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.

# DAMAGE DETECTION OF REDUCED ORDER SYSTEMS

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## SUMMARY

This paper provides new approaches to detect damage in reduced order systems which are insufficiently instrumented. In the framework of a state space formulation the paper will show that the use of the strain energy allows to define damage indexes able to provide information about damage to all the degrees of freedom with and without instrumentations. The key points of the approaches will be discussed with reference to the case of concentrated forces and seismic action. The efficacy of the approach will be showed by means of numerical applications. The effect of the noise will be finally investigated.

## INTRODUCTION

Reduced order systems are characterised by an incomplete set of input/output measurements. This means that some degrees of freedom are deficient of either a sensor or an actuator and, consequently, the mass, stiffness and damping matrices cannot be exactly identified; only the components of these matrices referred to the measured degrees of freedom will be known while the others components, those connected to the unmeasured degrees of freedom, will be dependent on a set of unknown factors. Some of the first studies on such systems are reported in [1] and [2] where the authors have developed a new methodology to identify the physical parameters of the second order model by using the solution of a symmetric complex eigenvalue problem starting from a mixed complete set of measurements. The minimum requirement for the methodology is that all the degrees of freedom should contain either a sensor or an actuator with at least one co-located sensor-actuator pair. An extension of the study to systems with some degrees of freedom without any measurements is reported in [3].

For reduced order systems the classical methods proposed for damage detection [4], [5] generally become inefficient and new approaches are required. In particular, the use of a strain energy-based approach allows, in some special circumstances, to provide information on damage in terms of both location and values. The present paper aims to extend the strain energy approach [3], [4] to the case of reduced order systems where the stiffness matrices before and after damage are not exactly identified. In particular the paper discusses two new approaches which are based on analysing changes of the strain energy of the entire system. The applicability of the approaches is referred to shear-type systems which are widely used

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in dynamical structural analyses. The result is that the use of the strain energy allows to locate and characterize the damage in each part of the systems included those which are not instrumented. The theoretical discussion is, first, referred to the case of concentrated time-dependent forces but it is, then, extended to the case of seismic action. The proposed approaches are compared and discussed by means of numerical examples which are developed in the case of noise-free and noise-polluted output. A remarkable result consists in the efficacy of the approach also in the presence of noise which makes the proposed methodology a valuable tool in locating and quantifying the damage in each parts of real structural systems with a reduced structural monitoring and a significant safe in terms of sensors and/or actuators.

## DYNAMICS OF LINEAR STRUCTURAL SYSTEMS

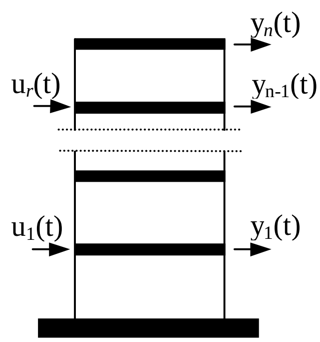


Figure 1- *N* degree-of-freedom structural system with concentrated forces

Let us consider an *N* degree-of-freedom structural system whose equations of motion are:

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{D}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{B}\mathbf{u}(t) \tag{1}$$

where $\mathbf{q}$(t) represents the $N\times 1$ vector of the nodal displacement and ( $\dot{}$ ) and ( $\ddot{}$ ) express its first and second derivatives with respect to time, $\mathbf{u}$(t) is the $r\times 1$ input vector containing the *r* external excitation and $\mathbf{B}$ is the $N\times r$ input matrix. $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are the $N\times N$ mass, damping and stiffness matrices of the system. The output equations can be expressed as:

$$\mathbf{y}(t)=\mathbf{C}_d\mathbf{q}(t) \tag{2}$$

where $\mathbf{y}$(t) is the $m\times 1$ output vector containing the time histories of the displacement measurements and $\mathbf{C}_d$ is the $m\times N$ output matrix which relates the *m* displacements to the nodal quantities. Similar expressions can be written for velocities and accelerations.

By introducing the state vector $\mathbf{z}(t)=\left[\mathbf{q}(t)^T \quad \dot{\mathbf{q}}(t)^T\right]^T$, equations (1) and (2) can be conveniently rewritten into a system of first-order differential equations as follows:

$$\begin{bmatrix}\mathbf{D} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}\dot{\mathbf{z}}(t)+\begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & -\mathbf{M}\end{bmatrix}\mathbf{z}(t)=\begin{bmatrix}\mathbf{B}\\ \mathbf{0}\end{bmatrix}\mathbf{u}(t) \tag{3a}$$

$$\mathbf{y}(t)=\left[\mathbf{C}_d \quad \mathbf{0}\right]\mathbf{z}(t) \tag{3b}$$

The equations (3) are the symmetric first-order state space representation of the dynamics of the system (1). The solution of the complex eigenvector problem associated with (3a) provides the complex eigenvalues and eigenvectors $\lambda_i$ e $\psi_i$, with i = 1, 2, …, 2N. The eigenvectors can be normalized according to the relation:

$$\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi\Lambda} \end{bmatrix}^T \begin{bmatrix} \mathbf{D} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi\Lambda} \end{bmatrix} = \mathbf{I} \qquad \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi\Lambda} \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi\Lambda} \end{bmatrix} = -\boldsymbol{\Lambda} \tag{4}$$

where $\Lambda$ is the 2N×2N diagonal matrix containing the eigenvalues and $\boldsymbol{\psi} = \left[\boldsymbol{\psi}_1 \ \boldsymbol{\psi}_2 \cdots \boldsymbol{\psi}_{2N}\right]$ is the *N×2N* matrix containing the eigenvectors. The normalization condition in (4) implies that, for the case of proportional damping, the eigenvectors will have the absolute values of the real and imaginary part identical. Using the transformation $\mathbf{z}(t) = \left[\boldsymbol{\psi}^T \ \left(\boldsymbol{\psi\Lambda}\right)^T\right]^T \xi(t)$ and pre-multiplied the eqn. (3a) by $\left[\boldsymbol{\psi}^T \ \left(\boldsymbol{\psi\Lambda}\right)^T\right]^T$, the equations (3) can be rewritten in modal coordinates as follows:

$$\dot{\xi}(t) = \boldsymbol{\Lambda}\xi(t) + \boldsymbol{\psi}^T\mathbf{B}\mathbf{u}(t) = \boldsymbol{\Lambda}\xi(t) + \mathbf{B}_u\mathbf{u}(t) \tag{5a}$$

$$\mathbf{y}(t) = \mathbf{C}_s\boldsymbol{\psi}\xi(t) = \mathbf{C}_y\xi(t) \tag{5b}$$

