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Wave dispersion in the discrete analysis and proposal of optimal mass modeling

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ABSTRACT: The discrete analysis methods are frequently used for the study of the structure and soil. However, the assumption of the displacement interpolation function makes the waves dispersive, which means the numerical dispersion. The wave dispersion induced by the discretization depends on the mass modeling. One dimensional periodic structure is adopted as an analysis model and the dynamic transfer matrix method is applied. A wave solution and a finite element solution are used for the transfer matrix. The phase and group velocities in the structure are explicitly represented. These values are compared among the continuum modeling and the discretization modeling in which several consistent mass ratios are adopted. The optimal consistent mass ratio,which makes the wave velocity of the discrete model same as that of the continuum model, is newly developed here. The validity of this mass modeling technique is presented by examining the frequency response function.

1. INTRODUCTION

Beam-like structures consist of a sequence of identical segments which are connected to each other. Such structures, which are called periodic structures, are frequently found both in high-rise buildings on earth and large space structures (LSS's) in space. Then, the dynamic properties of periodic structures is a fundamental research theme in structural dynamics.

On earth, disturbances due to traffic loads such as train and mobile become important in the design of facilities enclosing precision machines or buildings near station. In space, the impact loads such as an artificial debris are critical ones in the design of LSS's. Since these forces may have a wide band of frequencies, it is necessary to study wave propagation in periodic, beam-like structure at high frequencies.

This paper examines the effect of periodically spaced masses on propagating waves, the validity of the finite elements for the analysis in high frequency range, and the new approach of mass modeling. A simple mathematical model of the structure, composed of a continuous rod with periodically spaced lumped masses, is used. This model can represent super-high-rise buildings since the floors behave like lumped masses and the inter-story aseismic elements behave like shear beams. Also, the beam-like trusses in space structures composed of flexible pipes and rigid joints can be considered by the periodic model. Therefore, this fundamental study of wave propagation in one-dimensional periodic structures may be useful in understanding both the earthquake response of super-high-rise buildings and the dynamic response of LSS's.

Since the work by Brillouin, many studies on periodic structures have been presented. Periodic structures have many interesting features. They have pass and stop frequency bands, they act as wave guides that propagate many kinds of waves. There are two analysis approaches for periodic structures. One approach is based on wave-propagation theory (Mead) and the other is based on transfer-matrix methods (Lin, Yong et al.). The former uses a series expansion on the wave number, and its application tends to be limited to periodic structures with infinite length. The latter uses a transfer matrix which is determined from the fundamental repeating element of the structure.

In this paper, the wave propagation characteristics of a rod with periodically spaced masses is examined over a wide frequency range. The wave solution as well as finite element solution are applied to the transfer matrix method. Explicit expressions are developed for the pass and stop bands, the phase and group velocities, and an equivalent damping. These results give clear physical insight into the wave propagation and dynamic characteristics of the structure.

In order to grasp the effect of the discretization, wave propagation in a discrete model is compared with that of continuous model. The discrete model consists of massless springs and discrete or consistent masses. By examining the effect of mass modeling, it is clarified that the discrete solution becomes dispersive and this numerical dispersion depends on the mass modeling. Then, by matching the discrete and continuous results, an optimal discrete model is developed. This optimal mass model drastically improves the discrete solution. Many mass modeling techniques have been proposed up to now (Goudreau, Hughes, Melosh et al.) while the study on the wave propagation seems not to be exist.

2. TRANSFER-MATRIX METHOD OF CONTINUOUS MODEL

Consider the one-dimensional periodic structure composed of a continuum body with n+1 equally spaced lumped masses, as shown in Fig. 1. Denote the mass of each lumped mass by m, the distance between masses by l, and the elastic modulus, section area and mass density of the continuum by G, A and ρ. A prescribed displacement is applied to the left boundary.

