LAGRANGIAN APPROACH TO STRUCTURAL COLLAPSE SIMULATION

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SUMMARY

Computer analysis of structures has traditionally been carried out using the displacement method combined with an incremental iterative scheme for nonlinear problems. In this paper, a Lagrangian approach is developed, which is a mixed method, where besides displacements, the stress-resultants and other variables of state are primary unknowns. The method can potentially be used for the analysis of structures to collapse as demonstrated by numerical examples. The evolution of the structural state in time is provided a weak formulation using Hamilton's principle. It is shown that a certain class of structures, known as reciprocal structures has a mixed Lagrangian formulation in terms of displacements and internal forces. The form of the Lagrangian is invariant under finite displacements and can be used in geometric nonlinear analysis. For numerical solution, a discrete variational integrator is derived starting from the weak formulation. This integrator inherits the energy and momentum conservation characteristics for conservative systems and the contractivity of dissipative systems. The integration of each step is a constrained minimization problem and is solved using an Augmented Lagrangian algorithm. In contrast to the displacement-based method, the Lagrangian method clearly separates the modeling of components from the numerical solution. Phenomenological models of components essential to simulate collapse can therefore be incorporated without having to implement model-specific incremental state determination algorithms. The state determination is performed at the global level by the optimization method.

INTRODUCTION

Nonlinear analyses of structural response to hazardous loads such as earthquake and blast forces should include (i) the effects of significant material and geometric nonlinearities (ii) various phenomenological models of structural components and (iii) the energy and momentum transfer to different parts of the structure when structural components fracture. Computer analysis of structures has traditionally been carried out using the displacement method, wherein the displacements in the structure are treated as the primary unknowns, combined with an incremental iterative scheme for nonlinear problems. In this paper,

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an alternative method is proposed for the analysis of structures considering both material and geometric nonlinearities. The formulation attempts to solve problems using a force-based approach in which momentum appears explicitly and can be potentially used to deal with structures where deterioration and fracture occur before collapse. In conventional formulations, the response of the structure is considered as the solution of a set of differential equations in time. Since the differential equations hold at a particular instant of time, they provide a temporally local description of the response and are referred to as the *strong form*. In contrast, in this paper, a time integral of functions of the response over the duration of the response is considered. Such an approach presents a temporally global picture of the response and is referred to as the *weak form*.

The kernel of the integral mentioned above consists of two functions – the *Lagrangian* and the *dissipation* functions – of the response variables that describe the configuration of the structure and their rates. The integral is called the *action integral*. In elastic systems, the configuration variables are typically displacements. It is shown here, however, that in considering elastic-plastic systems it is natural to also include the time integrals of internal forces in the structure as configuration variables. The Lagrangian function is energy-like and describes the conservative characteristics of the system, while the dissipation function is similar to a flow potential and describes the dissipative characteristics. In a conservative system, the action integral is rendered stationary (maximum, minimum or saddle point) by the response. In analytical mechanics, this is called *Hamilton's principle* or more generally the *principle of least action*. For non-conservative systems such as elastic-plastic systems, such a variational statement is not possible, and only a weak form which is not a total integral is possible. It is shown moreover that the form of the Lagrangian is invariant under finite deformations. Such a weak formulation enables the construction of numerical integration schemes that inherit the energy and momentum conservation characteristics for conservative systems and the contractivity of dissipative systems.

An overview of variational methods that have been developed for plasticity is first presented in order to place the present work in context. The concept of reciprocal structures and their Lagrangian formulation is then explained using simple systems with springs, masses, dashpots and sliders. The Lagrangian formulation for skeletal structures is subsequently developed and treatment of geometric nonlinearity is shown. Some remarks are then made about the uniqueness of the solution and the extension of the approach to continua. A discrete variational integrator is derived starting from the weak formulation. The solution of each step is a constrained minimization problem and is solved using an Augmented Lagrangian algorithm. Numerical examples are then presented.

VARIATIONAL PRINCIPLES FOR PLASTICITY

Variational formulations of plasticity are based on the principle of maximum dissipation and the consequent normality rule Simo [1]. The local Gauss point level constitutive update has been ascribed a variational structure based on the concept of closest point projection (Simo [2] and Armero [3]). Various approaches have however been used for deriving global variational formulations for plasticity, each of which when discretized in time, leads to a constrained minimization problem in every step. These are base on (1) Complementarity and Mathematical Programming Approach (Maier [4]), (2) Variational Inequality Approach (Duvaut [5]) and (3) Convex Analysis and Monotone Operator Approach (Romano [6]). The most common procedure is to use the Backward Euler method to approximate the rate quantities in the variational statement leading, to a constrained minimization problem in each time increment (see Simo [1] for a detailed presentation). Variational formulation of dynamic plasticity was not extended beyond the variational inequality formulation of Duvaut [5]. In this work, a *weak formulation for dynamic plasticity* is attempted using Hamilton's principle. It can be shown that the Backward Euler method used for quasi-static plasticity is unsuitable for dynamic analysis because of its excessive numerical damping. A numerical integrator well-suited for dynamic analysis is developed herein discretizing the variational principle instead of the differential equations.

SIMPLE PHENOMENOLOGICAL MODELS OF RECIPROCAL STRUCTURES

Reciprocal structures are those structures characterized by convex potential and dissipation functions. In this section, the concept of reciprocal structures is explained using simple spring-mass-damper-slider models shown in Figure 1. Mixed Lagrangian and Dissipation functions of such systems are derived for various structural components.

