THEORETICAL PREDICTION OF SHEAR STRENGTH AND DUCTILITY OF REINFORCED CONCRETE BEAMS

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ABSTRACT

Kupfer (1991) proposed a shear model for calculating the shear capacity at the ultimate limit state of slender RC or PC beams. The compatibility of strains and the equilibrium of stresses of the cracked concrete and shear reinforcement were applied to the shear model using constitutive laws of materials taking into account the effect of aggregate interlocking. In this study an analytical method to predict the shear strength of RC beams is proposed with some modifications using incremental analysis based on Kupfer's model . A new proposal for the effective compressive strength of concrete and a method to classify theoretically the shear failure modes are included. The beam ductility is also predicted based on the degradation of diagonally compressed concrete due to the axial elongation of beam plastic hinges in post yield range.

KEYWORDS

Shear strength, reinforced concrete beam, failure mode, ductility

INTRODUCTION

In the early 70th, M. P. Nielsen (1971) proposed a method to predict shear strength of RC beams based on the lower bound of the theory of plasticity. The shear design method recommended in the AIJ (Architectural Institute of Japan) Design Guidelines (1990) is also based on the lower bound theory. These methods have been used for the prediction of shear strength of RC beams and considered to be more theoretical than empirical design equations proposed in the past. However the experimentally observed physical phenomena do not necessarily correspond to the predicted behaviors. For example, some tests (Watanabe et al., 1991; Iwai et al., 1991) indicated that the stress of shear reinforcement did not reach its yield strength even in case of beams which had the amount of shear reinforcement less than the upper limit in these methods. In this study an incremental analytical method to predict the shear strength and failure modes of RC beams are proposed based on the Kurfers' model (1991) with some modifications. An analytical model to predict the limit deflection of ductile beams under bending and shear is also proposed.

PREDICTION OF SHEAR STRENGTH

Basic Concept of Analysis

Kupfer (1991) proposed a shear model for calculating the shear capacity at the ultimate limit state of slender RC or PC beams. The compatibility of strains and the equilibrium of stresses of the cracked concrete and shear reinforcement were applied to the shear model using constitutive laws of materials taking into account

the effect of aggregate interlocking at the cracked concrete surface. Figures 1 and 2 show the shear model proposed by Kupfer et al. The stress state in diagonally cracked RC beams is characterized by a biaxial stress field in the concrete and a uniaxial tension field represented by the shear reinforcement.
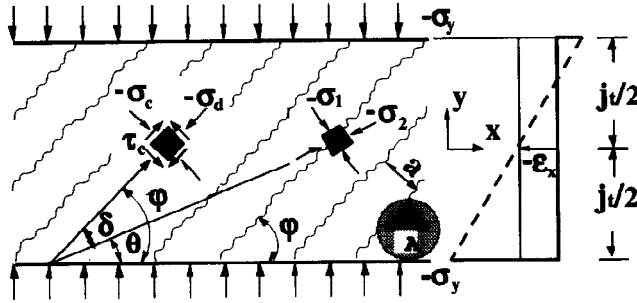

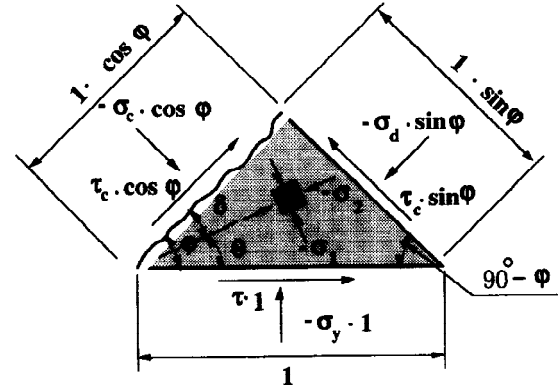Fig. 1. Stress state in the cracked concrete.

Fig. 2. Stress state of concrete element A in Fig. 1

τ_c, σ_c:shear stress and normal stress at cracked surface due to aggregate interlock, σ_d:normal stress in concrete parallel to the crack inclination, σ_1, σ_2:principal stresses in concrete itself, τ,:horizontal shear stress of concrete element, σ_y:vertical compressive stress of concrete element induced from shear reinforcement, φ: diagonal crack inclination, θ:inclination of principal compressive stress, δ:angle between φ and θ, a:average spacing of diagonal cracks, j_t:distance between upper and lower stringers.