Assuming that the *j*-dof is collocated with one co-located sensor-actuator pair, it results:

$$\mathbf{B}_u(:,j)^T = \mathbf{C}_y(j,:) \ \Rightarrow \ \left[\boldsymbol{\psi}^T\mathbf{B}(:,j)\right]^T = \mathbf{C}_d(j,:)\boldsymbol{\psi} \tag{6}$$

being $\mathbf{B}(:,j)^T = \mathbf{C}_d(j,:)$.
The normalization condition allows to express the mass, damping and stiffness matrices as function of the complex eigenvalues and eigenvectors as:

$$\mathbf{M} = \left(\boldsymbol{\psi\Lambda\psi}^T\right)^{-1}, \mathbf{K} = -\left(\boldsymbol{\psi\Lambda}^{-1}\boldsymbol{\psi}^T\right)^{-1}, \mathbf{D} = -\mathbf{M}\boldsymbol{\psi\Lambda}^2\boldsymbol{\psi}^T\mathbf{M} \tag{7}$$

with the eigenvectors satisfying the condition $\boldsymbol{\psi\psi}^T = \mathbf{0}$.
The assumption of proportional damping allows to express the mass normalized real eigenvectors, $\varphi$, in function of the complex eigenvectors, $\boldsymbol{\psi}$, as follows:

$$\varphi = \mathrm{real}\left(\tilde{\boldsymbol{\psi}}\right) \rightarrow \tilde{\boldsymbol{\psi}} = \boldsymbol{\psi}\left(\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^*\right)^{0.5} \tag{8}$$

## "FULL" AND "REDUCED" ORDER MODELS OF LINEAR STRUCTURAL SYSTEMS

Given the dynamics of the system by means of the second-order equations (1), it is straightforward and easy to obtain the first-order state space representation of the system provided by equations (3). On the other hand, the so called *inverse vibration problem* which derives the physical matrices **M**, **D** and **K** of the system from identified complex model data is more complex but also more common. In particular, following the work of [2], the first step is to identify the first-order state space representation of the system from general input/output data using the algorithm ERA/OKID [8] which provides the identified complex eigenvalues and eigenvectors. If the system has at least one co-located degree of freedom the complex eigenvectors can be normalized on the basis of the expressions (4) by using the equation (6). Finally the second order matrices can be derived by using the (7).

A "full order" problem is defined when each DOF is measured by a sensor and/or an actuator. The most common case studied in literature [9], [10] considers either a complete set of actuators or a complete set of sensors. Clearly, such condition constitutes a strong requirement because it involves a large amount of measurements. A more general case is analysed in [2] where each degree of freedom is instrumented by either a sensor or an actuator with only one co-located sensor-actuator pair, that is $m + r = N + 1$. In all the above cases the complex eigenvectors $\hat{\boldsymbol{\psi}}$ and the matrices of the second order model of the system ($\hat{\mathbf{M}}$, $\hat{\mathbf{D}}$ and $\hat{\mathbf{K}}$) are fully identified.

On the contrary, when the structural system is insufficiently instrumented, being some degrees of freedom deficient of either a sensor or an actuator, the problem becomes a "reduced order" one with $m + r < N + 1$. In this case the complex eigenvectors and the second order matrices cannot be fully retrieved.

In particular, by using the $m + r$ available input/output data with one co-located sensor-actuator pair and defining $n = m + r - 1$, only $n$ components of the complex eigenvector matrix $\hat{\boldsymbol{\psi}}$ can be identified while the remaining $p$ (with $p=N-n$) components of the matrix, remain unknown. Then, the eigenvector matrix is partitioned as follows:

$$\hat{\boldsymbol{\psi}} = \begin{bmatrix} \hat{\boldsymbol{\psi}}_1 \\ \hat{\boldsymbol{\psi}}_2 \end{bmatrix} \tag{9}$$

being $\hat{\boldsymbol{\psi}}_1$ the known partition of $\hat{\boldsymbol{\psi}}$ of dimension $n \times 2N$ and $\hat{\boldsymbol{\psi}}_2$ the unknown one of dimension $p \times 2N$. Then starting from equations (7) and using equation (9), the general form of the identified mass and stiffness matrices can be expressed in partitioned forms as functions of $\hat{\boldsymbol{\psi}}_1$ and $\hat{\boldsymbol{\psi}}_2$ as follows:

$$\hat{\mathbf{M}} = \left(\hat{\boldsymbol{\psi}}\Lambda\hat{\boldsymbol{\psi}}^{\mathrm{T}}\right)^{-1} = \left[\begin{array}{c|c} \hat{\boldsymbol{\psi}}_1\boldsymbol{\Lambda}\hat{\boldsymbol{\psi}}_1^{\mathrm{T}} & \hat{\boldsymbol{\psi}}_1\boldsymbol{\Lambda}\hat{\boldsymbol{\psi}}_2^{\mathrm{T}} \\ \hline \hat{\boldsymbol{\psi}}_2\boldsymbol{\Lambda}\hat{\boldsymbol{\psi}}_1^{\mathrm{T}} & \hat{\boldsymbol{\psi}}_2\boldsymbol{\Lambda}\hat{\boldsymbol{\psi}}_2^{\mathrm{T}} \end{array}\right]^{-1} \tag{10}$$

$$\hat{\mathbf{K}} = -\left(\hat{\boldsymbol{\psi}}\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\psi}}^{\mathrm{T}}\right)^{-1} = -\left[\begin{array}{c|c} \hat{\boldsymbol{\psi}}_1\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\psi}}_1^{\mathrm{T}} & \hat{\boldsymbol{\psi}}_1\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\psi}}_2^{\mathrm{T}} \\ \hline \hat{\boldsymbol{\psi}}_2\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\psi}}_1^{\mathrm{T}} & \hat{\boldsymbol{\psi}}_2\boldsymbol{\Lambda}^{-1}\hat{\boldsymbol{\psi}}_2^{\mathrm{T}} \end{array}\right]^{-1} \tag{11}$$

Of these two matrices, only the upper left portion will be uniquely determined where the other three partitions contain undetermined factors.

**Shear-type systems: the case of tri-diagonal stiffness matrix**

In some structural systems, it is possible to make the assumption that the mass matrix is diagonal and the stiffness and damping matrices are tri-diagonal. For these systems the expressions (10) and (11) can be simplified. In fact, it can be shown [3] that the unknown part $\hat{\boldsymbol{\psi}}_2$ of the identified complex eigenvector matrix contains only one undetermined factor $\alpha_i$ with $i = 1, 2, \ldots, p$ for each row. That means:

$$\hat{\boldsymbol{\psi}}_1 = \boldsymbol{\psi}_1 ; \qquad \hat{\boldsymbol{\psi}}_2 = \begin{bmatrix} \alpha_1 \boldsymbol{\psi}_2(1,:) \\ \alpha_2 \boldsymbol{\psi}_2(2,:) \\ \cdots \\ \alpha_p \boldsymbol{\psi}_2(p,:) \end{bmatrix} \tag{12}$$

referring $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ as the simulated parts of the eigenvector matrix.
Similarly, the mass-normalized eigenvector matrix $\hat{\varphi}$ can be expressed as follows:

$$\hat{\varphi} = \begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \end{bmatrix}; \quad \hat{\varphi}_1 = \varphi_1 ; \quad \hat{\varphi}_2 = \begin{bmatrix} \alpha_1 \varphi_2(1,:) \\ \alpha_2 \varphi_2(2,:) \\ \cdots \\ \alpha_p \varphi_2(p,:) \end{bmatrix} \tag{13}$$