Mass with Kelvin type Resisting System

Consider a spring-mass-damper system with the spring and the damper in parallel (Kelvin Model shown in Figure 1(a)) subjected to a time-varying force input $P(t)$. The equation of motion is given by:

$$m\ddot{u} + c\dot{u} + ku = P \tag{1}$$

where m is the mass, k is the modulus of the spring, c is the damping constant, u is the displacement of the mass and a superscripted "." denotes derivative with respect to time. The well known approach in Analytical Mechanics is to multiply equation (1) by a virtual displacement function δu, integrate over the time interval $[0,T]$ by parts to obtain the action integral, I, in terms of the Lagrangian function, Λ, and the dissipation function, φ, as shown below (see for example, José [7]):

$$\delta I = -\delta \int_0^T L\,(u,\dot{u})\,dt + \int_0^T \frac{\partial \varphi(\dot{u})}{\partial \dot{u}}\delta u\,dt - \int_0^T P\delta u\,dt = 0 \tag{2}$$

where δ denotes the variational operator, and the Lagrangian function, Λ, and the dissipation function, φ, of this system are given by:

$$\mathbf{L}\,(u,\dot{u}) = \frac{1}{2}m\dot{u}^2 - \frac{1}{2}ku^2 \text{ and } \varphi(\dot{u}) = \frac{1}{2}c\dot{u}^2 \tag{3}$$

Notice that due to the presence of the dissipation function and because the force $P(t)$ can in general be non-conservative, equation (2) defines δI and not I itself. Conversely, starting from (2), equation (1) can be obtained as the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathbf{L}}{\partial \dot{u}}\right) - \left(\frac{\partial \mathbf{L}}{\partial u}\right) + \frac{\partial \varphi}{\partial \dot{u}} = P \quad \Rightarrow \quad m\ddot{u} + c\dot{u} + ku = P \tag{4}$$

Thus, the Lagrangian and dissipation functions and the action integral determine the equation of motion.

Mass with Maxwell type Resisting System

Consider on the other hand, a spring-mass-damper system with the spring and the damper in series (Maxwell Model - shown in Figure 1(b)) subjected to a time varying base-velocity input, $v_{in}(t)$. The formulation requires to obtain a Lagrangian function and a dissipation function for this system that determine the equations of motion as above. Formulation of compatibility of deformations results in:

$$v_{in} + \frac{\dot{F}}{k} + \frac{F}{c} = \dot{u} \tag{5}$$

where F is the force in the spring and damper. Writing the equation of equilibrium of the mass, $m\ddot{u} + F = 0$, solving for the velocity $\dot{u}$ and substituting in equation (5), we have:

$$\frac{1}{k}\dot{F} + \frac{1}{c}F + \frac{1}{m}\int_0^t F\,d\tau = -v_{in} - v_0 \tag{6}$$

where v_0 is the initial velocity of the mass. Defining $J = \int_0^t F\,d\tau$ (as suggested by El-Sayed [8]), the *impulse* of the force in the spring and damper, equation (6) can be written as:

$$\frac{1}{k}\ddot{J} + \frac{1}{c}\dot{J} + \frac{1}{m}J = -v_{in} - v_0 \tag{7}$$

From the correspondence between equations (7) and (1), we conclude that the Lagrangian function, Λ, the dissipation function, φ and the action integral, δI of this system are given by:

$$\mathbf{L}\left(J, \dot{J}\right) = \frac{1}{2}\frac{1}{k}\dot{J}^2 - \frac{1}{2}\frac{1}{m}J^2, \ \varphi\left(\dot{J}\right) = \frac{1}{2}\frac{1}{c}\dot{J}^2 \text{ and } \delta I = -\delta\int_0^T \mathbf{L}\left(J, \dot{J}\right)dt + \int_0^T \frac{\partial\varphi\left(\dot{J}\right)}{\partial \dot{J}}\delta J dt + \int_0^T \left[v_{in}\left(t\right) + v_0\right]\delta J dt \tag{8}$$

Equation (7) can also be thought of as the equation of motion of the *dual* system shown in Figure 1(c). We observe that while the Lagrangian and Dissipation functions involve the displacement and the velocity for a parallel (Kelvin type) system, they involve the impulse and the force for a series (Maxwell type) system.
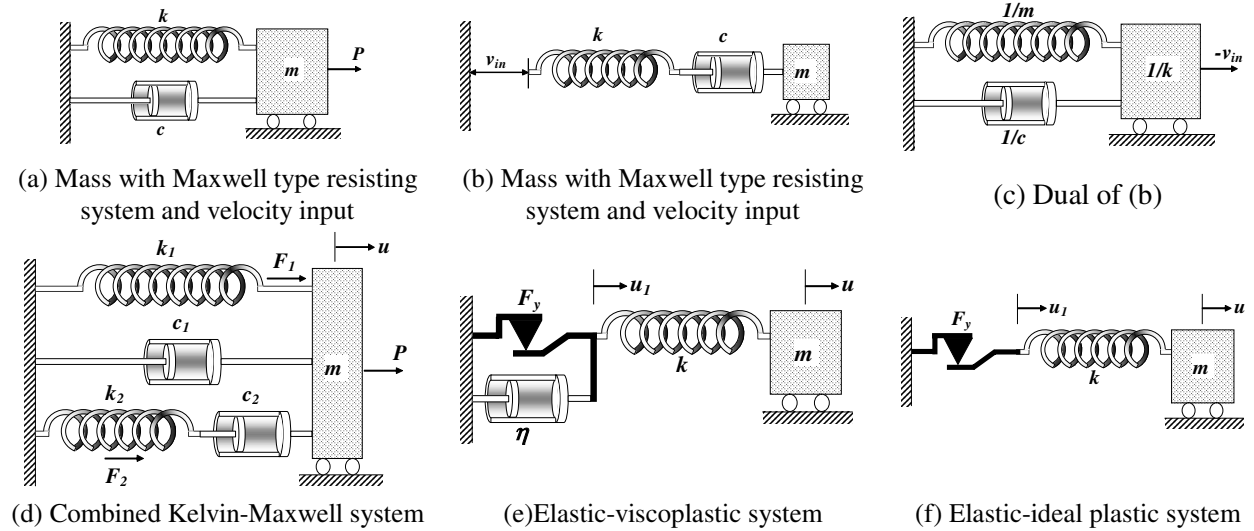

