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CALCULATION OF PROBABILISTIC FUZZY RESPONSE SPECTRA OF EARTHQUAKE

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SUMMARY

In this article, the effect of both sources of uncertainties on the response of dynamic systems is studied. They are randomness and fuzziness. Randomness is due to probabilistic nature of earthquake occurrence and the fuzziness is due to vague description of the earthquake and the system characteristic data. Assuming earthquake as a stationary Gaussian random process, the response spectra of a system could be obtained using extreme value theorem. On the other hand, the fuzzy mathematics is capable to consider the effect of different parameters vagueness. Conjunction of these two theorems enables us to calculate probabilistic fuzzy response spectra of the earthquake.

INTRODUCTION

Our knowledge about earthquake, which is considered as a natural hazard, is highly conjugated with uncertainty. This uncertainty could be categorized into two classes.

From one point of view, earthquake is a random event. It means that predicting the characteristics of the future seismic motion based on the current information is a matter of randomness, because there is not a comprehensive data from all influencing factors involved in generating seismic motion.

From other point of view, not only we have only very limited information, but also the available data are not precise. It means that our knowledge about earthquake is vague. This vagueness is not like randomness.

To illustrate these two kinds of uncertainty, we can consider the procedure of obtaining peak ground acceleration of a specified site, which is referred to as hazard analysis. In such analysis, according to seismic source characteristics such as fault length and width and the rupture mechanism in addition to source-site distance and geological properties of underneath soil, it is possible to obtain PGA of the site. Even If we assume that all the parameters involved in the analysis be precise, but the PGA of the site is a random variable, since there is not comprehensive information about all the parameters that have some

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influence on the value of PGA. In the other word, we only use few parameters to obtain the value of PGA and it is impossible to consider the influence of all relevant parameters. As a result, PGA is a random variable even if all the considering parameters be accurate.

On the other hand, it is impossible to determine the precise value of different parameters such as fault length and width. It means that the values associated to fault length and width are vague and are not accurate. This kind of uncertainty is different from randomness, since there is not any random event, but our knowledge is not complete and accurate.

For years, probability theory was the only mathematical framework for considering all kind of uncertainties. But by progress of engineering subjects and increasing the complexity of problems, it is necessary to study the influence of information fuzziness in the response of dynamic systems.

This article introduces a method for calculation of probabilistic fuzzy response spectra of earthquake. For this purpose we present a probabilistic fuzzy process. This process has probability in its occurrence and vagueness in interpreting its obtained values. Assuming the excitation as probabilistic fuzzy process and modeling the structure system with fuzzy parameters we can use a combination of probability mathematics and fuzzy logic methods.

In probabilistic approach the excitation is a stationary Gaussian process. On the other hand, the characteristic parameters of this process are fuzzy numbers, which represent our vague knowledge about the attributes of excitation.

The structural response is calculated in frequency domain using the Poisson probabilistic distribution of extreme values. We can have the mean and variance of maximum response in accordance of different vague natural frequencies and damping ratios. Finally we can calculate a response spectrum which has the probabilistic and fuzzy characteristics of earthquake in it.

In the first part of the paper, the fuzzy mathematics is introduced in the nutshell. In the proceeding section, the notion of fuzzy random variable and a practical procedure to perform calculation on fuzzy random variables are presented. Later, using the mentioned procedure, the fuzzy probabilistic response of a SDOF system subjected to fuzzy random excitation is calculated.

It must be mentioned that the method presented in this article to perform calculation with fuzzy random variables is a general mathematical framework. In this paper, we apply this method to the stationary Gaussian random process, however, it is possible to apply this method to other processes like non-stationary and non-Gaussian.