Kupfers' model only gives the shear capacity at the ultimate limit state. Figure 3 indicates the shear failure process of RC beams considered in this study. Horizontal axis indicates the strain of shear reinforcement and, vertical axes indicate resisting shear and principal compressive stress of concrete itself. According to Fig. 3, shear failure can be classified into three modes as,

STF : just after the yielding of shear reinforcement the beam shows its maximum strength due to excessive widening of diagonal crack without crushing of diagonally compressed concrete,.

SYCF: after the yielding of shear reinforcement, the resisting shear increases due to aggregate interlocking and then the beam reaches its maximum shear strength showing crushing of diagonally compressed concrete and

SCF : after forming diagonal cracks beam arrives at the maximum shear strength due to the crushing of diagonally compressed concrete without yielding of shear reinforcement.

In this study, the shear failure process is analysed by incremental method until the beam reaches its maximum shear capacity.

Equilibrium condition of stresses

From the equilibrium conditions σ_d, τ and σ_y are expressed as

Fig. 3 Failure modes

$$\sigma_d = -\frac{2\tau}{\sin 2\varphi} - 2\tau_c \cot 2\varphi + \sigma_c \qquad (1)$$

$$\tau = \tau_c + (\sigma_c - \sigma_y)\cot\varphi \qquad (2)$$

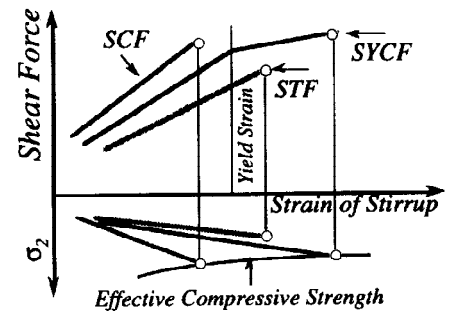$$\sigma_y = -p_w f_{ws} \qquad (3),$$

where, p_w, f_{ws} : ratio and yield strength of shear reinforcement. The principal stresses of concrete σ_1 and σ_2 are given by Eq.(4). The crack inclination φ can be given as Eq.(5) by the use of axial stress σ_{xf} and

shear stress τ_f due to load effect at the beginning of crack formation ($n = \sigma_{xf}/\tau_f$). In case of beams φ becomes 45 degree.

$$\sigma_{1,2} = \frac{1}{2}\left\{\sigma_d + \sigma_c \pm \sqrt{(\sigma_d - \sigma_c)^2 + 4\tau_c^2}\right\} \qquad (4) \qquad\qquad \cot\varphi = -\frac{n}{2} + \sqrt{1 + (\frac{n}{2})^2} \qquad (5)$$

The angle δ between φ and θ is calculated by Eq.(6).

$$\tan 2\delta = \frac{2\tau_c}{-\sigma_d + \sigma_c} \qquad (6) \qquad\qquad \theta = \varphi - \delta \qquad (7)$$

Compatibility condition of strains

In any considered direction the smeared strain of the web concrete results from the strain of concrete itself, ε_0, and the smeared strains, ε_w and ε_v, due to crack opening, w, and crack shear displacement, v.

$$\varepsilon = \varepsilon_o + \varepsilon_w + \varepsilon_v \qquad (8)$$

Then the smeared strains, ε_x and ε_y, in the considered directions, x and y, can be expressed as

$$\varepsilon_x = \varepsilon_{xo} + \varepsilon_{xw} + \varepsilon_{xv} \qquad (9) \qquad\qquad \varepsilon_y = \varepsilon_{yo} + \varepsilon_{yw} + \varepsilon_{yv} \qquad (10).$$

The strains of concrete itself, ε_{x0} and ε_{y0}, can be driven from the principal strains, ε_{10} and ε_{20}, of the concrete.

$$\varepsilon_{xo} = \varepsilon_{10}\sin^2\theta + \varepsilon_{20}\cos^2\theta \qquad (11) \qquad\qquad \varepsilon_{yo} = \varepsilon_{10}\cos^2\theta + \varepsilon_{20}\sin^2\theta \qquad (12)$$

The smeared uniaxial and shear strains due to diagonal crack in the directions x and y are expressed by Eqs.(13) through (16).

$$\varepsilon_{xw} = \sin^2\varphi \cdot w/a \qquad (13) \qquad\qquad \varepsilon_{yw} = \cos^2\varphi \cdot w/a \qquad (14)$$

$$\varepsilon_{xv} = -\sin\varphi \cdot \cos\varphi \cdot v/a \qquad (15) \qquad\qquad \varepsilon_{yv} = \sin\varphi \cdot \cos\varphi \cdot v/a \qquad (16)$$

where, a is average crack spacing.