The fundamental structural element of this problem is a continuum body of length l attached to lumped masses of mass $m/2$ at both ends. The dynamic stiffness matrix **S** is obtained by using the wave solution of the one-dimensional continuum:

$$\mathbf{S} = \frac{k_b \beta}{2 \sin\beta} \begin{bmatrix} -\alpha\beta \sin\beta + 2\cos\beta & -2 \\ -2 & -\alpha\beta\sin\beta + 2\cos\beta \end{bmatrix} \tag{1}$$

where the following quantities are introduced:

$$k_b = \frac{GA}{l}, \quad \alpha = \frac{m}{\rho A l}, \quad \beta = \frac{\omega}{V} l, \quad V = \sqrt{\frac{G}{\rho}} \tag{2}$$

Here, k_b is the static spring constant, α is the mass ratio of the lumped mass to the continuum, β is a nondimensional frequency and V is the body wave velocity.

The transfer matrix **T** is derived directly from **S**

$$\mathbf{T} = \begin{bmatrix} -\mathbf{S}_{12}^{-1}\mathbf{S}_{11} & \mathbf{S}_{12}^{-1} \\ -\mathbf{S}_{21}+\mathbf{S}_{22}\mathbf{S}_{12}^{-1}\mathbf{S}_{11} & -\mathbf{S}_{22}\mathbf{S}_{12}^{-1} \end{bmatrix} = \begin{bmatrix} \cos\beta - \frac{\alpha\beta}{2}\sin\beta & -\frac{1}{k_b}\frac{\sin\beta}{\beta} \\ k_b\left\{1-\left(\cos\beta - \frac{\alpha\beta}{2}\sin\beta\right)^2\right\} & \cos\beta - \frac{\alpha\beta}{2}\sin\beta \end{bmatrix} \tag{3}$$

The transfer matrix gives the relationship between the state vectors of node i and node i +1 as follows

$$[\, u_{i+1} \;\, f_{i+1} \,]^t = \mathbf{T}[\, u_i \;\, f_i \,]^t \tag{4}$$

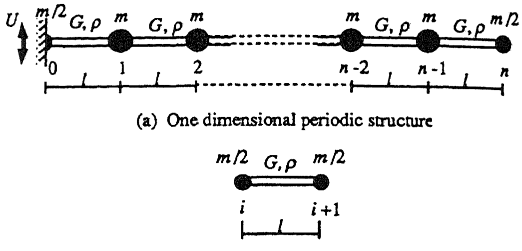
Here, u_i and f_i represent the displacement and force at node i. The eigenproblem of the transfer matrix is

$$\mathrm{T}\,\Phi = \Phi\,\Lambda \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \Phi = [\; \Phi_1 \quad \Phi_2 \;] \tag{5}$$

where the eigenvalues Λ and eigenvectors Φ yield

$$\lambda_i = \left(\cos\beta - \frac{\alpha\beta}{2}\sin\beta\right) \pm \sqrt{\left(\cos\beta - \frac{\alpha\beta}{2}\sin\beta\right)^2 - 1}$$

$$\Phi_i = \begin{bmatrix} -1 \\ \pm\frac{k_b\beta}{\sin\beta}\sqrt{\left(\cos\beta - \frac{\alpha\beta}{2}\sin\beta\right)^2 - 1} \end{bmatrix} \tag{6}$$



(a) One dimensional periodic structure

(b) Fundamental structural element

Figure 1. Figure of analysis.

An important feature of the eigenvalues of the transfer matrix is that they are reciprocals of each other, i.e., $\lambda_1 \lambda_2 = 1$. For the case of no material damping, the eigenvalues are real valued when the expression in the radical in Eq. 6 is positive, and the eigenvalues form a complex conjugate pair when this expression is negative. For consistency, we assign mode numbers such that $|\lambda_1| \le 1$. Then, λ_1 corresponds to a propagating wave and λ_2 corresponds to a reflecting wave. Since the eigenvalues are reciprocals, we can denote $\lambda_1=\lambda$ and $\lambda_2=1/\lambda$.