(a) Mass with Maxwell type resisting system and velocity input

(b) Mass with Maxwell type resisting system and velocity input

(c) Dual of (b)

(d) Combined Kelvin-Maxwell system

(e) Elastic-viscoplastic system

(f) Elastic-ideal plastic system

Figure 1. Simple phenomenological models

Mass with Combined Kelvin and Maxwell Resisting Systems

Consider now the combined Kelvin-Maxwell system shown in Figure 1(d) subject to a Force Input. (Note that the velocity input has been excluded for the sake of simplicity). The forces in the springs are denoted by F_1 and F_2 respectively and their impulses by J_1 and J_2. If we define the flexibilities of the springs as $a_1 = 1/k_1$ and $a_2 = 1/k_2$, then the equations of equilibrium and compatibility become respectively:

$$m\ddot{u} + c\dot{u} + k_1 u + \dot{J}_2 = P \text{ and } a_2\ddot{J}_2 + \frac{1}{c_2}\dot{J}_2 - \dot{u} = 0 \tag{9}$$

It is found that elimination of either u or J_2 results in a differential equation that does not have a weak formulation that separates into a Lagrangian part and a dissipation part. Such a formulation would therefore not lend itself to the derivation of the discrete variational integrators of the next section. Moreover, when considering plasticity, the dissipative term in equation the compatibility equation is not single valued and hence, elimination of J_2 would not be possible. It is therefore necessary to devise mixed Lagrangian and dissipation functions that contain u, J_2 and their time derivatives. Consider the following Lagrangian, dissipation functions and action integral:

$$\mathbf{L}\left(u, J_2, \dot{u}, \dot{J}_2\right) = \frac{1}{2}m\dot{u}^2 - \frac{1}{2}k_1 u^2 + \frac{1}{2}a_2 J_2^2 + J_2\dot{u}; \ \varphi\left(\dot{u}, \dot{J}_2\right) = \frac{1}{2}c_1\dot{u}^2 + \frac{1}{2}\frac{1}{c_2}\dot{J}_2^2;$$

$$\delta I = -\delta\int_0^T \mathbf{L}\left(u, \dot{u}, J_2, \dot{J}_2\right)dt + \int_0^T \frac{\partial\varphi\left(\dot{u}\right)}{\partial \dot{u}}\delta u dt + \int_0^T \frac{\partial\varphi\left(\dot{J}_2\right)}{\partial \dot{J}_2}\delta J_2 dt - \int_0^T P\delta u dt \tag{10}$$

It can be easily verified that the corresponding Euler-Lagrange equations are the equilibrium equation of the parallel subsystem and the compatibility equation of the series subsystem:

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{L}}{\partial \dot{u}}\right)-\left(\frac{\partial \boldsymbol{L}}{\partial u}\right)+\frac{\partial \varphi}{\partial \dot{u}}=P \Rightarrow m\ddot{u}+c\dot{u}+k_1 u+\dot{J}_2=P \; ; \; \frac{d}{dt}\left(\frac{\partial \boldsymbol{L}}{\partial \dot{J}_2}\right)-\left(\frac{\partial \boldsymbol{L}}{\partial J_2}\right)+\frac{\partial \varphi}{\partial \dot{J}_2}=0 \Rightarrow a_2\ddot{J}_2+\frac{1}{c_2}\dot{J}_2-\dot{u}=0 \quad (11)$$

Alternate formulation for Combined Kelvin and Maxwell System
It is found however, that it is more convenient for MDOF structural systems to use a Lagrangian function of all the spring forces as shown below, even though it is not minimal.

$$\boldsymbol{L}\left(J_2, \dot{u}, \dot{J}_2\right)=\frac{1}{2}m\dot{u}^2+\frac{1}{2}a_1\dot{J}_1^2+\frac{1}{2}a_2\dot{J}_2^2+\left(J_1+J_2\right)\dot{u} \qquad (12)$$

or in matrix notation:

$$\boldsymbol{L}\left(\mathbf{J}, \dot{u}, \dot{\mathbf{J}}\right)=\frac{1}{2}m\dot{u}^2+\frac{1}{2}\dot{\mathbf{J}}^{\mathrm{T}}\mathbf{A}\dot{\mathbf{J}}+\mathbf{J}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\dot{u} \qquad (13)$$

where $\mathbf{J}=[J_1 \; J_2]^{\mathrm{T}}$, $\mathbf{A}=\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$, the flexibility matrix and $\mathbf{B}=[1 \; 1]$, the *equilibrium matrix*. The equilibrium matrix operates on the vector of internal forces to produce the vector of nodal forces. The *compatibility matrix* operates on the velocity vector to produce the rate of change of deformation. As a consequence of the Principle of Virtual Work, the transpose of the compatibility matrix is the equilibrium matrix. The dissipation function and the action integral are still given by equation (10). The Euler-Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{L}}{\partial \dot{u}}\right)-\left(\frac{\partial \boldsymbol{L}}{\partial u}\right)+\frac{\partial \overline{\varphi}}{\partial \dot{u}}=P \Rightarrow m\ddot{u}+c\dot{u}+\mathbf{B}\dot{\mathbf{J}}=P \; ; \; \frac{d}{dt}\left(\frac{\partial \boldsymbol{L}}{\partial \dot{\mathbf{J}}}\right)-\left(\frac{\partial \boldsymbol{L}}{\partial \mathbf{J}}\right)+\frac{\partial \overline{\varphi}}{\partial \dot{\mathbf{J}}}=0 \Rightarrow \mathbf{A}\ddot{\mathbf{J}}+\frac{\partial \overline{\varphi}}{\partial \dot{\mathbf{J}}}-\mathbf{B}^{\mathrm{T}}\dot{u}=0 \quad (14)$$