By the use of the preceding Eqs. (9) through (16) the smeared strains due to diagonal crack, w/a and v/a, follow as Eqs.(17) and (18)

$$\frac{w}{a} = \varepsilon_x + \varepsilon_y - \varepsilon_{10} - \varepsilon_{20} \quad (17) \qquad \frac{v}{a} = \varepsilon_y \tan\varphi - \varepsilon_x \cot\varphi - \varepsilon_{10}\frac{\sin^2\varphi - \sin^2\theta}{\sin\varphi\cos\varphi} + \varepsilon_{20}\frac{\cos^2\varphi - \sin^2\theta}{\sin\varphi\cos\varphi} \quad (18)$$

Constitutive Laws

The resulting principal concrete strains, ε_{10} and ε_{20}, which correspond to the stresses, σ_1 and σ_2, are calculated according to the reference (Kupfer et al, 1973).

$$\varepsilon_{10} = \frac{\sigma_1 + \sigma_2}{9K_s} + \frac{2\sigma_1 - \sigma_2}{6G_s} \qquad (19) \qquad\qquad \varepsilon_{20} = \frac{\sigma_1 + \sigma_2}{9K_s} + \frac{2\sigma_2 - \sigma_1}{6G_s} \qquad (20)$$

where, the secant compressive modulus K_s and shear modulus G_s are given as follows.

$$K_s = 17000(1 - 1.6(\tau_o/\sqrt{f_c})^{1.8}) \text{ (MPa)} \quad (2) \qquad\qquad G_s = 13000(1 - 3.5(\tau_o/\sqrt{f_c})^{2.5}) \text{ (MPa)} \qquad (22)$$

Where, τ_0 : octahedra shear stress.

In this study it is assumed that the compressive failure of concrete occurs when the principal compressive stress , σ_2 , in the cracked concrete attain to the effective compressive strength vf'_c. Past researches by Nielsen (1984) , Collins (1982) and Hsu (1991) indicated that the coefficient v decreases as the increase of the compressive strength of concrete , f'_c, and the increase of smeared strain , w/a , due to crack widening. Considering their findings the effective strength of concrete is given by Eqs 23-1 and 23-2 based on the analysis on the past test results. Figure 4 shows the vf'_c-f'_c, relationship of Eqs. (23-1) and (23-2) when w/a takes 0.002 and 0.01. In Fig. 4, equations proposed by Collins and Hsu ($\varphi = \theta = 45°$) are also indicated.

Fig. 4. Comparison of Eq. 23 to Collins and Hsu equations

$$vf'_c = \frac{f'_c}{(1.0 + \frac{300w/a}{f'^{0.1}_c})} \quad \text{(MPa)} \qquad\qquad f'_c \le 70MPa \qquad (23\text{-}1)$$

$$vf'_c = \frac{70}{(1.0 + 230w/a)} + \frac{(f'_c - 70)}{(1.0 + 170(w/a)^{\sqrt{70/f'_c}})} \quad \text{(MPa)} \quad f'_c > 70MPa \qquad (23\text{-}2)$$

The shear and normal stresses due to aggregate interlock at cracked surface , τ_c and σ_c, are given by Eqs. 24 and 25 by Walraven (1979), and Eqs. 26 and 27 by Li-Maekawa (1991).

$$\tau_c = -\frac{f_{cube}}{30} + \left(1.8w^{-0.8} + (0.234w^{-0.707} - 0.2)f_{cube}\right)v \qquad (24)$$

$$\sigma_c = \frac{f_{cube}}{20} - \left(1.35w^{-0.63} + (0.191w^{-0.552} - 0.15)f_{cube}\right)v \qquad (25)$$

$$\tau_c = 3.83 \cdot f'_c \cdot v^2 /(w^2 + v^2) \qquad (26)$$

$$\sigma_c = -3.83 \cdot f'_c (0.5 \cdot \pi - \tan^{-1}(w/v) - wv/(w^2 + v^2)) \qquad (27)$$

The average spacing of shear cracks a in beams is given by Eq. 28 (Kupfer et al, 1972; Kupfer et al, 1986).