THE BASIC CONCEPTS OF FUZZY SETS

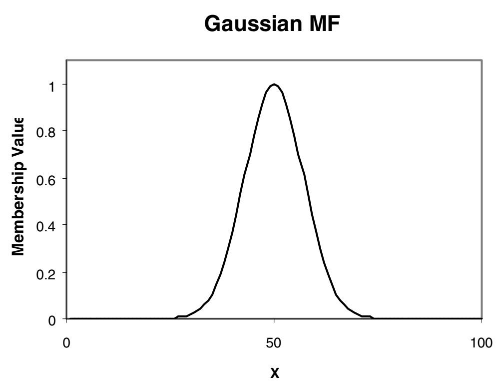
A *Fuzzy Set* **A** (Zimmermann [1]) , defined in a universe of discourse **X** is expressed by its membership function (MF) $\mu_A(x)$

$$\mu_A(x): \mathbf{X} \rightarrow [0,1] \qquad (1)$$

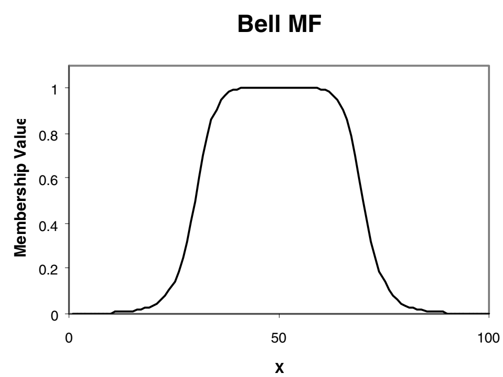
where the degree of membership $\mu_A(x)$ expresses the extent to which x fulfills the category described by A. We show fuzzy set A with a set of ordered pairs:

$$A = \{(x, \mu_A(x) | x \in X)\} \qquad (2)$$

Some kinds of common membership functions used frequently in fuzzy analysis are shown in figure 1.



(a)



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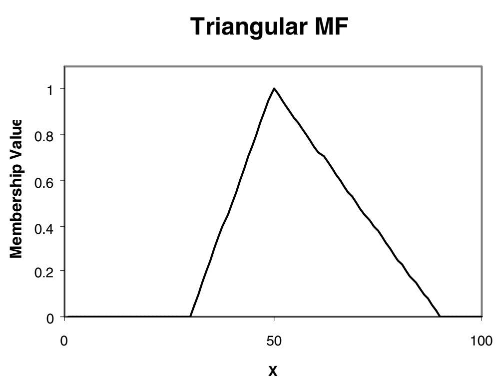

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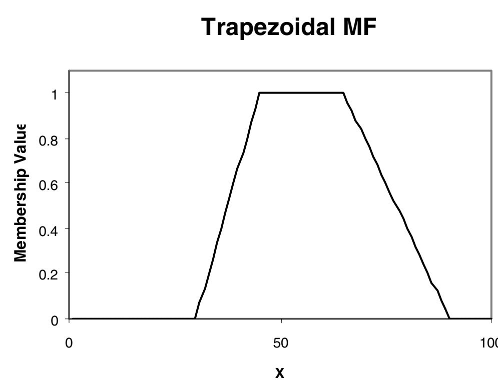

(d)

Fig 1: Some common membership functions. (a) Gaussian MF, (b) Bell MF, (c) Triangular MF, (d) Trapezoidal MF

The crisp set of elements that belong to fuzzy set A at least to the degree α is called $\alpha-$ *level set* or α-cut :

$$A_\alpha = \{ x \in \mathbf{X} \mid \mu_A(x) \geq \alpha \} \qquad (3)$$

Vertex method

According to the extension principle (Zadeh [2]), algebraic operations on real numbers can be extended to fuzzy numbers. However, the implementation of the computation is not trivial.

Vertex method (Dong, Shah [3]) is a simple and efficient method for calculating algebraic functions which is extended to fuzzy numbers. This method is based on the concept of α-cut and the "Simplex Method" in optimization.

By implementation the definition of α-cut, a fuzzy number A is discretized into n interval number $A_\alpha=[A_\alpha^-,A_\alpha^+]$. All interval variables form an n-dimensional rectangular $X_1{*}X_2{*}\ldots{*}X_n$ with 2^n vertices.

