi = \frac{m_a}{m_a + m_b} \qquad (15)$$

As an alternative, equation (12) can be analyzed to see if it eventually yields a zero estimate for the moment at a certain point X. However, in agreement with the fact that the spectral estimate is supposed to assess the maximum value of the corresponding variable, even in the cases where the magnitudes are bound to be very small, it will seldom, if ever, lead to an absolute zero.

On the other hand, considering that the expression given by equation (12) is actually the square of the estimate of the bending moment, and that the form adopted by the square of the moment is parabolic, as is also shown in Figure 1, it can be argued that double curvature should be associated with the positioning of a minimum of the second degree polynomial

$$M_x^2 = (M_a^2 + M_b^2 + 2\overline{M_a M_b})\xi^2 - 2(M_a^2 + \overline{M_a M_b})\xi + M_a^2 \qquad (16)$$

in the range spanning between the end points of the beam. From this expression, obtained by rearrangement of equation (12), it can be seen that the polynomial always has a minimum, since the coefficient of the quadratic term can be shown to be always positive. Indeed, by direct application of the basic equation (8), the coefficient is found to be the square of the estimate of the sum

$$m_{a+b} = m_a + m_b \qquad (17)$$

Hence, the estimate of the bending moment has a minimum value, and it is located at a point X given by

$$\xi_m = \frac{M_a^2 + \overline{M_a M_b}}{M_{a+b}} \qquad (18)$$

Consequently, the criterion to determine whether double curvature occurs is associated to establishing when this expression evaluates to a quantity that is both greater than zero, i.e.,

$$M_a^2 + \overline{M_a M_b} > 0 \qquad (19)$$

and less than one, i.e.,

$$M_a^2 + \overline{M_a M_b} < M_a^2 + M_b^2 + 2\overline{M_a M_b} \qquad (20)$$

This second condition can be reordered leading to equation (21). This simpler expression also reflects the symmetry that the results should have. Furthermore, it is interesting to note that the corresponding minimum value of equation (16) is given by (22). Its value is the estimate of the smallest possible bending moment, and obviously it can be zero only in the improbable case in which the numerator of the expression is itself zero.

$$M_b^2 + \overline{M_a M_b} > 0 \qquad (21)$$

$$M_m^2 = \frac{M_a^2 M_b^2 - \overline{M_a M_b}}{M_{a+b}} \qquad (22)$$

4 NONLINEAR INTERACTION PROVISIONS

Interaction curves or surfaces can be interpreted as the boundary of the region of admissible values of the design variables. For instance, in Figure 2 the interaction curve for a reinforced concrete column is schematically shown as the solid line, marking the boundary of the region where allowable combinations of axial force and positive bending moment must be plotted. Of course, in symmetrical cross-sections, the region for negative moments is an identical reflection of the one shown. Furthermore, it is well known (Heyman, 1971) that for ductile materials and cross-sections, the admissible or safe region is convex.

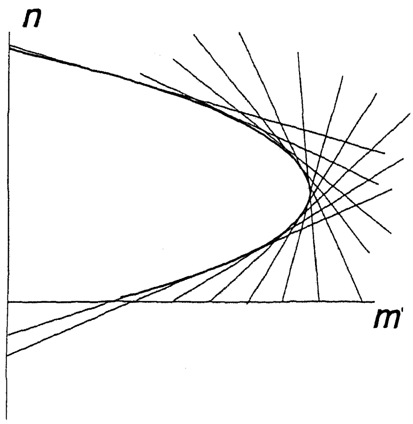

Figure 2. Interaction Curve as an Envelope

The interaction curve, as any other line, can be regarded as the envelope of its own tangents, as is insinuated in the same Figure 2. Henceforth, imposing safety with respect to each and all of the tangents, interpreted as representative of a mode of failure, is wholly equivalent to have the design variables confined within the safe region. Since the safe region is convex, there is no risk of conservatively chopping part of it off when using the alternative procedure.