From Eq. 5, the n-th power of the transfer matrix is given by

$$\mathrm{T}^n = \Phi\,\Lambda^n\,\Phi^{-1} \tag{7}$$

Combining Eqs. 4 and 7 yields the global equation of the periodic structure

$$\begin{bmatrix} u_n \\ f_n \end{bmatrix} = \Phi\,\Lambda^n\,\Phi^{-1}\begin{bmatrix} u_0 \\ f_0 \end{bmatrix} \tag{8}$$

One important property of Eq. 8 is that when $\lambda_2=1/\lambda > 1$, the matrix Λ^n may suffer numerical instability when n becomes large. To avoid this instability, we introduce generalized state vectors (Yong et al.) as

$$[\, u_i \;\, f_i \,]^t = \Phi[\, \xi_i \;\, \eta_i \,]^t \tag{9}$$

Substituting this relationship into Eq. 8 yields

$$\begin{bmatrix} \xi_n \\ \eta_n \end{bmatrix} = \begin{bmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{bmatrix}\begin{bmatrix} \xi_0 \\ \eta_0 \end{bmatrix} = \begin{bmatrix} \lambda^n \xi \\ \lambda^{-n}\eta \end{bmatrix} \tag{10}$$

Then, Eq. 10 yields four equations from the boundary conditions. When a displacement U is prescribed at node 0, and node n is free, the equations are

$$\begin{aligned} \Phi_{11}\xi + \Phi_{12}\eta &= U & \Phi_{11}\lambda^n\xi + \Phi_{12}\lambda^{-n}\eta &= u_n \\ \Phi_{21}\xi + \Phi_{22}\eta &= f_0 & \Phi_{21}\lambda^n\xi + \Phi_{22}\lambda^{-n}\eta &= 0 \end{aligned} \tag{11}$$

Equations 11 can be solved to obtain the state values at nodes 0 and n. The results are

$$f_0 = \frac{\Phi_{21}\Phi_{22}\left(1-\lambda^{2n}\right)}{\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21}\lambda^{2n}} U \tag{12}$$

$$u_n = \frac{\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21}}{\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21}\lambda^{2n}}\lambda^n U \qquad (13)$$

These expressions avoid numerical instability, since they do not contain the potentially large term λ^{-n}.

3. PHASE AND GROUP VELOCITIES, AND EQUIVALENT DAMPING

Equation 10 shows that the eigenvalues of the transfer matrix correspond to the amplification factors of the propagating waves in the fundamental element. The eigenvalues given by eq. 6 are either both real or complex valued. When the eigenvalues are complex, their absolute values are both unity and the wave propagates with a phase delay without decay. On the other hand, if the eigenvalues are real, the wave attenuates.

The condition for which the eigenvalues are complex is examined next. Consider the expression in the radical in Eq. 6. The eigenvalues are complex when this expression is negative. Let β_1 and β_2 denote the frequencies where this expression is zero. The closed-form expressions for these frequencies are

$$\sin\beta_1 = 0\,, \qquad \tan\beta_2 = \frac{4\alpha\beta_2}{\alpha^2\beta_2^2 - 4} \qquad (14)$$

Then, the eigenvalues are complex when

$$\beta_1 = j\pi < \beta < \beta_2 < (j+1)\pi \quad \text{for } j = 0, 1, 2, \cdots \qquad (15)$$

and become real for all other frequencies. Complex λ indicates waves propagating without decay and real λ indicates decaying waves. The frequency ranges which correspond to complex λ are called pass bands and all other frequency ranges are called stop bands.

Next, the velocities of the propagating waves are examined. The phase velocity is given in terms of the phase angle of the eigenvalue:

$$V_{Phase} = \frac{\omega l}{\arg(\lambda)} = \frac{V\beta}{\cos^{-1}\left(\cos\beta - \frac{\alpha\beta}{2}\sin\beta\right)} \qquad (16)$$

The corresponding group velocity is obtained from the derivative of the phase angle:

$$V_{Group} = \frac{l\,\partial\omega}{\partial\arg\lambda} = V\left|\frac{\sqrt{4 - \left(2\cos\beta - \alpha\beta\sin\beta\right)^2}}{(\alpha+2)\sin\beta + \alpha\beta\cos\beta}\right| \qquad (17)$$

These velocities are defined only in the pass band. This means that waves propagate only for frequencies in pass bands and that their phase and group velocities depend on the frequency and the mass ratio.