The mixed Lagrangian of equation (13) and the Dissipation function of equation (10) form the basis of further developments in this paper. Observe that the Lagrangian is not unique. For example, the Lagrangian:

$$\overline{\boldsymbol{L}}\left(u, \mathbf{J}, \dot{u}, \dot{\mathbf{J}}\right)=\frac{1}{2}m\dot{u}^2+\frac{1}{2}\dot{\mathbf{J}}^{\mathrm{T}}\mathbf{A}\dot{\mathbf{J}}-\mathbf{J}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}u \qquad (15)$$

would result in the same governing differential equations(14). In fact, any Lagrangian differing from that in (13) by only a *gauge transformation* of the form:

$$\overline{\boldsymbol{L}}\left(u, \mathbf{J}, \dot{u}, \dot{\mathbf{J}}\right)=\boldsymbol{L}\left(\mathbf{J}, \dot{u}, \dot{\mathbf{J}}\right)+\frac{d}{dt}\chi\left(u, \mathbf{J}\right) \qquad (16)$$

where $\chi(u,\mathbf{J})$ is any scalar function would result in identical Euler-Lagrange equations (see for example, Scheck [9]). The form (15) is obtained from a Legendre transformation of the potential energy in spring 2. However, we prefer the form (13) due to its following features: (1) It does not contain the displacement, u explicitly. Therefore the momentum, $\frac{\partial L}{\partial u}$ is conserved (see for example, Scheck [9]). This leads to the idea of the generalized momentum, $p_u=\frac{\partial L}{\partial u}=m\dot{u}+J_1+J_2$. (2) It extends to geometric nonlinear problems where the equilibrium matrix $\mathbf{B}$ is not constant, as shown in a later section.

Elastic-plastic Dynamic System
Consider the elastic-visco-plastic dynamic system of Figure 1(e). This is in fact a visco-plastic regularization of the elastic-ideal-plastic system of Figure 1(f). Let the yield force of the slider be F_y, so

that that force F_{slider} in the slider is such that $|F_{\text{slider}}| = F_y$. If the force in the spring is F and its impulse, $J = \int_0^t F d\tau$, then the rate of deformation of the slider-dashpot combination is:

$$\dot{u}_1 = \frac{1}{\eta}\left\langle |F| - F_y \right\rangle \operatorname{sgn}(F) = \frac{1}{\eta}\left\langle |\dot{J}| - F_y \right\rangle \operatorname{sgn}(\dot{J}) \tag{17}$$

where again, , $\langle x \rangle$ is the Mackaulay Bracket and $\operatorname{sgn}(x)$, the signum function. The above constitutive equation can be obtained as follows from a convex dissipation function:

$$\varphi(\dot{J}) = \frac{1}{2\eta}\left\langle |\dot{J}| - F_y \right\rangle^2 \;\Rightarrow\; \dot{u}_1 = \frac{\partial \varphi(\dot{J})}{\partial \dot{J}} = \frac{1}{\eta}\left\langle |\dot{J}| - F_y \right\rangle \operatorname{sgn}(\dot{J}) \tag{18}$$

The equations of equilibrium and compatibility are therefore:

$$m\ddot{u} + \dot{J} = P \text{ and } a\ddot{J} + \frac{\partial \varphi(\dot{J})}{\partial \dot{J}} - \dot{u} = 0 \tag{19}$$

where $a = 1/k$, and it is verified without difficulty that the Lagrangian function, the dissipation function and the action integral are respectively:

$$\mathbf{L}(J, \dot{u}, \dot{J}) = \frac{1}{2}m\dot{u}^2 + \frac{1}{2}a\dot{J}^2;\; \varphi(\dot{J}) = \frac{1}{2\eta}\left\langle |\dot{J}| - F_y \right\rangle^2;\; \delta I = -\delta\int_0^T \mathbf{L}(J, \dot{u}, \dot{J})dt + \int_0^T \frac{\partial \varphi(\dot{J})}{\partial \dot{J}}\delta J dt - \int_0^T P\delta u dt \tag{20}$$

Figure 1(f) shows an elastic-ideal plastic dynamic system. As noted above, this system is obtained from the viscoplastic one in the limit of the regularizing viscous coefficient, η, going to zero. The dissipation function φ of equation (18) then becomes:

$$\varphi(\dot{J}) = \begin{cases} 0 \text{ if } |\dot{J}| \le F_y \\ \infty \text{ if } |\dot{J}| > F_y \end{cases} \tag{21}$$

i.e., $\varphi(\dot{J}) = \mathsf{U}_C(\dot{J})$ where C is the elastic domain, $C = \{x : |x| < F_y\}$ and U_C is the so called indicator function of the set C. The Lagrangian formulation of the elastic-ideal plastic system is then the same as that of the elastic viscoplastic system, i.e. equations (20), with the dissipation function suitably interpreted.

Structures composed entirely of components whose potential as well as dissipation functions are convex functions have a Lagrangian Formulation. The systems discussed in the previous sub-sections are of this type. This class also includes a wider variety of other behavior such as hyperelasticity, rate-independent plasticity, viscoelasticity, viscoplasticity and tension- or compression-only resistance.