Then, the identified “full order” mass and stiffness matrices become:

$$\hat{\mathbf{M}} = \left(\hat{\psi}\Lambda\hat{\psi}^{T}\right)^{-1} = \left[\begin{array}{c|c} \hat{\mathbf{M}}_{11} & \mathbf{0} \\ \hline \mathbf{0} & \hat{\mathbf{M}}_{22} \end{array}\right] = \left[\begin{array}{c|ccc} \mathbf{M}_{11} & 0 & \cdot\cdot & 0 \\ \hline 0 & (\alpha_1\alpha_1)^{-1} M_{22} & \cdot\cdot & 0 \\ : & : & : & : \\ 0 & 0 & \cdot\cdot & (\alpha_{N-1}\alpha_{N-1})^{-1} M_{NN} \end{array}\right] \tag{14}$$

$$\hat{\mathbf{K}} = -\left(\hat{\psi}\Lambda^{-1}\hat{\psi}^{T}\right)^{-1} = \left[\begin{array}{c|c} \hat{\mathbf{K}}_{11} & \hat{\mathbf{K}}_{12} \\ \hline \hat{\mathbf{K}}_{21} & \hat{\mathbf{K}}_{22} \end{array}\right] = \left[\begin{array}{c|ccc} \mathbf{K}_{11} & \alpha_1^{-1} K_{12} & \cdot\cdot & 0 \\ \hline \alpha_1^{-1} K_{21} & (\alpha_1\alpha_1)^{-1} K_{22} & \cdot\cdot & 0 \\ : & : & : & : \\ 0 & 0 & \cdot\cdot & (\alpha_{N-1}\alpha_{N-1})^{-1} K_{NN} \end{array}\right] \tag{15}$$

being $\hat{\mathbf{M}}_{11} = \mathbf{M}_{11}$ and $\hat{\mathbf{K}}_{11} = \mathbf{K}_{11}$ the known partitions of the mass and stiffness matrices while $\hat{\mathbf{M}}_{22}$, $\hat{\mathbf{K}}_{12}$ and $\hat{\mathbf{K}}_{22}$ are the unknown parts of them. For the details about these derivations, the reader is referred to the work by Yu [3].

## DETECTION OF DAMAGE: NEW STRAIN ENERGY - BASED APPROACHES

Damage in civil engineering structures may generally alter significantly the stiffness and the modal parameters but not the mass of the system. Therefore we assume that the mass matrix remains constant before and after damage, i.e. $\mathbf{M} = \mathbf{M}^d$, while the stiffness matrices before the damage ($\mathbf{K}$) and after damage ($\mathbf{K}^d$) are different. Obviously, in the case of “reduced order models”, since the identified stiffness matrix depends on undetermined factors, the comparison between the identified matrices $\hat{\mathbf{K}}$ and $\hat{\mathbf{K}}^d$ does not provide any information about damage at the missed degrees of freedom and, hence, alternative approaches to detect damage are needed. In what follows a strain energy based approach is discussed. In particular we initially try to locate damage by investigating the changes in the modal strain energy. Then, we consider the changes in terms of the displacement-based-strain energy. The procedure is applied to systems with tri-diagonal sparse stiffness and damping matrices.

**The modal strain energy approach**

Starting from the identified mass, stiffness and damping matrices $\hat{\mathbf{M}}$, $\hat{\mathbf{K}}$, $\hat{\mathbf{D}}$ and the mass-normalized eigenvector matrix $\hat{\varphi}$ of a reduced order system, the modal strain energy stored in the mass-normalized mode shapes is:

$$\hat{U} = \sum_{r=1}^{n} \hat{U}_r = \frac{1}{2}\sum_{r=1}^{n} \hat{\varphi}(:,r)^T \hat{K} \hat{\varphi}(:,r) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \hat{K}_{ij} \sum_{r=1}^{n} \hat{\varphi}(i,r)\hat{\varphi}(j,r) \tag{16}$$

Considering that:

$$\sum_{r=1}^{n} \hat{\varphi}(i,r)\hat{\varphi}(j,r) = \hat{M}_{ij}^{-1} \tag{17}$$

the expression (16) can be considered a mass-normalized modal strain energy and, hence, can be expressed by the following form:

$$\hat{U} = \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{U}_{ij} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \hat{K}_{ij}\hat{M}_{ij}^{-1} \tag{18}$$

Analogous results are obtained for the identified system with damage whose modal strain energy $\hat{U}^d$ will assume the expression:

$$\hat{U}^{(d)} = \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{U}_{ij}^{(d)} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \hat{K}_{ij}^{(d)} (\hat{M}_{ij}^{(d)})^{-1} \tag{19}$$

It is noteworthy that, whatever is the number of the missing coordinates in the reduced order system, the modal strain energy (16) will not depend on the undetermined factors $\alpha_k$ which cancel out because of the expressions (13) and (14). Analogous considerations are valid for the strain energy of the system after damage which will be independent on the damaged factors $\alpha_k^d$.

Now, with reference to each term of the modal strain energy expressions $\hat{U}$ and $\hat{U}^d$, we introduce a *modal damage index* defined as:

$$\beta_{ij} = \frac{\hat{U}_{ij}^{(d)}}{\hat{U}_{ij}} = \frac{\hat{K}_{ij}^{(d)} \sum_{r=1}^{n} \hat{\varphi}^{(d)}(i,r)\hat{\varphi}^{(d)}(j,r)}{\hat{K}_{ij} \sum_{r=1}^{n} \hat{\varphi}(i,r)\hat{\varphi}(j,r)} \qquad i,j = 1.....N \tag{20}$$

which is still independent on the undetermined factors $\alpha_k$ and $\alpha_k^d$.

By using the equation (18) we can observe that for $i \neq j$ the terms $\beta_{ij}$ are zero while for i = j the terms $\beta_{ii}$ are equal to the changes occurred in the diagonal terms of the stiffness matrices of the configuration before and after damage.