Finally, the attenuation of waves is examined. To make an analogy with damped vibrations of an oscillator, an equivalent damping ratio is used. In a structure, a wave decays as it propagates from one end of a segment to the next by the value $|\lambda|$. Therefore, an equivalent damping ratio h_e is obtained

$$h_e = \frac{1}{\beta}\log|\lambda| = \left|\frac{1}{\beta}\cosh^{-1}\left(\cos\beta - \frac{\alpha\beta}{2}\sin\beta\right)\right| \qquad (18)$$

This result implies that wave attenuation occurs only in the stop bands. The equivalent damping depends on the frequency and increases with the mass ratio, α. This is physically explained by the fact that the added masses cause reflections which attenuates wave through the structure. When $\alpha = 0$, there are no lumped masses, the equivalent damping is zero, and no wave attenuation occurs.

Figure 2 illustrates the main points of this section. Plots are shown for the phase and group velocities and the equivalent damping. In the pass bands, the phase and group velocities show how the lumped masses cause frequency dependence or dispersion. For nonzero α, the waves propagate with velocities less than that of the body wave. The phase and group velocities

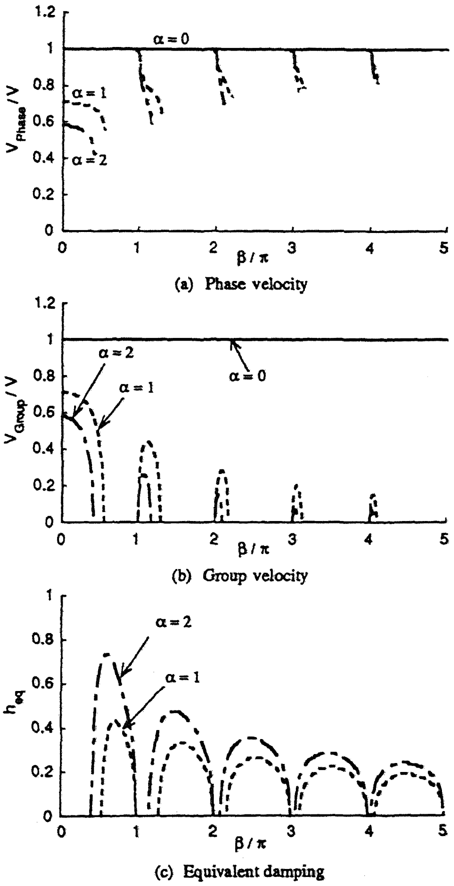


Figure 2. Phase and group velocities and equivalent damping of continuous model.

are less than the body wave velocity by a factor of $1/\sqrt{1+\alpha}$ at the beginning of the first pass band. At the beginning of all other pass bands, the phase velocity is equal to the body wave velocity and the group velocity is equal to zero. Within each pass band, the phase velocities decrease as the frequency increases. Both the group and phase velocities become slower for increasing mass ratios. In the stop bands, waves do not propagate, and a decaying effect becomes important. The equivalent damping increases with the mass ratio.

4. WAVE PROPAGATION OF DISCRETE MODEL

This section analyzes a discrete model of the periodic structure. In the discrete analysis such as finite element method, a structure is modeled by stiffness and mass matrices. If consistent and discrete mass modelings are combined, the dynamic stiffness matrix of the fundamental structural element becomes

$$\overline{S} = k_b \begin{bmatrix} 1-\frac{\beta^2}{2}\left\{(1+\alpha)-\frac{\theta}{3}\right\} & -1-\frac{\theta}{6}\beta^2 \\ -1-\frac{\theta}{6}\beta^2 & 1-\frac{\beta^2}{2}\left\{(1+\alpha)-\frac{\theta}{3}\right\} \end{bmatrix} \tag{19}$$

Here, θ is the mass ratio of the consistent mass to the total mass of the continuum. For the discrete mass model, $\theta = 0$. The analysis of the consistent mass model requires re-evaluation of the eigenvalue problem.