When $y = f(x_1,x_2,\ldots,x_n)$ is continuous in the n-dimensional rectangular region, and also no extreme point exists in this region (including the boundaries), then the value of interval function can be obtained by

$$Y=f(X_1,\ldots,X_n) = \left[\min (f(c_i)) , \max(f(c_i)) \right] ,\ \ i = 1,\ldots,n \qquad (4)$$

where c_i is the ordinate of the i-th vertex.

The algebra of interval numbers

The notion of vertex method which is based on constructing possible combinations with interval numbers boundaries at each α-level and deriving the boundaries of result, is very innovative and applicable to every function. However, the amount of calculation increases as the number of parameters involved in the function increases (Ansari [4]).

On the other hand, if the function is a combination of basic algebraic operations, ie, addition, subtraction, multiplication and division, it is possible to determine which vertex is the boundary of result. It can be achieved by the algebra of the interval numbers.

Let $A_\alpha = [a_\alpha^- , a_\alpha^+]$ and $B_\alpha= [b_\alpha^- , b_\alpha^+]$ be two interval numbers which are identical to α-cuts of two fuzzy numbers. Fuzzy basic algebraic operations can be expressed as follow:

1/ Addition:

$$A_\alpha + B_\alpha = [a_\alpha^- , a_\alpha^+] + [b_\alpha^- , b_\alpha^+] = [a_\alpha^- + b_\alpha^- , a_\alpha^+ + b_\alpha^+] \qquad (5)$$

2/ Subtraction:

$$A_\alpha - B_\alpha = [a_\alpha^- , a_\alpha^+] - [b_\alpha^- , b_\alpha^+] = [a_\alpha^- - b_\alpha^+ , a_\alpha^+ - b_\alpha^-] \qquad (6)$$

3/ Multiplication: There is not a general expression for multiplication. It is essential to distinguish between following cases:

a/ $a_\alpha^- , a_\alpha^+ , b_\alpha^- , b_\alpha^+ \geq 0$

$$Aa . Ba = [a_\alpha^- , a_\alpha^+] . [b_\alpha^- , b_\alpha^+] = [a_\alpha^- . b_\alpha^- , a_\alpha^+ . b_\alpha^+] \qquad (7)$$

b/ $a_\alpha^- , a_\alpha^+ \geq 0 \quad b_\alpha^- , b_\alpha^+ \leq 0$

$$Aa . Ba = [a_\alpha^- , a_\alpha^+] . [b_\alpha^- , b_\alpha^+] = [a_\alpha^+ . b_\alpha^- , a_\alpha^- . b_\alpha^+] \qquad (8)$$

c/ $a_\alpha^- , a_\alpha^+ , b_\alpha^- , b_\alpha^+ \leq 0$

$$Aa . Ba = [a_\alpha^- , a_\alpha^+] . [b_\alpha^- , b_\alpha^+] = [a_\alpha^+ . b_\alpha^+ , a_\alpha^- . b_\alpha^-] \qquad (9)$$

d/ $a_\alpha^-, a_\alpha^+ \leq 0 \quad b_\alpha^-, b_\alpha^+ \geq 0$

$$Aa \,.\, Ba = [a_\alpha^-, a_\alpha^+] \,.\, [b_\alpha^-, b_\alpha^+] = [a_\alpha^+ \,.\, b_\alpha^+ \,,\; a_\alpha^- \,.\, b_\alpha^-] \tag{10}$$

e/ In other cases , it is necessary to use vertex method.

$$A_\alpha \,.\, B_\alpha = [a_\alpha^-, a_\alpha^+] \,.\, [b_\alpha^-, b_\alpha^+] =$$
$$[\,\mathrm{Min}\,(a_\alpha^- . b_\alpha^- \,,\; a_\alpha^- . b_\alpha^+ , a_\alpha^+ . b_\alpha^- \,,\; a_\alpha^+ . b_\alpha^+)\,,$$
$$\mathrm{Max}\,(a_\alpha^- . b_\alpha^- \,,\; a_\alpha^- . b_\alpha^+ , a_\alpha^+ . b_\alpha^- \,,\; a_\alpha^+ . b_\alpha^+)] \tag{11}$$

It is worth mentioning that the amount of calculation with the help of these expressions is the minimum value which can be achieved.