In fact, considering first the elements which do not connect the missing degrees of freedom, the identified stiffness and mass terms equal the simulated ones, i.e. $\hat{K}_{ii} = K_{ii}$, $\hat{K}_{ii}^d = K_{ii}^d$, $\hat{M}_{ii} = M_{ii}$, $\hat{M}_{ii}^d = M_{ii}^d$. Having assumed that the mass remains constant, i.e. $M_{ii}^d = M_{ii}$, it implies that:

$$\beta_{ii} = \frac{\hat{U}_{ii}^{(d)}}{\hat{U}_{ii}} = \frac{\hat{K}_{ii}^{d}(\hat{M}_{ii}^{d})^{-1}}{\hat{K}_{ii}\hat{M}_{ii}^{-1}} = \frac{K_{ii}^{d}(M_{ii}^{d})^{-1}}{K_{ii}M_{ii}^{-1}} = \frac{K_{ii}^{d}}{K_{ii}} \tag{21}$$

In the case of elements connecting the missing degrees of freedom the identified stiffness and mass terms are related to the simulated ones through the expressions (14) and (15). Thus, being $M_{ii}^{d} = M_{ii}$, it results:

$$\beta_{ii} = \frac{\hat{U}_{ii}^{(d)}}{\hat{U}_{ii}} = \frac{\hat{K}_{ii}^{d}\left(\hat{M}_{ii}^{d}\right)^{-1}}{\hat{K}_{ii}\hat{M}_{ii}^{-1}} = \frac{K_{ii}^{d}\left(\alpha_{i-1}^{d}\alpha_{i-1}^{d}\right)^{-1}\dfrac{1}{M_{ii}^{d}\left(\alpha_{i-1}^{d}\alpha_{i-1}^{d}\right)^{-1}}}{K_{ii}\left(\alpha_{i-1}\alpha_{i-1}\right)^{-1}\dfrac{1}{M_{ii}\left(\alpha_{i-1}\alpha_{i-1}\right)^{-1}}} = \frac{K_{ii}^{d}\dfrac{1}{M_{ii}^{d}}}{K_{ii}\dfrac{1}{M_{ii}}} = \frac{K_{ii}^{d}}{K_{ii}} \tag{22}$$

showing that the index $\beta_{ii}$ is an indicator of the changes occurred at the diagonal elements of the stiffness matrix due to damage.

**The displacement-based-strain energy approach**

The modal strain energy approach is effective in locating and quantifying the damage in the system only with reference to the diagonal terms while it does not provide any information about the off-diagonal terms which result zero. To add more information to the off-diagonal terms, an alternative approach based on a displacement-based-strain energy is suggested.

The total strain energy stored in the reduced identified model of the system before damage when subjected to the “identified” global displacement vector $\hat{x}$ is given by:

$$\hat{U} = \frac{1}{2}\sum_{i}\sum_{j} x_i^{T}\hat{K}_{ij}\hat{x}_j \tag{23}$$

The “identified” global displacements vector contains the “simulated” displacements obtained using the identified reduced-order model subjected to an input excitation.

For the reduced identified system after damage, the strain energy is defined by:

$$\hat{U}^{d} = \frac{1}{2}\sum_{i}\sum_{j} x_i^{T}\hat{K}_{ij}^{d}\hat{x}_j \tag{24}$$

Considering the transformation:

$$\mathbf{x} = \mathbf{M}^{-0.5}\mathbf{q} \tag{25}$$

the strain energy can be conveniently rewritten into a mass normalized expression as follows:

$$\hat{U} = \sum_{i=1}^{n}\sum_{j=1}^{n}\hat{U}_{ij} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\hat{M}_{ii}^{-0.5}\hat{K}_{ij}\hat{M}_{jj}^{-0.5} \tag{26a}$$

for the system before damage, and, similarly:

$$\hat{U}^{(d)} = \sum_{i=1}^{n}\sum_{j=1}^{n}\hat{U}_{ij}^{(d)} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\hat{M}_{ii}^{(d)-0.5}\hat{K}_{ij}^{(d)}\hat{M}_{jj}^{(d)-0.5} \tag{26b}$$

for the system after damage.
For each term of the modal strain energy expressions (26a and b), it is possible to define a *displacement-based-damage index* as:

$$d_{ij} = \frac{\hat{U}_{ij}^{(d)}}{\hat{U}_{ij}} = \frac{\sum_i\sum_j x_i^T\hat{K}_{ij}^{(d)}\hat{x}_j}{\sum_i\sum_j x_i^T\hat{K}_{ij}\hat{x}_j} = \frac{\hat{K}_{ij}^{(d)}}{\hat{K}_{ij}}\cdot\sqrt{\frac{\hat{M}_{ii}\hat{M}_{jj}}{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}} \qquad i,j=1.....N \tag{27}$$

which allows to estimate the amount of damage also between the i-th and j-th degree of freedom.
The statement is briefly presented for the elements of the system which do not connect the missing degrees of freedom having the identified stiffness and mass terms equal the simulated ones, i.e. $\hat{K}_{ij} = K_{ij}$, $\hat{K}_{ij}^{d} = K_{ij}^{d}$, $\hat{M}_{ij} = M_{ij}$ and $\hat{M}_{ij}^{d} = M_{ij}^{d}$. In this case, for the undamaged elements, being $K_{ij}^{d} = K_{ij}$ and the mass terms constant, the displacement-based-damage index is given by:

$$d_{ij} = \frac{\hat{K}_{ij}^{d}}{\hat{K}_{ij}}\cdot\sqrt{\frac{\hat{M}_{ii}\hat{M}_{jj}}{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}} = \frac{K_{ij}^{d}}{K_{ij}}\cdot\sqrt{\frac{M_{ii}M_{jj}}{M_{ii}M_{jj}}} = 1 \qquad i,j=1.....N \tag{28}$$

For the damaged elements, the index is given by:

$$d_{ij} = \frac{\hat{K}_{ij}^{d}}{\hat{K}_{ij}}\cdot\sqrt{\frac{\hat{M}_{ii}\hat{M}_{jj}}{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}} = \frac{K_{ij}^{d}}{K_{ij}}\cdot\sqrt{\frac{M_{ii}M_{jj}}{M_{ii}^{d}M_{jj}^{d}}} = \frac{K_{ij}^{d}}{K_{ij}} \qquad i,j=1.....N \tag{29}$$

being the mass terms of these elements affecting only by the damage and not by the undetermined factors $\alpha_k$ and $\alpha_k^d$.
Different considerations require the elements of the system which connect the missing degrees of freedom. Then, for the undamaged elements characterized by $K_{ij}^{d} = K_{ij}$, the index is given by:

$$d_{ij} = \frac{\hat{K}_{ij}^{d}}{\hat{K}_{ij}}\cdot\sqrt{\frac{\hat{M}_{ii}\hat{M}_{jj}}{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}} = \frac{K_{ij}^{d}}{K_{ij}}\frac{\left(\alpha_{i-1}^{d}\cdot\alpha_{j-1}^{d}\right)^{-1}}{\left(\alpha_{i-1}\cdot\alpha_{j-1}\right)^{-1}}\sqrt{\frac{\hat{M}_{ii}\hat{M}_{jj}}{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}} = \frac{\left(\alpha_{i-1}^{d}\cdot\alpha_{j-1}^{d}\right)^{-1}}{\left(\alpha_{i-1}\cdot\alpha_{j-1}\right)^{-1}}\sqrt{\frac{\hat{M}_{ii}\hat{M}_{jj}}{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}} = \sqrt{\frac{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}{\hat{M}_{ii}\hat{M}_{jj}}}\sqrt{\frac{\hat{M}_{ii}\hat{M}_{jj}}{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}} = 1 \tag{30}$$