COMPATIBILITY EQUATIONS OF A FRAME ELEMENT

In order to obtain a Lagrangian formulation for a frame structure, the compatibility equations need to be expressed in a form similar to Eq. (14). Consider the beam element with rigid plastic hinges at the two ends. From Figure 2(b), the compatibility of deformations in the element gives:

$$\begin{aligned} &\left\{ \dot{\varepsilon}_{hinge1} \quad 0 \quad -\dot{\theta}_{hinge1}^y \quad \dot{\theta}_{hinge1}^z \quad 0 \quad 0 \right\}^{\mathbf{T}} + \left\{ \dot{q}_{beam}^1 \quad \dot{q}_{beam}^2 \quad \dot{q}_{beam}^3 \quad \dot{q}_{beam}^4 \quad \dot{q}_{beam}^5 \quad \dot{q}_{beam}^6 \right\}^{\mathbf{T}} \\ &+ \left\{ \dot{\varepsilon}_{hinge2} \quad 0 \quad 0 \quad 0 \quad \dot{\theta}_{hinge2}^y \quad -\dot{\theta}_{hinge2}^z \right\}^{\mathbf{T}} - \left\{ \dot{q}^1 \quad \dot{q}^2 \quad \dot{q}^3 \quad \dot{q}^4 \quad \dot{q}^5 \quad \dot{q}^6 \right\} = \mathbf{0} \end{aligned} \tag{22}$$

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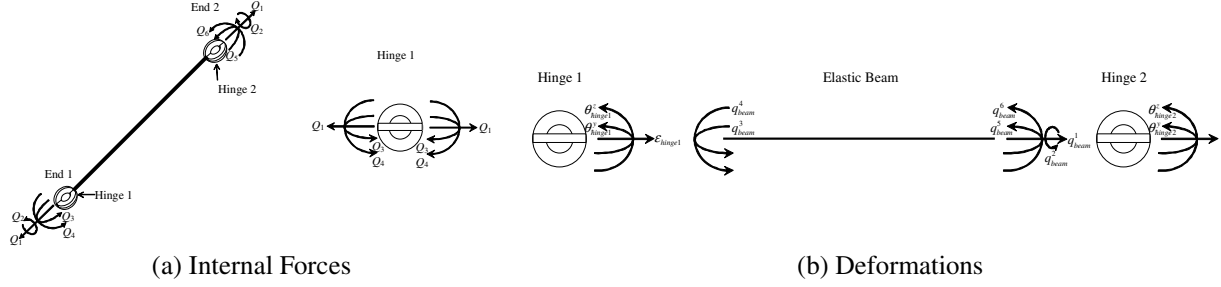(a) Internal Forces   (b) Deformations

Figure 2. Beam element with rigid-plastic hinges

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Let $\mathbf{A}^e$ be the elastic flexibility matrix of the element. Then $\dot{\mathbf{q}}_{beam} = \mathbf{A}^e \dot{\mathbf{Q}}$, where $\mathbf{q}$ is defined as $\mathbf{q}_{beam} = \left\{ q_{beam}^1 \quad q_{beam}^2 \quad q_{beam}^3 \quad q_{beam}^4 \quad q_{beam}^5 \quad q_{beam}^6 \right\}^{\mathbf{T}}$ and $\mathbf{Q}$ is the element independent end force vector. Let φ_{hinge1} and φ_{hinge2} be the dissipation functions of hinges 1 and 2 respectively. Then we have:

$$\left\{ \dot{\varepsilon}_{hinge1} \quad \dot{\theta}_{hinge1}^y \quad \dot{\theta}_{hinge1}^z \right\}^{\mathbf{T}} = \frac{\partial \varphi_{hinge1}}{\mathbf{F}_{hinge1}} \text{ and } \left\{ \dot{\varepsilon}_{hinge2} \quad \dot{\theta}_{hinge2}^y \quad \dot{\theta}_{hinge2}^z \right\}^{\mathbf{T}} = \frac{\partial \varphi_{hinge2}}{F_{hinge2}} \tag{23}$$

where $\mathbf{\Phi}_{hinge1}$ and $\mathbf{\Phi}_{hinge1}$ are the stress-resultants in hinges 1 and 2 respectively. Note that the end forces are: $\mathbf{F}_{hinge1} = \left\{ Q_1 \quad -Q_3 \quad Q_4 \right\}^{\mathbf{T}}$ and $\mathbf{F}_{hinge1} = \left\{ Q_1 \quad Q_5 \quad -Q_6 \right\}^{\mathbf{T}}$. Define a dissipation function $\varphi^e = \varphi_{hinge1} + \varphi_{hinge2}$; Combining these results gives the element equation:

$$\mathbf{A}^e \dot{\mathbf{Q}} + \frac{\partial \varphi^e}{\partial \mathbf{Q}} - \mathbf{B}^{\mathbf{T}} \dot{\mathbf{u}}^e = \mathbf{0} \tag{24}$$

GOVERNING EQUATIONS OF SKELETAL STRUCTURES

The governing equations of the structure consist of the equilibrium equations, the compatibility equations and the constitutive equations. The equilibrium equations are:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{B}\dot{\mathbf{J}} - \mathbf{P} = \mathbf{0} \tag{25}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{B}$ are the mass, damping and equilibrium matrices respectively $\mathbf{F}$ is the vector of element internal forces, $\mathbf{F} = \left\{ \left(\mathbf{Q}^1 \right)^{\mathbf{T}} \quad \left(\mathbf{Q}^2 \right)^{\mathbf{T}} \quad \cdots \quad \left(\mathbf{Q}^{N_{elem}} \right)^{\mathbf{T}} \right\}^{\mathbf{T}}$, $\mathbf{J} = \int_0^t \mathbf{F} d\tau$ is the impulse vector and $\mathbf{P}$ is the external force vector. Let $\mathbf{A}$ be the block diagonal assembly of the element elastic flexibility matrices. The the compatibility equation is:

$$\mathbf{A}\ddot{\mathbf{J}} + \frac{\partial \varphi(\dot{\mathbf{J}})}{\partial \dot{\mathbf{J}}} - \mathbf{B}^{\mathbf{T}} \dot{\mathbf{u}} = \mathbf{0} \tag{26}$$