$$1/a = 5p_w/d_s + 2j_t \qquad (28)$$

Stress strain relation of stirrup is assumed as shown in Fig. 5 taking into account the tension stiffening effect due to surrounding concrete The solid line shows average stress strain relation of stirrup over its whole length. Stirrup stress and strain at the point A , which corresponds to the diagonal cracking , are given by Eqs. 29 and 30.

$$f_{dia} = Q_s/(p_w \cdot b \cdot j_t \cdot \cot\varphi) \qquad (29)$$

$$\varepsilon_{dia} = f_{dia}A_{sw}/(E_c b \cdot s + E_s A_{sw}) \qquad (30)$$

where, Q_s :the shear at diagonal cracking, E_c :elastic modulus of concrete, and E_s :elastic modulus of shear reinforcement.
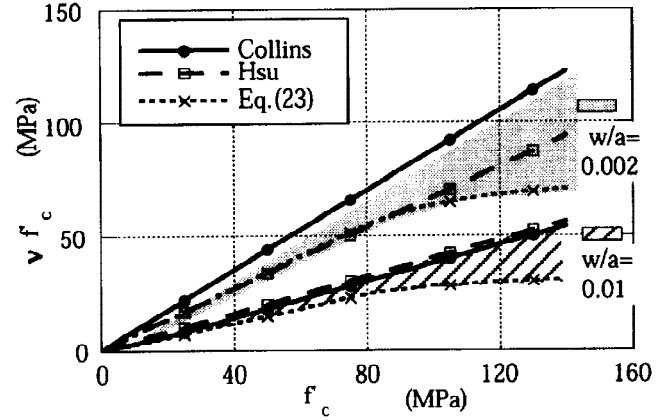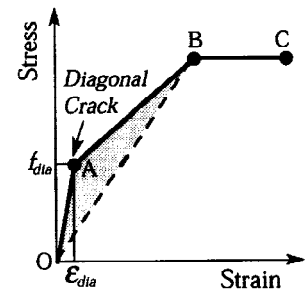

Fig. 5 Stress-strain curve of shear reinforcement

Longitudinal strain at the centroidal beam axis , ε_x , is given by Eq. 31.

$$\varepsilon_x = \frac{\tau \cdot b \cdot j_t \cdot \cot\theta}{2(E_c b \cdot 2(h-d) + E_s A_{sw})} \quad (31)$$

where, ε_x:axial strain, τ: shear stress, b:width of section, j_t: distance between upper and lower stringer, θ: inclination of principal compressive stress, E_c: elastic modulus of concrete, h: total depth of section, d: effective depth of section, $(h-d)$: distance from extreme tension fiber to centroid of tensile reinforcement, E_s: elastic modulus of shear reinforcement, A_{sw}:sectional area of shear reinforcement.

Calculation procedure of shear strength

The strain of stirrup is gradually increased with an increment of $\Delta\varepsilon_y$ until the resisting shear becomes maximum as shown in Fig. 6.