The individual tangent safety condition can be written as the linear relationship of equation (23). In this expression, the coefficients of the axial force and the bending moment are the reciprocals of the corresponding intercepts as shown for generic tangents in Figure 3.

$$\mu m + \nu n \le 1 \qquad (23)$$

This suggests the definition of a linear combination of the axial force and the bending moment as the new "safety" variable

$$q = \mu m + \nu n \qquad (24)$$

that certainly has an estimate that can be calculated using equations (8), (9) and (10). The resulting expressions are the evaluation formula

$$Q^2 = \mu^2 M^2 + 2\mu\nu\overline{MN} + \nu^2 N^2 \qquad (25)$$

together with the definitions

$$M^2 = \Sigma\Sigma\rho_{ij} m_i m_j \qquad N^2 = \Sigma\Sigma\rho_{ij} n_i n_j \qquad (26)$$

and

$$\overline{MN} = \Sigma\Sigma\rho_{ij} m_i n_j \qquad (27)$$

These relationships allow the calculation of the safety variable for any tangent whose intercepts are known. From a practical point of view, the interaction curve can be approximated through a discrete number of tangents, as shown in Figure 3, and the evaluation of safety variables limited to those cases. Again, given the convexity of the safe region, such an approximation is conservative.

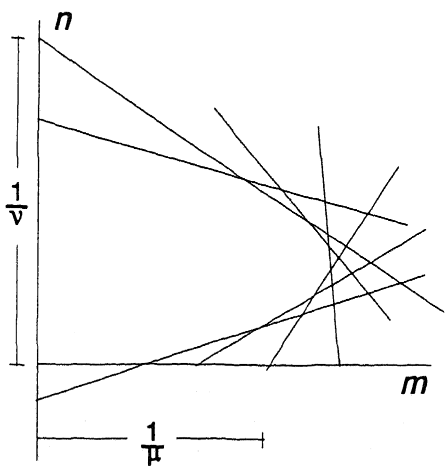

Figure 3. Interaction Curve Tangent Approximation

Of course allowance must be made to consider the effect of static forces. This can be achieved by directly reducing from the safety condition associated to a given tangent, the capacity required by the bending moment and the axial force originated in static loading. The conditions implicit in both equations (23) and (25) is then to be generalized to

$$Q \le 1 - \mu m_{stat} - \nu n_{stat} \qquad (28)$$

as the actual safety criterion with respect to a specific tangent mode of failure.

5 SOME NUMERICAL APPLICATIONS

Analysis for a code design of a 17-story reinforced concrete building gives (Hidalgo et al. ,1990) for a column in the ninth story the following axial force and bending moment at its top end

1.	-324.1 [kN]	50.189 [kN-m]
2.	229.7	-17.455
3.	-8.3	10.265

for each of the three modes most significant for the particular member. The CQC coefficients for these three modes, shown in matrix layout, are

1.000	0.064	0.004
0.064	1.000	0.010
0.004	0.010	1.000

indicating that the case is one of fairly negligible modal coupling.

The formulas of equation (26) give the standard estimates of the axial force and the bending moment as the values

385.1 [kN] 53.1 [kN-m]

while equation (27) renders

-1925. [kN^2-m]

as the corresponding cross-estimator.

The design using the combination of static loadings with the standard earthquake estimates, considering a 0.25[m] square cross-section, was performed for the following two static loading cases

A.	600. [kN]	16.15 [kN-m]
B.	900.	11.60

For comparison purposes, the static loadings were adapted so as to have both in cases A and B the same reinforcement ratio of 0.06. Earthquake load was factored by 1.43. Case A was controlled by the combination corresponding to the difference between the static axial load and the earthquake load, while case B was controlled by the sum of the axial loads. In both cases the controlling bending moment was the sum of the static and earthquake loadings.

Application of the criterion of equation (28) to the verification of the design in case A shows that there is an over-strength that can be accounted for by reducing the reinforcement in 6.8 percent. The overstrength of the design in case B is much slighter, so that the possible reduction in the reinforcement would be a meager 3.5 percent.

The overstrength implicit in design using only the standard estimators is indeed highly variable. For instance, if case B is modified by suppressing the load factor of the earthquake forces and increasing in magnitude the static forces so as to attain a level C given by

C.	1066. [kN]	34.42 [kN-m]

so as to retain as design limit condition the same reinforcement ratio, an overstrength of about 20 percent will be found. Actually, under load condition C the reinforcement obtained from the proposed design criterion is 0.05 rather than 0.06.