Internal imposed displacements within elements, such as resulting from pre-stressing or thermal loads have been neglected here for the sake of simplicity - no forcing term in equation (26). Pre-multiplying equation (25) by a kinematically admissible virtual displacement $\delta \mathbf{u}$ (satisfying compatibility), and equation (26) by a statically admissible virtual impulse $\delta \mathbf{J}$ (satisfying equilibrium), we have:

$$\delta \mathbf{u}^{\mathbf{T}} \mathbf{M}\ddot{\mathbf{u}} + \delta \mathbf{u}^{\mathbf{T}} \mathbf{C}\dot{\mathbf{u}} + \delta \mathbf{u}^{\mathbf{T}} \mathbf{B}\dot{\mathbf{J}} - \delta \mathbf{u}^{\mathbf{T}} \mathbf{P} = \mathbf{0} \text{ and } \delta \mathbf{J}^{\mathbf{T}} \mathbf{A}\ddot{\mathbf{J}} + \delta \mathbf{J}^{\mathbf{T}} \frac{\partial \varphi(\dot{\mathbf{J}})}{\partial \dot{\mathbf{J}}} - \delta \mathbf{J}^{\mathbf{T}} \mathbf{B}^{\mathbf{T}} \dot{\mathbf{u}} = \mathbf{0} \tag{27}$$

Equilibrium is considered in the undeformed configuration, so that the equilibrium matrix B is a constant. Geometric nonlinearity, where $\mathbf{B}$ is a function of $\mathbf{u}$, is considered in the next section. Adding equations (27), integrating over the time interval [0,T] and making use of integration by parts, we obtain:

$$\delta I = -\delta \int_0^T \left[\frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2} \dot{\mathbf{J}}^T \mathbf{A} \dot{\mathbf{J}} + \mathbf{J}^T \mathbf{B}^T \mathbf{u} \right] dt + \int_0^T \delta \mathbf{u}^T \mathbf{C} \dot{\mathbf{u}} dt + \int_0^T \delta \mathbf{J}^T \frac{\partial \varphi(\dot{\mathbf{J}})}{\partial \dot{\mathbf{J}}} dt - \int_0^T \delta \mathbf{u}^T \mathbf{P} = \mathbf{0} \tag{28}$$

With the Lagrangian and the dissipation function are then given by:

$$L\left(\mathbf{J}, \dot{\mathbf{u}}, \dot{\mathbf{J}}\right) = \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2} \dot{\mathbf{J}}^T \mathbf{A} \dot{\mathbf{J}} + \mathbf{J}^T \mathbf{B}^T \dot{\mathbf{u}} \text{ and } \bar{\varphi}\left(\dot{\mathbf{u}}, \dot{\mathbf{J}}\right) = \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{C} \dot{\mathbf{u}} + \varphi\left(\dot{\mathbf{J}}\right) \tag{29}$$

Conversely equations (25) and (26) can be obtained from the relation (28) as Euler-Lagrange equations.

EFFECT OF GEOMETRIC NONLINEARITY ON THE LAGRANGIAN FUNCTION

Having examined the structural dynamic problem under small deformations, it is now desired to consider equilibrium in the deformed configuration. The effect of large structural displacements can be considered, while that of large deformations within the corotational frames of elements can be ignored. This seems to be justified for elastic-plastic frame elements where significant displacements occur after yielding when hinges form, thus not accompanied by large deformations within the element corotational frame. The effect of the change of length on the flexibility coefficients of beam-column members is also neglected since this is a higher order effect. Large deformations may be included by proceeding from the Lagrangian density and performing spatial discretization such as by the Finite Element Method. Some remarks on this are made in the next section.

The difference in formulation from the previous case is only the fact that the equilibrium matrix, $\mathbf{B}$, is a function of displacement, $\mathbf{B}(\mathbf{u})$. However, the equilibrium equations (25) being in global coordinates and the compatibility equations (26) being incremental (compatibility of deformation and displacement rates) must both remain unchanged by this additional consideration. It is demonstrated (see Sivaselvan [10]) that the spatially pre-discretized Lagrangian of equation (29) holds in the deformed configuration as well:

$$L\left(\mathbf{u}, \mathbf{J}, \dot{\mathbf{u}}, \dot{\mathbf{J}}\right) = \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2} \dot{\mathbf{J}}^T \mathbf{A} \dot{\mathbf{J}} + \mathbf{J}^T \left[\mathbf{B}(\mathbf{u})\right]^T \dot{\mathbf{u}} \tag{30}$$

Since all other terms of the Euler-Lagrange equations remain unaffected, it is sufficient to examine: .

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{u}}}\right) - \left(\frac{\partial L}{\partial \mathbf{u}}\right) = \mathbf{M} \ddot{\mathbf{u}} + \frac{d}{dt}\left(\mathbf{B}^T \mathbf{J}\right) - \frac{\partial}{\partial \mathbf{u}}\left(\dot{\mathbf{u}}^T \mathbf{B}^T \mathbf{J}\right) = \mathbf{M} \ddot{\mathbf{u}} + \mathbf{B}^T \dot{\mathbf{J}} + \left[\left(\frac{d\mathbf{B}^T}{dt}\right) - \frac{\partial}{\partial \mathbf{u}}\left(\dot{\mathbf{u}}^T \mathbf{B}^T\right)\right] \mathbf{J} \tag{31}$$

Let the structure have a total of N_ε deformations (and hence N_ε internal forces). The matrix $\mathbf{B}$ therefore has N_ε columns. Let $\mathbf{B}_i$ represent the ith column of $\mathbf{B}$ (Notice that the meaning of $\mathbf{B}_i$ here is different from that in the last section, where it denoted the ith column-wise partition of $\mathbf{B}$). Consider the ith column of the term $\left(\frac{d\mathbf{B}}{dt}\right) - \frac{\partial}{\partial \mathbf{u}}\left(\dot{\mathbf{u}}^T \mathbf{B}\right)$.