FUZZY RANDOM VARIABLES

The notion of a fuzzy random variable will be introduced as follow (Kwakernaak [6]) . Let (Ω, F, P) be a probability triple. Suppose that U is a random variable defined on this triple. Assume now that we perceive this random variable through a set of windows $W_i, i \in J$, with J a finite or countable set, each representing an interval of the real line, such that $W_i \bigcap W_j = \varnothing$ for $i \neq j$ and $\bigcup_{i \in J} W_i = R$. Perceiving the random variable through these windows means that for each ω we can only establish whether $U(\omega) \in W_i$ for some $i \in J$.

Let us define the function $I_i : R \to [0,1]$ as the characteristic function of the set W_i. Also let S be the space of all piecewise continues functions mapping $R \to [0,1]$. we then define the perception of the random variable U , as described above, as the mapping $X : \Omega \to S$ given by

$$\omega \xrightarrow{X} X_\omega \tag{12}$$

with $X_\omega = I_i$ if and only if $U(\omega) \in W_i$. This means that we associate with each $\omega \in \Omega$ not a real number $U(\omega)$, as in the case of ordinary random variable, but a characteristic function X_ω which is an element of S .

The map $X : \Omega \to S$ described above characterizes a special type of fuzzy random variable. The random variable U , of which this fuzzy random variable is a perception, is called an *original* of the fuzzy random variable.

At this point we generalize and define a fuzzy random variable as a map $\xi : \Omega \to F$, where F is a set of all fuzzy numbers. Denote the image of ω in F under ξ as $\xi(\omega) = (R, X_\omega, a_\omega)$ with $X_w \in S$ and $a_\omega : R \to P$. The map $X : \Omega \to S$ and $a_\omega : R \to P$. The map $X : \Omega \to S$ specified by

$$\omega \xrightarrow{X} X_\omega$$

is required to be such that for each $\mu \in (0,1]$ both U_μ^-, U_μ^+, defined by

$$U_{\mu}^{-}(\omega)=\inf\{x\in R|X_{\omega}(x)\geq\mu\}$$
$$U_{\mu}^{+}(\omega)=\sup\{x\in R|X_{\omega}(x)\geq\mu\} \quad (13)$$

are finite real-valued random variables defined on (Ω, F, P) satisfying

$$(\forall\omega\in\Omega) \qquad X_{\omega}(U_{\mu}^{-}(\omega))\geq\mu, \qquad X_{\omega}(U_{\mu}^{+}(\omega))\geq\mu \quad (14)$$

For each $\omega\in\Omega$ and each $x\in R$, $a_{\omega}(x)$ is the statement

$$a_{\omega}(x)=(\textit{the original assumes the value x at the po}\operatorname{int}\omega) \quad (15)$$

While fuzzy random variable U is broken to its α-cut random variable $U_{\mu}^{-}(\omega), U_{\mu}^{+}(\omega)$, it is possible to perform different calculation using Vertex Method or the algebra of interval numbers, which were described in the previous section. It is possible to summarize this method in the following steps:

Step1. Choose a level $\mu\in[0,1]$.
Step2. Determine $U_{\mu}^{-}(\omega), U_{\mu}^{+}(\omega)$ from (13).
Step3. Calculate desired relationship $f_{\mu}^{-}(\omega; U_{\mu}^{-}, U_{\mu}^{+}), f_{\mu}^{+}(\omega; U_{\mu}^{-}, U_{\mu}^{+})$ using Vertex Method (4) or algebra of interval numbers (5~11).
Step4: Repeat steps 1 through 3 for sufficient number of values of μ

Using this procedure, in the next section the probabilistic fuzzy response of a SDOF system is calculated.