The corresponding transfer matrix is obtained as:

$$\overline{T} = \begin{bmatrix} \frac{6-\beta^2\{3(1+\alpha)-\theta\}}{6+\beta^2\theta} & -\frac{1}{k_b}\frac{6}{6+\beta^2\theta} \\ \frac{(1+\alpha)\beta^2 k_b}{6+\beta^2\theta}\left\langle 6-\frac{\beta^2}{2}\{3(1+\alpha)-2\theta\}\right\rangle & \frac{6-\beta^2\{3(1+\alpha)-\theta\}}{6+\beta^2\theta} \end{bmatrix} \tag{20}$$

The eigenvalues of this transfer matrix are

$$\overline{\lambda}_i = \frac{6-\beta^2\{3(1+\alpha)-\theta\}}{6+\beta^2\theta} \pm i\sqrt{1-\left\langle\frac{6-\beta^2\{3(1+\alpha)-\theta\}}{6+\beta^2\theta}\right\rangle^2} \tag{21}$$

Using the definitions in Eqs. 16 and 17, the phase and group velocities are determined from Eq. 21

$$\overline{V}_{Phase} = \frac{\beta V}{\cos^{-1}\left\langle\frac{6-\beta^2\{3(1+\alpha)-\theta\}}{6+\beta^2\theta}\right\rangle} \tag{22}$$

$$\overline{V}_{Group} = \frac{V\left(6+\beta^2\theta\right)^2}{36(1+\alpha)\beta}\sqrt{1-\left(\frac{6-\beta^2\{3(1+\alpha)-\theta\}}{6+\beta^2\theta}\right)^2} \tag{23}$$

Equations 22 and 23 show that wave propagates in the frequency range

$$\beta \le 2\sqrt{\frac{3}{3(1+\alpha)-2\theta}} \tag{24}$$

Finally, the equivalent damping ratio is given

$$\overline{h}_e = \frac{1}{\beta}\cosh^{-1}\left|\frac{6-\beta^2\{3(1+\alpha)-\theta\}}{6+\beta^2\theta}\right| \tag{25}$$

These results show that the wave velocities of the discrete model vary with β, α and θ that is, the frequency, consistent mass ratio and mass ratio of lumped mass.

Figure 3 shows the phase velocity, group velocity and equivalent damping ratio for a homogeneous continuum ($\alpha = 0$), modeled with various consistent mass ratios. This case corresponds to a problem such as a response of a homogeneous soil subjected to a vertically incident S wave. Unlike the exact continuum model, where the phase and group velocities are always equal to the body wave velocity, the discrete and consistent mass models show dispersion. In the discrete mass model ($\theta = 0$), the wave velocities decrease as the frequency increases and waves propagate only in the frequency range $\beta \le 2$. In the consistent mass model (θ =1), the wave velocities first increase, then decrease as the frequency increases, and waves propagate for a frequency range that is wider than that of the discrete mass model ($\beta \le 2\sqrt{3}$). If the average of the discrete and consistent masses are used (θ=1/2), the wave velocities are nearly equal to the body wave velocity in the frequency range $\beta \le 1$. Therefore, the consistent mass ratio of 1/2, which has been recommended by Hughes and Goudreau, seems to be a good choice for discrete models in case of the homogeneous continuum. Figure 4 shows the corresponding results for a continuum with added masses ($\alpha = 1$). This case corresponds to problems such as a super-high-rise building and a beam-like trusses in space structure. Unlike the homogeneous continuum, the consistent mass model shows a good correspondence with the continuum model. Only one pass band exists for the discrete model while the pass and stop bands emerges repeatedly for the continuum model.

To improve the accuracy of the discrete model, the consistent mass ratio θ_{opt} which satisfies that eq. 22 equals to eq. 16 is evaluated below.

$$\theta_{opt} = 6\,\frac{\beta^2 - 2 + 2\cos\beta - \alpha\beta\sin\beta + \alpha\beta^2}{\beta^2(2 - 2\cos\beta + \alpha\beta\sin\beta)} \tag{26}$$

When the above consistent mass ratio is used, the pass band is defined by

$$\frac{\alpha\beta}{2}\tan\frac{\beta}{2} = 1 \qquad (0 < \beta < \pi) \tag{27}$$