Comparison between predicted and observed shear strengths

Fig. 7 shows the comparison between theoretically predicted shear strength and experimentally observed shear strength. In the figure predicted shear strength by ACI Code (1989) is also indicated. In Fig.7, Y axis indicates the experimentally observed shear strength , V_{exp} , and X axis the predicted shear strength , V_{ana}, where both axes were normalized by the shear force , V_f , at which the bending moment at critical section could reach the theoretical flexural capacity. In these data only the beams which showed shear failure are useful for the evaluation of the prediction method. Therefore the mean value, the standard deviation, and the variance indicated in the figures were calculated only for the data points which have $V_{exp}/V_f \le 0.9$ in order to avoid the beams failing in flexure. From Figs. 7-b and 7-c it is seen that the shear strength of RC beams can be predicted with enough accuracy by applying the method proposed in this paper. Figures 7-b and 7-c also show that the Li-Maekawas' equations for aggregate interlocking action give better results than Walraven's equations. It is noted that when Li-maekawas' model was used the STF failure mode did not appear in the analysis. This is due to the higher agregate interlocking action of Li-maekawas' model than Walraven's model.
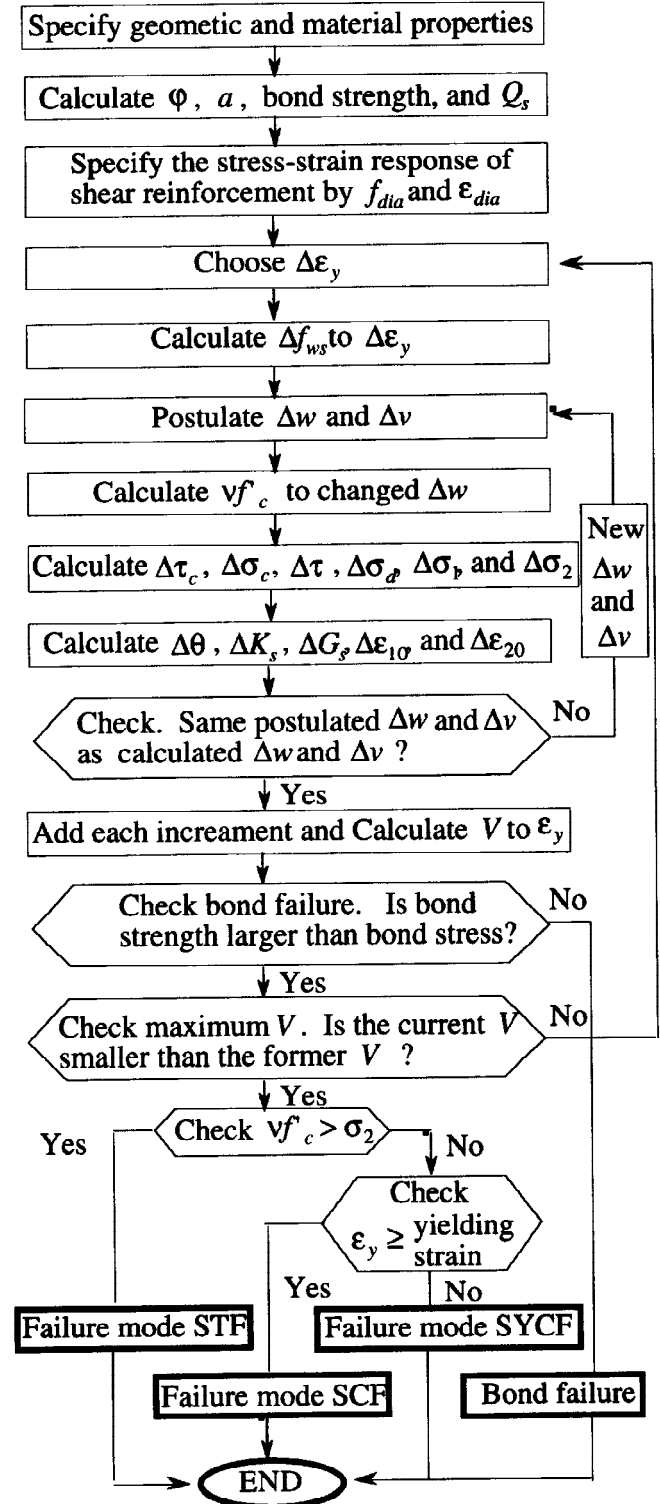
EVALUATION OF BEAM DUCTILITY
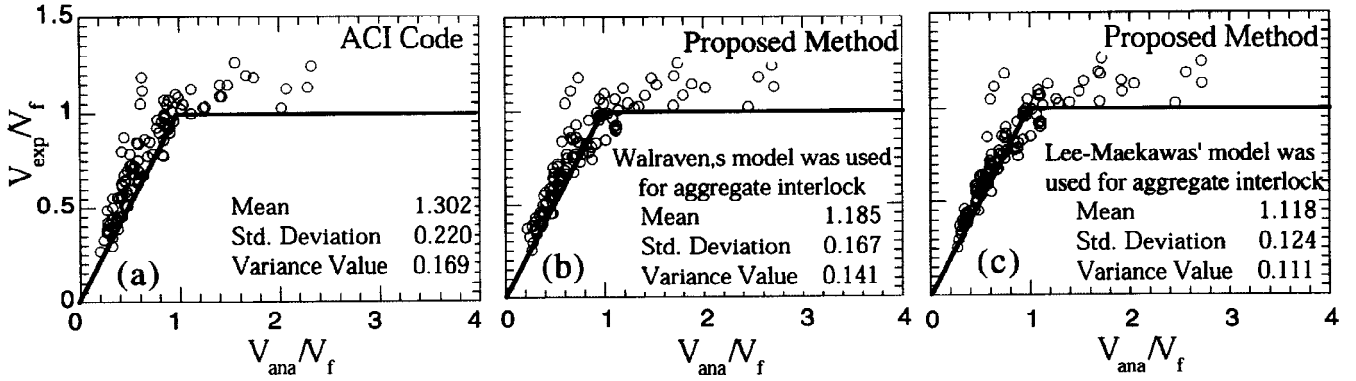