CALCULATION OF PROBABILISTIC FUZZY RESPONSE SPECTRA

In this section, the probabilistic fuzzy response spectrum of earthquake is calculated. For this purpose, it is assumed that the earthquake excitation is stationary Gaussian process. Consider a SDOF specified by the equation (16).

$$m\ddot{q}+c\dot{q}+kq=p \quad (16)$$

if $p(t), q(t)$ assumed to be stationary random processes with $S_p(\Omega), S_q(\Omega)$ as their power spectrum density(PSD) function respectively, it is possible to express the power spectrum density of response, $S_q(\Omega)$ as a function of excitation PSD, $S_p(\Omega)$ and system parameters as follow:

$$S_q(\Omega)=\frac{S_x(\Omega)}{(\omega^2-\Omega^2)^2+4\xi^2\omega^2\Omega^2} \quad (17)$$

where $\omega=\sqrt{\frac{k}{m}}$ is the natural frequency of system and $\xi=\frac{c}{2\sqrt{km}}$ is the damping ratio.

To obtain the spectrum, it is essential to calculate the extreme values of the process $S_q(\Omega)$. It is assumed that the probability distribution of extremes be Poisson. As a result, displacement, velocity and acceleration spectra can be calculated as follow (Pfaffinger [7]):

$$S_d(\omega,\xi)=\sigma_q(\sqrt{2\ln(2f_q.T_q)}+\frac{\gamma}{\sqrt{2\ln(2f_q.T_q)}} \tag{18}$$

$$S_v(\omega,\xi)=\omega\sqrt{1-\xi^2}S_d(\omega,\xi)=\omega\sqrt{1-\xi^2}\sigma_q(\sqrt{2\ln(2f_q.T_q)}+\frac{\gamma}{\sqrt{2\ln(2f_q.T_q)}}) \tag{19}$$

$$S_a(\omega,\xi)=\omega^2\sqrt{1-\xi^2}S_d(\omega,\xi)=\omega^2\sqrt{1-\xi^2}\sigma_q(\sqrt{2\ln(2f_q.T_q)}+\frac{\gamma}{\sqrt{2\ln(2f_q.T_q)}}) \tag{20}$$

where T_q is the duration of response q and

$$\sigma_q^2=\frac{1}{2\pi}\int_0^\infty S_q(\Omega)d\Omega \tag{21}$$

$$f_q=\frac{1}{2\pi}[\frac{\int_0^\infty \Omega^2.S_q(\Omega)d\Omega}{\int_0^\infty S_q(\Omega)d\Omega}]^{\frac{1}{2}} \tag{22}$$

Equation 18 to 22 represent the required relationships for calculating the probabilistic response spectrum of a single degree of freedom system. However, the involving parameters are not crisp values. In the other word, ω,ξ are fuzzy numbers. Thus, it is necessary to study the influence of different parameters' fuzziness on the response spectrum of the system.

Parametric Study of Probabilistic Fuzzy Response Spectra

After investigating sources of different parameters' fuzziness, it is necessary to present such uncertainties through different membership functions.

It must be noticed that although fuzziness, by itself, is objective but presentation of this objectivity through membership function is subjective. It means that there is not a deterministic rule for determination of membership function. Fortunately, in fuzzy mathematics, the results are not sensitive to the details of different membership functions. This is because in fuzzy arithmetic, the $\alpha-$ cut sets are of great importance and different membership functions that have identical $\alpha-$ cut sets have identical influence on the final results.

On the other hand, determination the range of variation of different membership functions is very important. This determination is mainly done from our knowledge of different parameters' accuracy. For example, the range of variation of geotechnical parameters of soil is clear according to different experiments. There is a range of highest possibility and a range of least possibility. The most common and simplest shape for representing such variations is trapezoid. For this reason, in this paper, we use trapezoidal membership function for the calculations.

For concluding remark of this section, it must mention that there are different methods for constructing membership functions that are introduced in fuzzy literature and most of them are based on *Fuzzy Statistics.* Some of these methods are presented by (Li, Chen, Hung [5]).