For the damaged elements, characterized by $K_{ij} \neq K_{ij}^{d}$ , the index is equal to:

$$d_{ij} = \frac{\hat{K}_{ij}^{d}}{\hat{K}_{ij}}\cdot\sqrt{\frac{\hat{M}_{ii}\hat{M}_{jj}}{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}} = \frac{K_{ij}^{d}}{K_{ij}}\frac{\left(\alpha_{i-1}^{d}\cdot\alpha_{j-1}^{d}\right)^{-1}}{\left(\alpha_{i-1}\cdot\alpha_{j-1}\right)^{-1}}\sqrt{\frac{\hat{M}_{ii}\hat{M}_{jj}}{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}} = \frac{K_{ij}^{d}}{K_{ij}}\sqrt{\frac{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}{\hat{M}_{ii}\hat{M}_{jj}}}\sqrt{\frac{\hat{M}_{ii}\hat{M}_{jj}}{\hat{M}_{ii}^{d}\hat{M}_{jj}^{d}}} = \frac{K_{ij}^{d}}{K_{ij}} \tag{31}$$

which provides the change in stiffness due to damage for each term of the stiffness matrix.

## NUMERICAL EXAMPLES

Let us analyse the three dofs shear-type system reported in Figure 2. The system (0) has a diagonal mass matrix ($m_i$ = 1, with $i$ = 1, 2, 3) and a tri-diagonal stiffness matrix ($k_1 = k_3 = 3$, $k_2 = 1$). The system is assumed to be classically damped. In the same figure is reported the system with three different damage patterns: the system (1) obtained by considering a stiffness reduction at the first floor, i.e. $k_1^{(d)} = 1.5$, the system (2) with a stiffness reduction at the third floor, $k_3^{(d)} = 1.5$, and the system (3) with a stiffness reduction simultaneously at the first and the second floors, $k_1^{(d)} = 1.5$ e $k_2^{(d)} = 0.5$.
The aim of the identification is to localize and characterize the damage by comparing the stiffness matrices before and after damage, in the case of "full order models and "reduced order models".

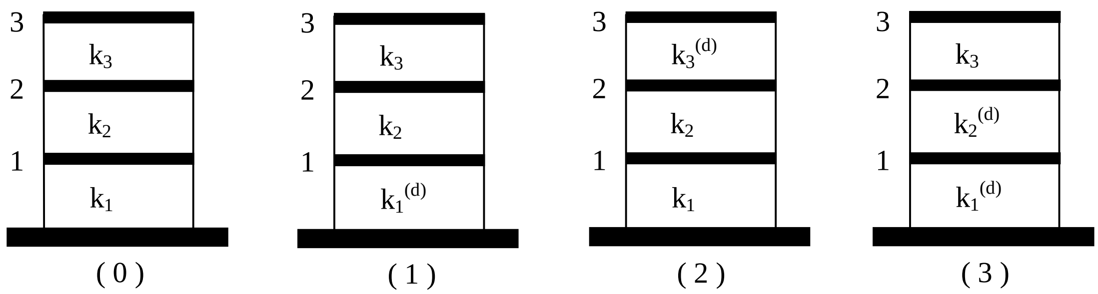


Figure 2- Three DOFs system: ( 0 ) no damage, ( 1 ), ( 2 ) and ( 3 ) with damage

*"Full Order System" (no noise)*: $\mathbf{u} = u_1$;  $\mathbf{y} = [\, y_1 \;\; y_2 \;\; y_3 \,]^T$ ;

For the "full order model" the outputs are the time histories of three acceleration measurements (one for each floor), $\mathbf{y} = [\, y_1 \;\; y_2 \;\; y_3 \,]^T$; the input is a random force applied at the first floor, $\mathbf{u} = u_1$. In this case the physical matrices of the system before and after damage are fully identified and their values are reported in (32a, b, c and d). It is evident that, in this case, the damage characterization and localization is directly obtained by comparing the stiffness matrices of the undamaged system, $\hat{\mathbf{K}}_{(0)}$, and the damaged systems, $\hat{\mathbf{K}}_{(1)}$, $\hat{\mathbf{K}}_{(2)}$ and $\hat{\mathbf{K}}_{(3)}$.

$$\hat{\mathbf{M}}_{(0)} = \begin{bmatrix} +1.0000 & +0.0000 & +0.0000 \\ +0.0000 & +1.0000 & -0.0000 \\ +0.0000 & -0.0000 & +1.0000 \end{bmatrix}; \quad \hat{\mathbf{K}}_{(0)} = \begin{bmatrix} +4.0000 & -1.0000 & +0.0000 \\ -1.0000 & +4.0000 & -3.0000 \\ +0.0000 & -3.0000 & +3.0000 \end{bmatrix} \tag{32a}$$

$$\hat{\mathbf{M}}_{(1)} = \begin{bmatrix} +1.0000 & +0.0000 & +0.0000 \\ +0.0000 & +1.0000 & -0.0000 \\ +0.0000 & -0.0000 & +1.0000 \end{bmatrix}; \quad \hat{\mathbf{K}}_{(1)} = \begin{bmatrix} +2.5000 & -1.0000 & +0.0000 \\ -1.0000 & +4.0000 & -3.0000 \\ +0.0000 & -3.0000 & +3.0000 \end{bmatrix} \tag{32b}$$

$$\hat{\mathbf{M}}_{(2)} = \begin{bmatrix} +1.0000 & +0.0000 & +0.0000 \\ +0.0000 & +1.0000 & -0.0000 \\ +0.0000 & -0.0000 & +1.0000 \end{bmatrix}; \quad \hat{\mathbf{K}}_{(2)} = \begin{bmatrix} +4.0000 & -1.0000 & +0.0000 \\ -1.0000 & +2.5000 & -1.5000 \\ +0.0000 & -1.5000 & +1.50000 \end{bmatrix} \tag{32c}$$

$$\hat{\mathbf{M}}_{(3)} = \begin{bmatrix} +1.0000 & +0.0000 & +0.0000 \\ +0.0000 & +1.0000 & -0.0000 \\ +0.0000 & -0.0000 & +1.0000 \end{bmatrix}; \quad \hat{\mathbf{K}}_{(3)} = \begin{bmatrix} +2.0000 & -0.5000 & +0.0000 \\ -0.5000 & +3.5000 & -3.0000 \\ +0.0000 & -3.0000 & +3.00000 \end{bmatrix} \tag{32d}$$

*"Reduced Order System" (no noise)*: $\mathbf{u} = u_1$;     $\mathbf{y} = y_1$ ;

For the "reduced order model" the output measurements at the two upper floors are missing, $\mathbf{y} = y_1$; the input is still a random force applied at the first floor, $\mathbf{u} = u_1$. In this case the physical matrices of the system cannot be fully identified. In the expressions (33) only the parts $\hat{\mathbf{M}}_{11} = \mathbf{M}_{11}$ and $\hat{\mathbf{K}}_{11} = \mathbf{K}_{11}$ are known while the others $\hat{\mathbf{M}}_{22}$, $\hat{\mathbf{K}}_{12}$ and $\hat{\mathbf{K}}_{22}$, are undetermined.