This condition is identical to the second equation of eq. 14. At zero frequencies, eq.27 becomes

$$\lim_{\beta\to 0}\theta_{opt} = \frac{2\alpha+1}{2(\alpha+1)} \tag{28}$$

In case of the homogeneous continuum model ($\alpha = 0$),

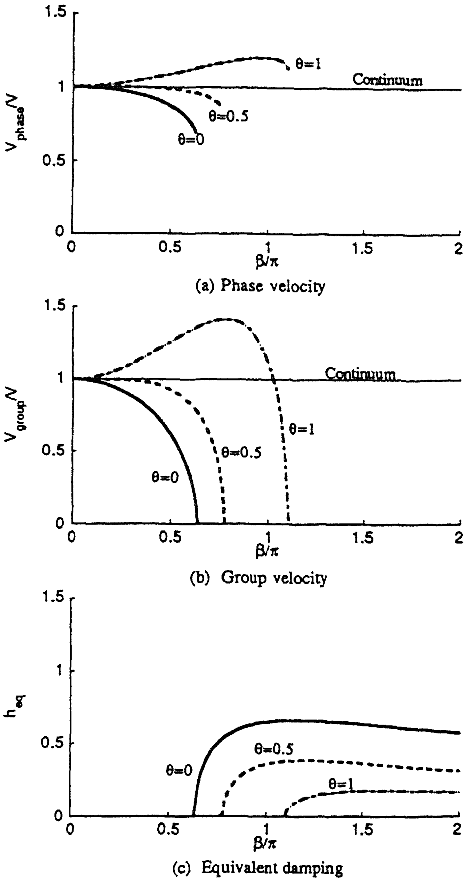


Figure 3. Phase velocity, Group velocity and Equivalent damping of discrete model ($\alpha = 0.0$)

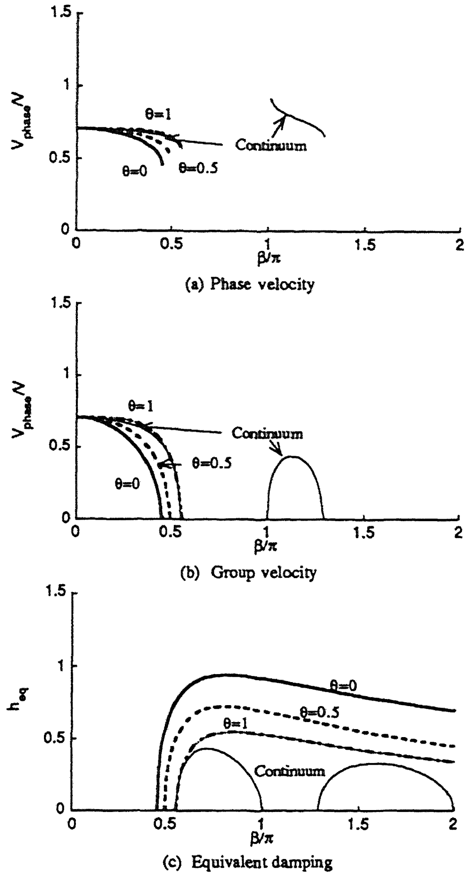


Figure 4. Phase velocity, Group velocity and Equivalent damping of discrete model ($\alpha = 1.0$)

this consistent mass ratio is equal to 1/2 at zero frequencies and increase with frequency, as shown in Fig. 5. However, when the lumped masses are added ($\alpha > 0$), the optimal consistent mass ratio becomes large. Also, the optimal consistent mass ratio increases with the nondimensional frequency β, and the frequency range of pass band decreases when a lumped mass ratio α increases. This implies that the consistent mass ratio should be selected with respect to the structural property and the frequency.