In figure 2, the common shape of membership function which is used in this paper to calculate response spectrum is shown.

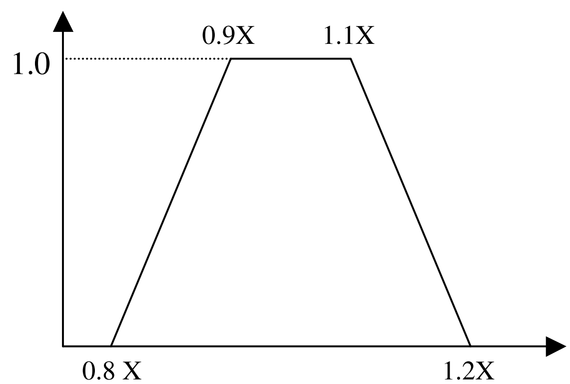

Fig 2: Membership Function used in this paper to calculate probabilistic fuzzy respond spectrum of SDOF

Using this membership function and the procedure described in previous section, it is possible to calculate probabilistic fuzzy response spectrum of earthquake. In figures 3-5, the probabilistic fuzzy response spectrum of 16/09/1978 Tabas earthquake with Mw=7.4 is calculated and the influence of different parameters' fuzziness is presented.

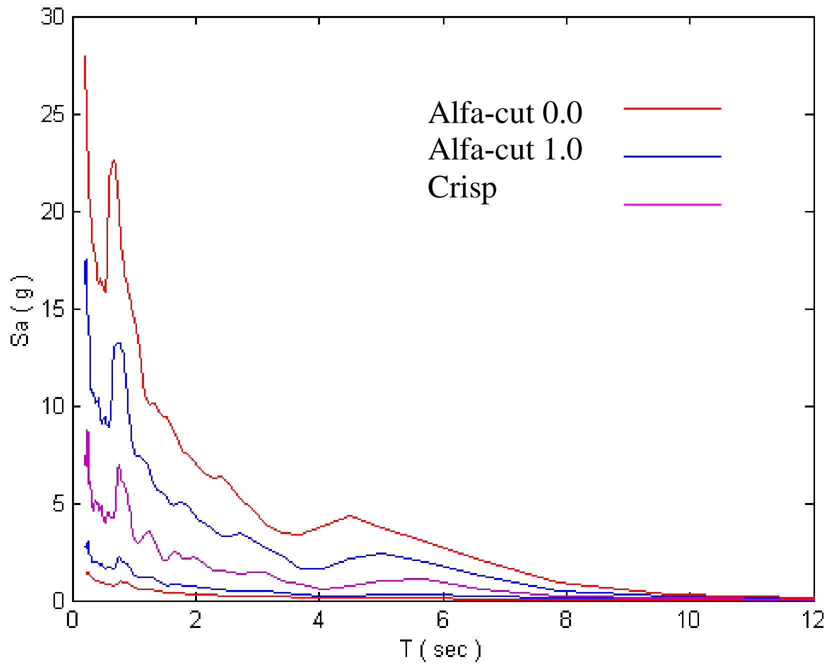

Fig3: Probabilistic fuzzy response spectrum of Tabas earthquake with ω as fuzzy number