$$\hat{\mathbf{M}}_{(0)} = \left[\begin{array}{c|cc} +1.0000 & +0.0000 & +0.0000 \\ \hline +0.0000 & +2.6808 & -0.0000 \\ +0.0000 & -0.0000 & +2.0531 \end{array}\right]; \quad \hat{\mathbf{K}}_{(0)} = \left[\begin{array}{c|cc} +4.0000 & -1.6373 & +0.0000 \\ \hline -1.6373 & 10.7231 & -7.0381 \\ +0.0000 & -7.0381 & +6.1593 \end{array}\right] \tag{33a}$$

$$\hat{\mathbf{M}}_{(1)} = \left[\begin{array}{c|cc} +1.0000 & +0.0000 & +0.0000 \\ \hline +0.0000 & +0.3414 & -0.0000 \\ +0.0000 & -0.0000 & +0.4089 \end{array}\right]; \quad \hat{\mathbf{K}}_{(1)} = \left[\begin{array}{c|cc} +2.5000 & -0.5843 & +0.0002 \\ \hline -0.5843 & +1.3656 & -1.1209 \\ +0.0002 & -1.1209 & +1.2267 \end{array}\right] \tag{33b}$$

$$\hat{\mathbf{M}}_{(2)} = \left[\begin{array}{c|cc} +1.0000 & +0.0000 & +0.0000 \\ \hline +0.0000 & +0.5572 & +0.0000 \\ +0.0000 & +0.0000 & +1.1519 \end{array}\right]; \quad \hat{\mathbf{K}}_{(2)} = \left[\begin{array}{c|cc} +4.0000 & -0.7465 & +0.0000 \\ \hline -0.7465 & +1.3930 & -1.2002 \\ +0.0000 & -1.2002 & +1.7279 \end{array}\right] \tag{33c}$$

$$\hat{\mathbf{M}}_{(3)} = \left[\begin{array}{c|cc} +1.0000 & +0.0000 & +0.0000 \\ \hline +0.0000 & +0.0343 & +0.0000 \\ +0.0000 & +0.0000 & +0.0299 \end{array}\right]; \quad \hat{\mathbf{K}}_{(3)} = \left[\begin{array}{c|cc} +2.0000 & -0.0926 & +0.0000 \\ \hline -0.0926 & +0.1201 & -0.0961 \\ +0.0000 & -0.0961 & +0.0897 \end{array}\right] \tag{33d}$$

It is evident that the comparison among the stiffness matrices (33) does not allow to localize and characterize the damage. Thus, it is necessary to consider the alternative procedures.

Preliminary information can be provided by the frequencies. In particular the identified frequencies for the undamaged system are $f_1^{(0)} = 4.1373 \cdot 10^{-1}$ , $f_2^{(0)} = 3.1434 \cdot 10^{-1}$, $f_3^{(0)} = 9.2995 \cdot 10^{-1}$; those for the damaged systems ( 1 ), ( 2 ) and ( 3 ) are respectively:

$f_1^{(1)} = 4.1143 \cdot 10^{-1}$,    $f_2^{(1)} = 2.5432 \cdot 10^{-1}$,    $f_3^{(1)} = 8.1730 \cdot 10^{-1}$;
$f_1^{(2)} = 3.4428 \cdot 10^{-1}$,    $f_2^{(2)} = 2.7566 \cdot 10^{-1}$,    $f_3^{(2)} = 9.0109 \cdot 10^{-1}$;
$f_1^{(3)} = 3.9922 \cdot 10^{-1}$,    $f_2^{(3)} = 2.2687 \cdot 10^{-1}$,    $f_3^{(3)} = 6.6767 \cdot 10^{-1}$

It is evident that the change in frequencies indicates the presence of damage in the system but nothing can be said about its location and characterization. More detailed information are provided by the suggested approaches based on the strain energy.

To this purpose the damage indexes $\beta^{(i)}$ and $d^{(i)}$, with i = 1, 2 and 3, for the damaged systems (1), (2) e (3) are reported in (34), (35) and (36).

$$\beta^{(1)} = \begin{bmatrix} +0.6250 & & \\ & +1.0000 & \\ & & +1.0000 \end{bmatrix}; \quad d^{(1)} = \begin{bmatrix} +0.6250 & +1.0000 & - \\ +1.0000 & +1.0000 & +1.0000 \\ - & +1.0000 & +1.0000 \end{bmatrix} \tag{34}$$

$$\beta^{(2)} = \begin{bmatrix} +1.0000 & & \\ & +0.6250 & \\ & & +0.5000 \end{bmatrix}; \quad d^{(2)} = \begin{bmatrix} +1.0000 & +1.0000 & - \\ +1.0000 & +0.6250 & +0.5000 \\ - & +0.5000 & +0.5000 \end{bmatrix} \tag{35}$$

$$\beta^{(3)} = \begin{bmatrix} +0.5000 & & \\ & +0.8750 & \\ & & +1.0000 \end{bmatrix}; \quad d^{(3)} = \begin{bmatrix} +0.5000 & +0.5000 & - \\ +0.5000 & +0.8750 & +1.0000 \\ - & +1.0000 & +1.0000 \end{bmatrix} \tag{36}$$

It can be observed that:

- the index $\beta$ provides the ratio $K_{ii}^{(d)} \big/ K_{ii}$ with reference to the terms of the diagonal stiffness matrix but nothing can be said about the cross terms which result zero. This means that it allows to localize the damage whatever is the damage pattern but it is able to characterize the damage only in some cases, such as system (1) but not systems (2) and (3);
- the index d provides the ratio $K_{ij}^{(d)} \big/ K_{ij}$ for each term of the stiffness matrix and, hence, allows to localize and characterize the damage in every case.

**The effect of the noise**

In what follows are reported the results of the damage identification related to the systems (0), (1) and (2) considering that the output measurements are affected by two different levels of noise. It is worthy to notice that the suggested approaches are still valid also in presence of noise. In fact the damage indexes (20) and (27) evaluated for the damaged systems (1) and (2) in the case of noise polluted output allow to localize and characterize the damage with a satisfactory approximation.