This section concludes by comparing the frequency response function for different models of the periodic structure. Results are computed for displacement response at the right end of the structure resulting from a unit harmonic displacement prescribed at the left end, where $n = 10$, material damping ratio $h = 0.01$ and $\alpha = 0$ and 1. Figure 6 compares the results of the continuous model with those of the discrete models with consistent mass ratios $\theta = 0$, 1/2, and 1. The comparison shows that the discrete model with $\theta = 1/2$ gives results which are closest to those of the continuous model when the homogeneous continuum ($\alpha = 0$) is considered. The re-

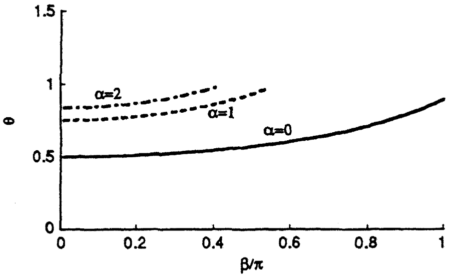


Figure 5. Optimal consistent mass ratios.

sults of the discrete mass model gives response peaks at lower frequencies compared with those of the continuous model, and the peaks of the consistent mass model are at higher frequencies. On the other hand, the discrete model with $\theta = 1$ gives the best result for the continuum with added masses ($\alpha = 1$). These can be expected from the behavior of the phase and group velocities, shown in Fig. 4. Figure 7 compares the results using the optimal consistent mass ratios defined by Eq. 26 with those of the continuum model. The results are more accurate in wide frequency range than those of the discrete models shown in Fig. 6. The pass band is also extended when using optimal consistent mass ratio.

6. CONCLUDING REMARKS

Wave propagation in a one-dimensional periodic structure is studied using the wave solution of the continuum and the transfer-matrix method. Wave propagation in a discrete model is also examined. The main results and conclusions of the study are:

1. The eigenvalues of the transfer matrix are used to determine explicit expressions for the phase and group wave velocities. Furthermore, to quantify wave attenuation, an equivalent damping ratio is developed.
2. The existence of lumped masses in the periodic structure results in stop and pass bands and a dispersion of the wave velocities.
3. For frequencies in the first pass band, the continuous model can be approximated by a discrete model. The discrete model shows a numerical dispersion which depends on mass modeling. When a homogeneous continuum such as a soil is considered, an average of discrete and consistent mass shows good results while a consistent mass model is adequate when the lumped masses exist as in a case of building.
4. By fitting the discrete results to the continuous results, an optimal mass model is proposed. The resulting consistent mass ratios yield the most accurate discrete model.

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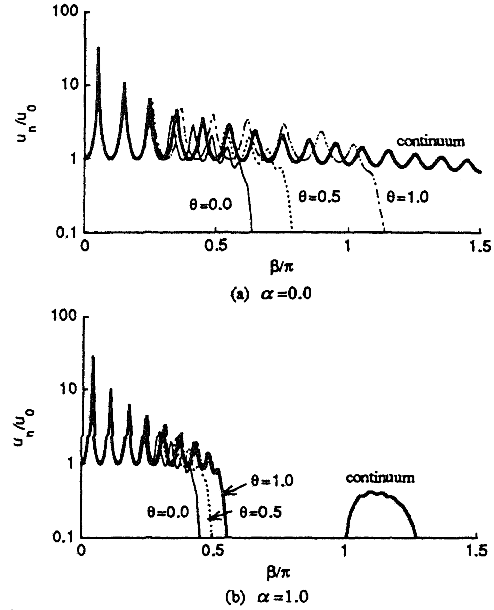

Figure 6. Frequency response of discrete model.

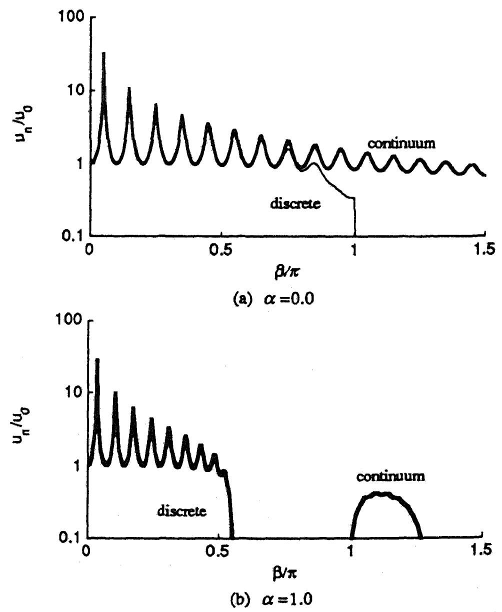

Figure 7. Frequency response of optimal mass model.