6 EQUIVALENCE TO GRAPHICAL METHOD

Gupta's graphical method is based in the notion that through the estimates of equation (2) the modal component response variables define a metric space vector. As mentioned before, an independent derivation of his findings is suitable to clarify the discussion. For such purpose, it is convenient to rewrite equation (2) as

$$R^2 = \Sigma\Sigma\rho_{ij} r^i r^j \qquad (29)$$

showing the contravariant nature that must be attributed in this context to the components of a response variable. Consequently, the covariant component, that has to be written as (Brillouin, 1964)

$$r_i = \Sigma\rho_{ij} r^j \qquad (30)$$

can be regarded as proportional to the set of normal coordinates. This is recognized by comparing the equation expressing the norm (29) in terms of both covariant and contravariant components, and the modal equation (1) evaluated at the time when the response variable under consideration has its maximum, i.e.,

$$\begin{aligned} R^2 &= \Sigma\, r_i r^i \\ R &= \Sigma \xi_i(t_r) r^i \end{aligned} \qquad (31)$$

Hence, any other response variable at the same instant of time can be assumed to have a non-maximal value of

$$s(t_r) = \frac{1}{R}\Sigma\Sigma\rho_{ij} r^j s^i \qquad (32)$$

and specifically, when the linear combinational variable of equation (24) attains its maximum value, given by equation (25), the synchronous values of the basic variables of which it is a combination, should be estimated as

$$m(t_r) = \frac{1}{Q}(\mu M^2 + \nu\overline{MN}) \qquad (33)$$
$$n(t_r) = \frac{1}{Q}(\mu\overline{MN} + \nu N^2)$$

These equations can be regarded as parametric expressions of the locus of the pairs of values of moment and force that maximize any linear combination of them. It is easy to eliminate simultaneously both parameters, since actually only their ratio is relevant. Thus, the quadratic form

$$N^2m^2 - 2\overline{MN}mn + M^2n^2 = M^2N^2 - \overline{MN}^2 \qquad (34)$$

is derived. The corresponding curve is an ellipse. Clearly, the ellipse is inscribed within a rectangle with half-sides the standard estimates of equation (26). The safety condition is then expressed stating that the ellipse is to be entirely within the safe region.

This graphical criterion is seen to be equivalent to the one proposed in Section 4, when observing that from equations (33), the linear combination

$$q(t_r) = \mu m(t_r) + \nu n(t_r) \qquad (35)$$

can be written as

$$q(t_r) = \frac{1}{Q}(\mu^2M^2 + 2\mu\nu\overline{MN} + \nu^2N^2) \qquad (36)$$

The criterion of having each and all points of the ellipse within the safe region is obviously equivalent to the one here proposed, namely, through equation (28). Of course, the ellipse has to be shifted through a displacement of its center to the position representative of the static load.

The analytical procedure is of certainly easier to use. In the graphical method, the ellipse may be drawn by finding its axes, both in position and magnitude, from equation (34). In such a case, attention must be paid to drawing scales used. The curve can also be drawn point by point, as is insinuated by Gupta, before proposing an approximation. The parametric equations (33) serve that purpose, and no equivalent modal response, as the one defined by the author, is needed.

7 CONCLUSIONS

The supposed inability of the spectral modal superposition technique to provide the information required by code provisions is shown to be removable. The most important tool for doing so is the formula derived that provides an estimate of a response variable that is a linear combination of two basic variables, the estimators of which, it is supposed, are obtained during the main analysis process. This formula's estimates, that can render the information that in static or time-history analyses is ordinarily obtained form equilibrium or small deformation geometry, are of the same level of quality as those of the basic variables. The only additional information that is required for its use at the post-processor stage is a cross-estimator that involves the modal components of the two basic variables. The case of several basic variables is not discussed, but it can easily be seen to be a direct extension of the two variable problem.

The use of this formula in design problems that are not directly linear in nature can also be achieved. Two such cases were discussed. In the first one, a method for the discrimination about the pattern of the curvature of a flexural member was developed. In a second and more important one, its use in establishing a criterion for nonlinear interaction design requirements was achieved. Through this criterion, overstrength that may be unavoidable when only the standard estimators are considered, can be suppressed. This overstrength, as is appreciated in the example presented, can vary from a very minor level, to rather significant values that should be considered for a reduction of costs. The interaction problem analyzed was a particular one, but again, its application to other cases is obviously identical.

The proposed criterion is found to be coincident with the graphical one proposed by Gupta, but its use is thought to be easier, particularly, as its coding into an algorithm is bound to be quite straightforward.

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