*"Full Order System" ( noise 2%)*: $\mathbf{u} = u_1$; $\mathbf{y} = [\, y_1 \;\; y_2 \;\; y_3 \,]^T$ ;

$$\hat{\mathbf{M}}_{(0)} = \begin{bmatrix} +1.0010 & +0.0001 & +0.0005 \\ +0.0001 & +0.9915 & -0.0017 \\ +0.0005 & -0.0017 & +0.9960 \end{bmatrix}; \quad \hat{\mathbf{K}}_{(0)} = \begin{bmatrix} +4.0038 & -0.9949 & +0.0032 \\ -0.99499 & +3.9605 & -2.9704 \\ +0.0032 & -2.9704 & +2.9782 \end{bmatrix} \tag{37a}$$

$$\hat{\mathbf{M}}_{(1)} = \begin{bmatrix} +0.9991 & +0.0005 & +0.0006 \\ +0.0005 & +0.9877 & +0.0171 \\ +0.0006 & +0.0171 & +0.9940 \end{bmatrix}; \quad \hat{\mathbf{K}}_{(1)} = \begin{bmatrix} +2.4923 & -0.9882 & -0.0145 \\ -0.9882 & +3.9122 & -2.9208 \\ -0.0145 & -2.9208 & +2.9367 \end{bmatrix} \tag{37b}$$

$$\hat{\mathbf{M}}_{(2)} = \begin{bmatrix} +1.0003 & -0.0001 & +0.0002 \\ -0.0001 & +1.0052 & -0.0068 \\ +0.0002 & -0.0068 & +1.0082 \end{bmatrix}; \quad \hat{\mathbf{K}}_{(2)} = \begin{bmatrix} +4.0168 & -1.0102 & -0.0075 \\ -1.0102 & +2.5209 & -1.5018 \\ -0.0075 & -1.5018 & +1.5104 \end{bmatrix} \tag{37c}$$

*"Reduced Order System" ( noise 2%)*: $\mathbf{u} = u_1$; $\mathbf{y} = y_1$ ;

$$\hat{\mathbf{M}}_{(0)} = \left[\begin{array}{c|cc} +1.0014 & +0.0003 & +0.0007 \\ \hline +0.0003 & +1.0351 & -0.0105 \\ +0.0007 & -0.0105 & +1.0316 \end{array}\right]; \quad \hat{\mathbf{K}}_{(0)} = \left[\begin{array}{c|cc} +4.0175 & -1.0379 & +0.0110 \\ \hline -1.0379 & +4.1872 & -3.1430 \\ +0.0110 & -3.0000 & +3.0316 \end{array}\right] \tag{38a}$$

$$\hat{\mathbf{M}}_{(1)} = \left[\begin{array}{c|cc} +0.9997 & +0.0003 & +0.0002 \\ \hline +0.0003 & +0.4794 & -0.0004 \\ +0.0002 & -0.0004 & +0.4956 \end{array}\right]; \quad \hat{\mathbf{K}}_{(1)} = \left[\begin{array}{c|cc} +2.5168 & -0.7081 & +0.0002 \\ \hline -0.7081 & +1.9601 & -1.4598 \\ +0.0002 & -1.4598 & +1.4522 \end{array}\right] \tag{38b}$$

$$\hat{\mathbf{M}}_{(2)} = \left[\begin{array}{c|cc} +1.0009 & +0.0005 & +0.0002 \\ \hline +0.0005 & +1.2396 & +0.0011 \\ +0.0002 & +0.0011 & +0.7966 \end{array}\right]; \quad \hat{\mathbf{K}}_{(2)} = \left[\begin{array}{c|cc} +4.0117 & -1.1146 & -0.0001 \\ \hline -1.1146 & +3.0813 & -1.4893 \\ -0.0001 & -1.4893 & +1.2008 \end{array}\right] \tag{38c}$$

$$\beta^{(1)} = \begin{bmatrix} +0.6283 & & \\ & +1.0132 & \\ & & +0.9910 \end{bmatrix}; \quad d^{(1)} = \begin{bmatrix} +0.6283 & +1.0103 & - \\ +1.0103 & +1.0132 & +1.0023 \\ - & +1.0023 & +0.9910 \end{bmatrix} \tag{39}$$

$$\beta^{(2)} = \begin{bmatrix} +1.0002 & & \\ & +0.6160 & \\ & & +0.5098 \end{bmatrix}; \quad d^{(2)} = \begin{bmatrix} +1.0002 & 0.9883 & - \\ 0.9883 & +0.6160 & +0.5016 \\ - & +0.5016 & +0.5098 \end{bmatrix} \tag{40}$$

*“Full Order System” ( noise 5%)*: $\mathbf{u} = u_1$; $\mathbf{y} = [\ y_1\ \ y_2\ \ y_3\ ]^T$

$$\hat{\mathbf{M}}_{(0)} = \begin{bmatrix} +0.9968 & +0.0008 & +0.0007 \\ +0.0008 & +1.0193 & +0.0742 \\ +0.0007 & +0.0742 & +1.0421 \end{bmatrix}; \quad \hat{\mathbf{K}}_{(0)} = \begin{bmatrix} +3.9688 & -1.0118 & -0.0798 \\ -1.0118 & +3.8721 & -2.8160 \\ -0.0798 & -2.8160 & +2.8935 \end{bmatrix} \tag{41a}$$

$$\hat{\mathbf{M}}_{(1)} = \begin{bmatrix} +0.9984 & +0.0007 & +0.0011 \\ +0.0007 & +0.9778 & +0.0381 \\ +0.0011 & +0.0381 & +0.9918 \end{bmatrix}; \quad \hat{\mathbf{K}}_{(1)} = \begin{bmatrix} +2.44841 & -0.9788 & -0.0341 \\ -0.9788 & +3.8227 & -2.8340 \\ -0.0341 & -2.8340 & +2.8710 \end{bmatrix} \tag{41b}$$

$$\hat{\mathbf{M}}_{(2)} = \begin{bmatrix} +0.9895 & +0.0005 & +0.0007 \\ +0.0005 & +1.0268 & +0.0042 \\ +0.0007 & +0.0042 & +1.0500 \end{bmatrix}; \quad \hat{\mathbf{K}}_{(2)} = \begin{bmatrix} +3.9565 & -1.0182 & -0.0409 \\ -1.0182 & +2.5125 & -1.4661 \\ -0.0409 & -1.4661 & +1.5104 \end{bmatrix} \tag{41c}$$

*“Reduced Order System” ( noise 5%)*: $\mathbf{u} = u_1$; $\mathbf{y} = y_1$ ;

$$\hat{\mathbf{M}}_{(0)} = \left[\begin{array}{c|cc} +0.9967 & +0.0001 & +0.0003 \\ \hline +0.0001 & +6.0619 & +0.0017 \\ +0.0003 & +0.0017 & +4.4983 \end{array}\right]; \quad \hat{\mathbf{K}}_{(0)} = \left[\begin{array}{c|cc} +3.9544 & -2.4387 & -0.0004 \\ \hline -2.4387 & +23.195 & -15.933 \\ -0.0004 & -15.933 & +14.543 \end{array}\right] \tag{42a}$$

$$\hat{\mathbf{M}}_{(1)} = \left[\begin{array}{c|cc} +0.9984 & -0.0007 & +0.0001 \\ \hline -0.0007 & +26.605 & +0.0002 \\ +0.0001 & +0.0002 & +0.5132 \end{array}\right]; \quad \hat{\mathbf{K}}_{(1)} = \left[\begin{array}{c|cc} +2.4870 & -5.0949 & +0.0004 \\ \hline -5.0949 & +103.71 & -11.162 \\ +0.0004 & -11.162 & +1.6003 \end{array}\right] \tag{42b}$$