Fig. 6 Calculation procedure

Fig. 7 Comparison between predicted and experimentally observed shear strength

Basic concept

When a beam is subjected to cyclic load in post yield range of deformation, it sometimes shows the reduction of resisting shear at a certain deflection limit due to crushing of diagonally compressed web concrete. This can be explained that the diagonally cracked concrete degrades due to the axial elongation of beam plastic hinge in post yield range. That is, the effective compressive strength of concrete decreases as the increase of axial strain. Fig.8 shows the above concept where X axis indicates the hinge rotation angle , R , and Y axes indicate the decrease of potential shear strength due to hinge elongation and the increase of axial strain of hinge , ε_x , (hinge elongation). In the figure the envelop curve of beam hysteretic restoring force characteristics is also shown and at a point which indicated by solid circle gives the deformation limit of the beam. To calculate the

Fig. 8 Concept of the degradation of shear capacity in post yield range

potential shear strength at a certain hinge rotation angle, the axial strain of hinge , ε_x, is given as the function of the hinge rotation angle and the number of loading cycles up to the considered hinge rotation angle. Then obtained value of ε_x is substituted into Eqs. 17 and 18 for the calculation of potential shear strength.
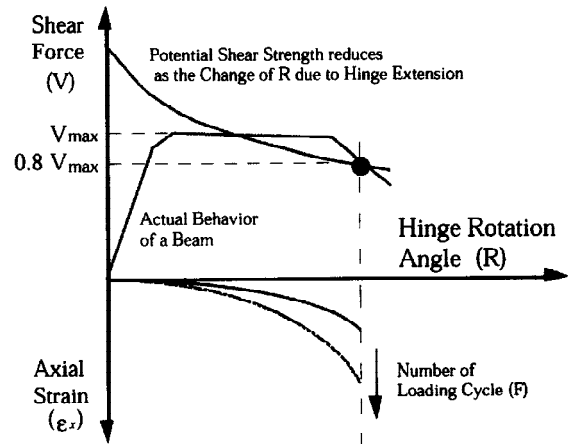
Axial strain ε_x and hinge rotation angle R

The axial strain , ε_x , changes as the increase of the number of loading cycles and the amplitude of hinge rotation angle , R. In this paper the axial strain of hinge region is given by following equations based on the flexural analysis on several types of beam sections. Equation 31 is for monotonic loading and Eq. 32 for cyclic loading.

$$\varepsilon_x = \frac{\varphi \cdot j_t}{2} = \frac{R \cdot j_t}{2l_h} \qquad (31) \qquad \varepsilon_x = \frac{R_n \cdot j_t}{2l_h} + \sum_{i=1}^{n}\left(\frac{R_i \cdot j_t}{2l_h}\right)^2 7F_i \qquad (32)$$

where R:hinge rotation angle, φ:curvature of hinge, l_h:hinge length j_t:distance between upper and lower stringer R_n:hinge rotation amplitude at last loading cycle R_i:hinge rotation amplitude at i-th loading cycle F_i:number of loading cycles with hinge rotation amplitude of R_i

Experimental verification

To verify the proposed evaluation method of ductility, three RC beams tested by the authors. These beams were designed according to the AIJ Guidelines (1990). Details of beams are indicated in Tab. 1. Beams B2 and B4 had non-uniformly distributed shear reinforcement (larger amount of shear reinforcement was provided within hinge regions). Beam B3 was designed to have same amount of shear reinforcement in hinge and outside hinge regions. In the test shear failure due to crushing of diagonally compressed concrete in plastic hinge region was expected. Therefore, supplemental cross ties were arranged in outside hinge regions of beams B3 and B4 to avoid the splitting bond failure. Sectional dimension of beams was 20 × 30 cm and three 16 mm in diameter deformed bars with yield strength of 435 MPa were arranged both in tension and compression side of the beam. Compressive strength of concrete at the time of loading tests was 70 MPa. The ratio of shear span length to beam total depth was 2. One full loading cycle at diagonal cracking load was first applied to each beam spec-

imen. Second loading cycle consisting two full cycles with the deflection amplitude of R=±1/200 was followed by a series of deflection controlled cycles comprising two full cycles to each of the deflection amplitude of R=±1/100, R=±3/200, R=±1/50, R=±1/40 and R=±3/100 where R is the rotation angle of beam. In the tests, loading was stopped when the maximum load after flexural yielding