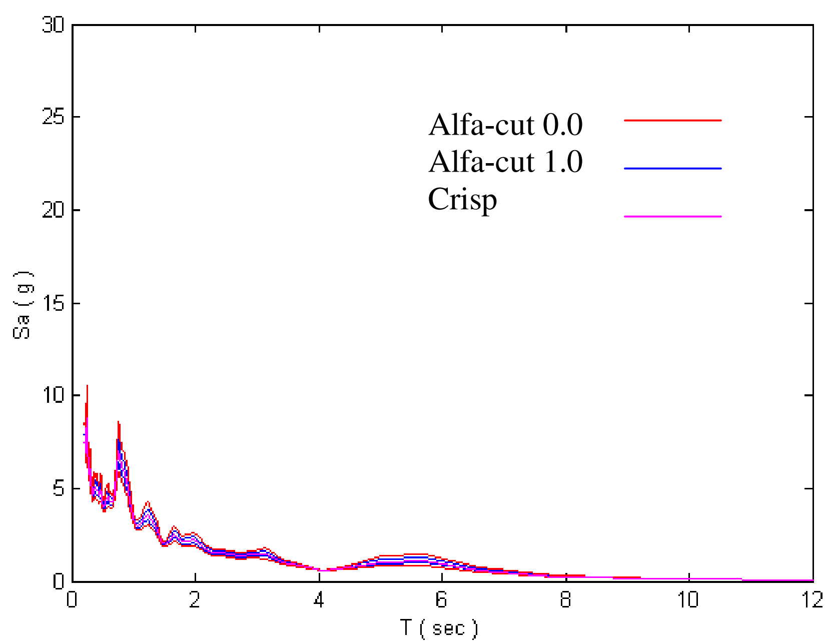

Fig4: Probabilistic fuzzy response spectrum of Tabas earthquake with ξ as fuzzy number

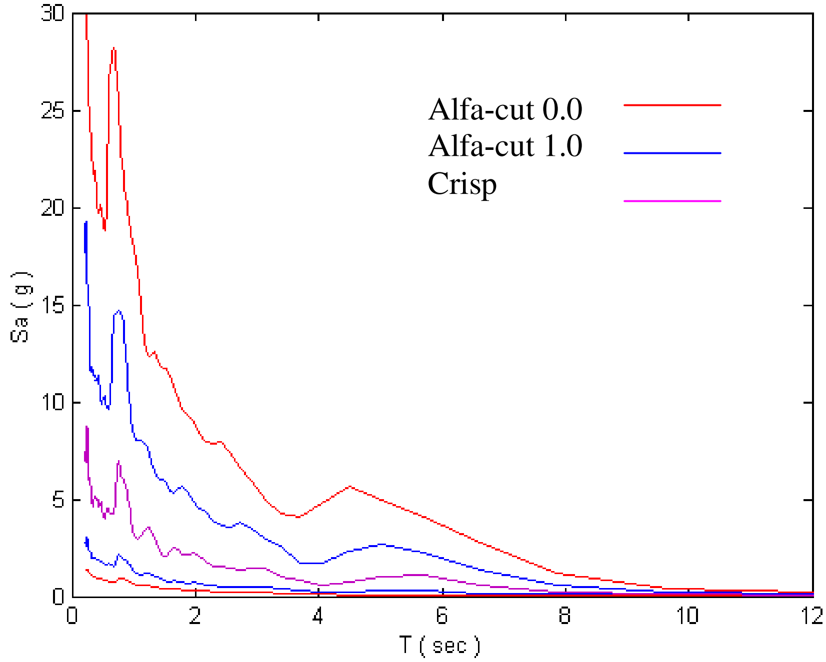

Fig5: Probabilistic fuzzy response spectrum of Tabas earthquake with ω, ξ as fuzzy numbers

The effect of natural frequency vagueness

As it is shown throughout these figures, the fuzziness of the natural frequency mainly affects the high frequency portion of the response spectra. In other words, the fuzziness of the stiffness and the mass are more important for the structures of short period. This fact is very important in the case of establishing seismic codes, for in these codes, determination of natural frequency of the structures is performed roughly and this vagueness of natural frequency has a great influence on the response of the structure.

The effect of damping vagueness

In comparison with the effect of natural frequency vagueness, the response spectrum has less sensitivity due to damping fuzziness. This means that approximate estimation of damping ratio does not affect the response of the structures a lot.

In figure 5, the combined effect of natural frequency and damping ratio fuzziness is shown.

CONCLUSIONS

In this paper, the mathematical framework for considering two sources of uncertainty, that are randomness and fuzziness, in the calculation of response spectrum is presented. The parametric study of the results shows that the effect of natural frequency fuzziness is very important. On the contrary, the effect of the damping ratio vagueness is not as important.

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