$$\hat{\mathbf{M}}_{(2)} = \left[\begin{array}{c|cc} +0.9995 & +0.0004 & -0.0002 \\ \hline +0.0004 & +0.6761 & +0.0001 \\ -0.0002 & +0.0001 & +0.7834 \end{array}\right]; \quad \hat{\mathbf{K}}_{(2)} = \left[\begin{array}{c|cc} +3.9539 & -0.8256 & +0.0009 \\ \hline -0.8256 & +1.6601 & -1.1013 \\ +0.0009 & -1.1013 & +1.2161 \end{array}\right] \tag{42c}$$

$$\beta^{(1)} = \begin{bmatrix} +0.6279 & & \\ & +1.0188 & \\ & & +0.9626 \end{bmatrix}; \quad d^{(1)} = \begin{bmatrix} +0.6279 & +0.9965 & - \\ +0.9965 & +1.0188 & +0.9890 \\ - & +0.9890 & +0.9626 \end{bmatrix} \tag{43}$$

$$\beta^{(2)} = \begin{bmatrix} +0.9972 & & \\ & +0.6085 & \\ & & +0.5249 \end{bmatrix}; \quad d^{(2)} = \begin{bmatrix} +0.9972 & 0.9970 & - \\ 0.9970 & +0.6085 & +0.5064 \\ - & +0.5064 & +0.5249 \end{bmatrix} \tag{44}$$

Similarly to the case of noise free measurements, it can be observed that both the index β and d reported in the previous cases of noise polluted output provide information on damage in the system in terms of both location and values with a satisfactory approximation.

## THE SEISMIC ACTION

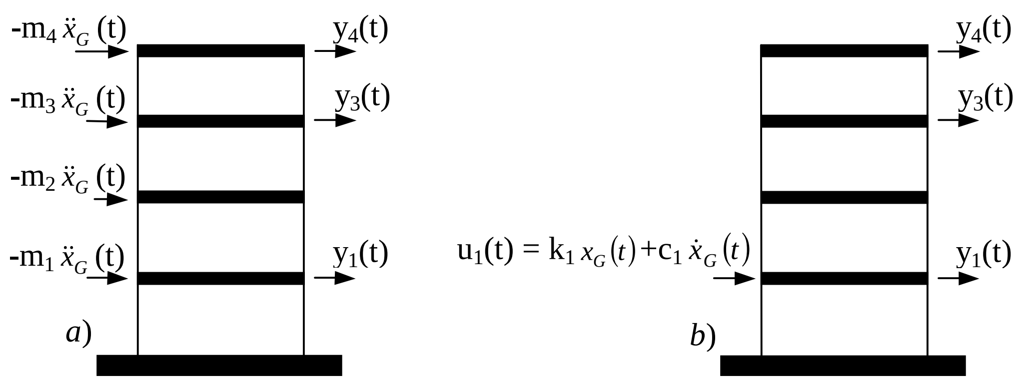


Figure-3 *N* dof structural system with seismic action: *a*) relative motion, *b*) absolute motion.

When the system is subjected to the seismic action the eqn. (1) becomes:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = -\mathbf{M}\boldsymbol{\tau}\,\ddot{x}_G(t) \tag{45}$$

where $\boldsymbol{\tau}$ is the *influence vector* with each element equal to unity and $\ddot{x}_G(t)$ is the ground acceleration.
In the case of seismic action the equation of motion can be also referred to the absolute motion. To this purpose, defining $\mathbf{x}_a(t)$ as the absolute displacement vector given by:

$$\mathbf{x}_a(t) = \mathbf{x}(t) + \boldsymbol{\tau}\,x_G(t) \tag{46}$$

the eqn. (45) can be rewritten as follows:

$$\mathbf{M}\ddot{\mathbf{x}}_a(t) + \mathbf{D}\dot{\mathbf{x}}_a(t) + \mathbf{K}\mathbf{x}_a(t) = \mathbf{K}\boldsymbol{\tau}\,x_G(t) + \mathbf{D}\boldsymbol{\tau}\,\dot{x}_G(t) \tag{47}$$

The equations of motion expressed in modal coordinates become:

$$\dot{\xi}(t) = \mathbf{\Lambda}\xi(t) - \boldsymbol{\psi}^{T}\mathbf{M\tau}\,\ddot{x}_{G}(t) \tag{48}$$

in the case of relative reference system, and:

$$\dot{\xi}_{a}(t) = \mathbf{\Lambda}\xi_{a}(t) + \boldsymbol{\psi}^{T}\mathbf{K\tau}\,x_{G}(t) + \boldsymbol{\psi}^{T}\mathbf{D\tau}\,\dot{x}_{G}(t) \tag{49}$$

in the case of absolute reference system.
The difference between the case of concentrated forces and the seismic action is significant. In the first case the forces applied at the degrees of freedom are independent (Figure 1) and represent different time-dependent histories of loading. On the contrary, the seismic action is a load acting at the same time to all degrees of freedom with an intensity proportional to the mass, $\mathbf{M\tau}$, and characterized by a unique time-dependent history given by $\ddot{x}_{G}(t)$ as reported in Figure 3a. Consequently, since in the case of seismic action there is no co-located degree of freedom, the complex eigenvector cannot be normalized by the conditions (4) and (6). The exception is given by shear-type systems where it is still possible to guaranty the existence of a co-located degree of freedom at the first floor if the motion is referred to an absolute reference system (Figure 3b). In fact, the input terms included in eqn. (47) are:

$$\mathbf{K\tau}\,x_{G} = \begin{bmatrix} K_{1} & \dots & \end{bmatrix}^{T} x_{G} \tag{50}$$

$$\mathbf{D\tau}\,\dot{x}_{G} = \begin{bmatrix} C_{1} & \dots & \end{bmatrix}^{T} \dot{x}_{G} \tag{51}$$

and, from eqn. (49), it follows:

$$\boldsymbol{\psi}^{T}\mathbf{K\tau} = k_{1}\hat{\boldsymbol{\psi}}_{1}^{T} \quad \text{e} \quad \boldsymbol{\psi}^{T}\mathbf{D\tau} = c_{1}\hat{\boldsymbol{\psi}}_{1}^{T} \tag{52}$$

The above results allow to extend the procedure of damage detection discussed in this paper also to shear-type systems subjected to seismic action.

## CONCLUSIONS

In this paper the authors have presented two new strain energy-based approaches to detect damage in reduced order models of linear structural systems where the incompleteness of the input/output measurements does not allow to identify properly each component of the stiffness, mass and damping matrices. The first approach, based on evaluating changes in the modal strain energy and valid only for tri-diagonal sparse stiffness and damping matrices, provides information with reference to the diagonal terms of the stiffness matrix. The second approach, based on the displacement-based-strain energy and valid again for shear-type systems, provides information with reference to both diagonal and off-diagonal terms of the stiffness matrix. The study has been, finally, supplemented by numerical examples which illustrated the validity of the approaches in the case of noise-free and noise polluted output.

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