Table 1 List of Tested beams

Specimen	Shear reinforcement (D6, f_{wy}=267 MPa)					
	Hinge region			Outside hinge region		
	n	s (cm)	p_w (%)	n	s (cm)	p_w (%)
B2	2	6.8	0.625	2	13.6	0.315
B3	2	11.0	0.389	3	11.0	0.582
B4	2	6.8	0.625	3	13.6	0.470

n: number of legs of shera reinforcement, p_w: shear reinforcement ratio
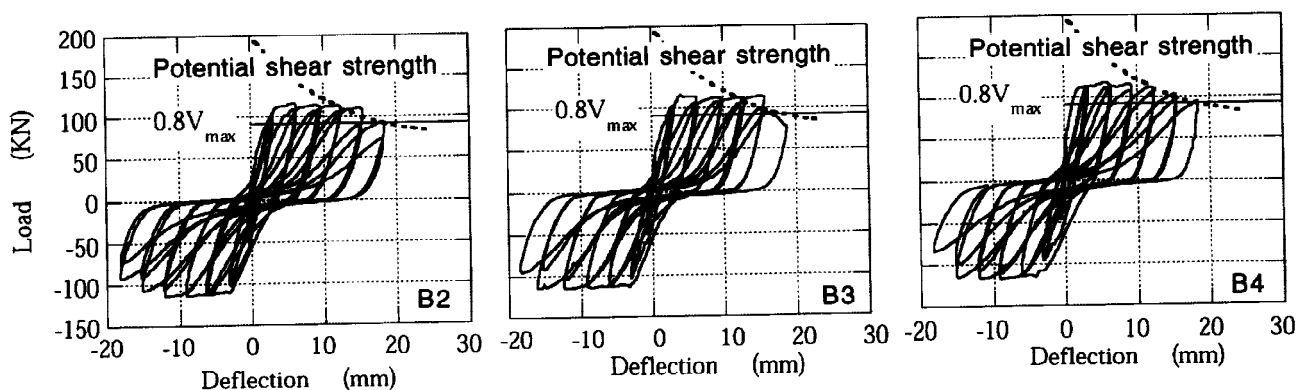s: spacing of shear reinforcement

Fig. 9 Load deflection curves and reductions of potential shear strength due to hinge elongation

dropped to the 80% of maximum load. This point was defined as the ductility limit of the beam. In the analysis, Li-Maekawas' equations were used for the aggregate interlocking action because it showed better results than Walraven's equations in the strength analysis mentioned earlier. Fig. 9 shows the experimentally observed load deflection curves of tested beams. In the figure theoretically predicted reduction of shear strength due to beam elongation is also indicated. Fig. 9 clearly indicates that the proposed method can be used to predict the ductility of beams under bending and shear. As a reference, an example of axial elongation of the beam plastic hinge is shown in Fig. 10, where the dotted line gives theoretically predicted one using Eq. 32 and the solid line gives experimentally observed one.

CONCLUSIONS

Shear strength of RC beams can be predicted by the incremental analysis proposed in this paper, where three failure modes are automatically distinguished. When evaluating shear strength of RC beams by proposed method, Li-Maekawas' equations for aggregate interlocking action gave better results than Walraven's
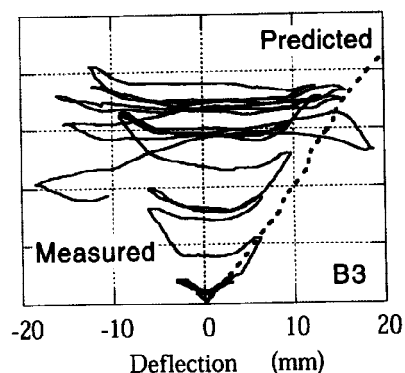

Fig. 10 Observed and theoretically predicted hinge elongation

equations. Effective strength of concrete proposed in this study covers high strength concrete and it gives good results for shear strength evaluation. For the ductility evaluation of beams the key is the elongation of plastic hinge region. Based on the strength degradation of concrete due to hinge elongation, beam ductility can be evaluated with enough accuracy.

ACKNOWLEDGMENT

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