$$\frac{d\mathbf{B}_i}{dt} = \frac{d}{dt} \begin{bmatrix} B_{1i} \\ B_{2i} \\ \vdots \\ B_{N_u i} \end{bmatrix} = \begin{bmatrix} \frac{\partial B_{1i}}{\partial u_1} & \frac{\partial B_{1i}}{\partial u_2} & \cdots & \frac{\partial B_{1i}}{\partial u_{N_u}} \\ \frac{\partial B_{2i}}{\partial u_1} & \frac{\partial B_{2i}}{\partial u_2} & \cdots & \frac{\partial B_{2i}}{\partial u_{N_u}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial B_{N_u i}}{\partial u_1} & \frac{\partial B_{N_u i}}{\partial u_2} & \cdots & \frac{\partial B_{N_u i}}{\partial u_{N_u}} \end{bmatrix} \dot{\mathbf{u}} \tag{32}$$

$$\begin{aligned} \dot{\mathbf{u}}^T \mathbf{B}^T &= \dot{\mathbf{u}}^T \begin{bmatrix} \mathbf{B}_1^T & \mathbf{B}_2^T & \cdots & \mathbf{B}_i^T & \cdots & \mathbf{B}_{N_u}^T \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{B}_1 \dot{\mathbf{u}} & \mathbf{B}_2 \dot{\mathbf{u}} & \cdots & \mathbf{B}_i \dot{\mathbf{u}} & \cdots & \mathbf{B}_{N_u} \dot{\mathbf{u}} \end{bmatrix}, \text{ a row vector} \end{aligned} \tag{33}$$

Therefore,

$$\left[\frac{\partial}{\partial \mathbf{u}}\left(\dot{\mathbf{u}}^{\mathrm{T}}\mathbf{B}\right)\right]_{i^{\mathrm{th}}\text{ column}} = \frac{\partial}{\partial \mathbf{u}}\left(\mathbf{B}_i^{\mathrm{T}}\dot{\mathbf{u}}\right) = \begin{bmatrix} \frac{\partial B_{1i}}{\partial u_1} & \frac{\partial B_{2i}}{\partial u_1} & \cdots & \frac{\partial B_{N_u i}}{\partial u_1} \\ \frac{\partial B_{1i}}{\partial u_2} & \frac{\partial B_{2i}}{\partial u_2} & \cdots & \frac{\partial B_{N_u i}}{\partial u_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial B_{1i}}{\partial u_{N_u}} & \frac{\partial B_{2i}}{\partial u_{N_u}} & \cdots & \frac{\partial B_{N_u i}}{\partial u_{N_u}} \end{bmatrix}\dot{\mathbf{u}} \tag{34}$$

It is postulated that the ith *deformation component, $\varepsilon_i(\mathbf{u})$, is a twice continuously differentiable function of the deformed configuration*. Then $\mathbf{B}_i = \left(\dfrac{\partial \varepsilon_i}{\partial \mathbf{u}}\right)^{\mathrm{T}}$ is the Jacobian of the deformation function, and

$\dfrac{\partial \mathbf{B}_i}{\partial \mathbf{u}} = \left(\dfrac{\partial \mathbf{B}_i}{\partial \mathbf{u}}\right)^{\mathrm{T}} = \dfrac{\partial^2 \varepsilon_i}{\partial \mathbf{u}^2}$, the Hessian is symmetric. Hence the right hand sides of equations (32) and (34) are

equal, implying that $\left(\dfrac{d\mathbf{B}}{dt}\right) - \dfrac{\partial}{\partial \mathbf{u}}\left(\dot{\mathbf{u}}^{\mathrm{T}}\mathbf{B}\right) = \mathbf{0}$. Having recognized the symmetry in $\mathbf{B}$, the above result may also

be proved using index notation as follows:

$$\left(\frac{d\mathbf{B}}{dt}\right) - \frac{\partial}{\partial \mathbf{u}}\left(\dot{\mathbf{u}}^{\mathrm{T}}\mathbf{B}\right) = \dot{B}_{ij} - B_{ij,p}\dot{u}_p = B_{ij,p}\dot{u}_p - B_{ij,p}\dot{u}_i = B_{pj,i}\dot{u}_p - B_{ij,p}\dot{u}_i = \mathbf{0} \tag{35}$$

Thus the formulation remains unchanged when geometric nonlinearity is included.

EXTENSION TO CONTINUUM FORMULATION

It is shown in Sivaselvan [10] that weak formulations analogous to equations (28) through (29) can be obtained for continua. The final results are presented here. For a three dimensional continuum, the Lagrangian formulation is given by:

$$L = \frac{1}{2}\dot{u}_k\dot{u}_k + \frac{1}{2}A_{ijkl}J_{ij}J_{kl} + \frac{1}{\rho_0}J_{ij}B_{ijk}^*\dot{u}_k \; ; \; \varphi\left(\dot{\mathbf{J}},\dot{\mathbf{u}}\right) = \mathsf{U}_C\left(\dot{\mathbf{J}}\right) + \frac{1}{2}c_{ij}\dot{u}_i\dot{u}_j \; ;$$

$$\delta I = -\delta \int_0^T\int_\Omega \rho_0 L\, d\Omega dt + \int_0^T\int_\Omega \rho_0 \frac{\partial \varphi}{\partial \dot{u}_k}\delta u_k\, d\Omega dt + \int_0^T\int_\Omega \rho_0 \frac{\partial \varphi}{\partial \dot{J}_{ij}}\delta J_{ij}\, d\Omega dt - \int_0^T\int_\Omega \rho_0 f_k \delta u_k\, d\Omega dt - \int_0^T\int_\Gamma \tau_k \delta u_k\, d\Gamma dt \tag{36}$$

and for a beam-column with finite deformation, by:

$$L = \frac{1}{2}\rho_0 \dot{\mathbf{u}}^{\mathrm{T}}\dot{\mathbf{u}} + \frac{1}{2}\dot{\mathbf{J}}^{\mathrm{T}}\mathbf{a}\dot{\mathbf{J}} + \mathbf{J}^{\mathrm{T}}\mathbf{B}^*\dot{\mathbf{u}} \; ; \; \varphi\left(\dot{\mathbf{J}},\dot{\mathbf{u}}\right) = \mathsf{U}_C\left(\dot{\mathbf{J}}\right) + \frac{1}{2}\dot{\mathbf{u}}^{\mathrm{T}}\mathbf{c}\dot{\mathbf{u}} \tag{37}$$

$$\delta I = -\delta \int_0^T\int_0^L L\, dxdt + \int_0^T\int_0^L \frac{\partial \varphi}{\partial \dot{\mathbf{u}}}\delta \mathbf{u}\, dxdt + \int_0^T\int_0^L \frac{\partial \varphi}{\partial \dot{\mathbf{J}}}\delta \mathbf{J}\, dxdt - \int_0^T\int_0^L \mathbf{f}^{\mathrm{T}}\delta \mathbf{u}\, dxdt - \int_0^T \mathbf{Q}^{\mathrm{T}}\delta \mathbf{q}\, dt \tag{38}$$

The analogy with equations (28) and (29) is seen easily. The integral over time can be discretized to obtain action sums from which discrete variational integrators can be obtained as shown in the next section. These can then be discretized in space using, for example, the finite element method.

TIME DISCRETIZATION - DISCRETE CALCULUS OF VARIATIONS

The numerical integration of the Lagrangian equations by *discrete variational integrators* is developed next for the time integration of the governing equations (39) and (40) of the structure. This development consists of two stages:

1. Following Kane [11], the action integral of equation (41) is discretized in time to obtain an action sum. Using discrete calculus of variations, finite difference equations are obtained, which are the discrete counterpart s of the Euler-Lagrange equations. It is seen that the numerical method obtained in this fashion conserves energy and momentum for a Lagrangian system and inherits the contractivity (stability in the energy norm) of dissipative systems.
2. The task in each time step is shown to be the solution of a constrained minimization problem for which an Augmented Lagrangian algorithm is developed.

The action integral of equation (42) is discretized using the midpoint rule and a time step h, using central differences. It is assumed in this process, that the $\mathbf{J}$ and $\mathbf{u}$ are twice continuously differentiable functions and $\mathbf{P}$ is a once continuously differentiable function of time, and that the dissipation function is continuously differentiable with respect to $\dot{\mathbf{J}}$. It is shown by Simo [12] using geometric arguments that the $O(h^2)$ accuracy holds in the limiting case of rate-independent plasticity when the viscous coefficient $\eta \rightarrow 0$ as well. The resulting action sum is given by:

$$
\begin{aligned}
-\delta \sum_{k=0}^{n-1} h & \left\{ \frac{1}{2}\left(\frac{\mathbf{u}_{k+1}-\mathbf{u}_k}{h}\right)^{\mathbf{T}} \mathbf{M}\left(\frac{\mathbf{u}_{k+1}-\mathbf{u}_k}{h}\right) + \frac{1}{2}\left(\frac{\mathbf{J}_{k+1}-\mathbf{J}_k}{h}\right)^{\mathbf{T}} \mathbf{A}\left(\frac{\mathbf{J}_{k+1}-\mathbf{J}_k}{h}\right) \right. \\
& \left. + \left(\frac{\mathbf{J}_{k+1}+\mathbf{J}_k}{2}\right)^{\mathbf{T}} \mathbf{B}^{\mathbf{T}}\left(\frac{\mathbf{u}_{k+1}-\mathbf{u}_k}{h}\right) \right\} \\
+ \sum_{k=0}^{n-1} h & \left\{ \left(\frac{\delta\mathbf{u}_{k+1}+\delta\mathbf{u}_k}{2}\right)^{\mathbf{T}} \mathbf{C}\left(\frac{\mathbf{u}_{k+1}-\mathbf{u}_k}{h}\right) \right\} \\
+ \sum_{k=0}^{n-1} h & \left\{ \left(\frac{\delta\mathbf{J}_{k+1}+\delta\mathbf{J}_k}{2}\right)^{\mathbf{T}} \left.\frac{\partial\varphi}{\partial\dot{\mathbf{J}}}\right|_{k+\frac{1}{2}} \right\} \\
+ \sum_{k=0}^{n-1} h & \left\{ \left(\frac{\delta\mathbf{u}_{k+1}+\delta\mathbf{u}_k}{2}\right)^{\mathbf{T}} \mathbf{P}_{k+\frac{1}{2}} \right\} \quad + \quad O\left(h^2\right)
\end{aligned}
\tag{43}
$$

where $nh = T$ and subscript k denotes the approximation at time $t = kh$. The time integration problem may now be stated as: *Given* $\{\mathbf{u}_0, \mathbf{u}_n\}$ *and* $\{\mathbf{J}_0, \mathbf{J}_n\}$, *find the sequences* $\{\mathbf{u}_1, \mathbf{u}_2, \dots \mathbf{u}_{n-1}\}$ *and* $\{\mathbf{J}_1, \mathbf{J}_2, \dots \mathbf{J}_{n-1}\}$ *that make the action sum of equation* (43) *stationary*. This is the *discrete variational problem* (Cadzow [13] and Marsden [14]). Using a procedure called summation by parts, analogous to integration by parts in the continuous case, the flowing result can be obtained (Sivaselvan [10]):

$$
\mathbf{M}\left(\frac{\mathbf{u}_{k+1}-2\mathbf{u}_k+\mathbf{u}_{k-1}}{h^2}\right)+\mathbf{C}\left(\frac{\mathbf{u}_{k+1}-\mathbf{u}_{k-1}}{2h}\right)+\mathbf{B}\left(\frac{\mathbf{J}_{k+1}-\mathbf{J}_k}{2h}\right)=\left(\frac{\mathbf{P}_{k+\frac{1}{2}}+\mathbf{P}_{k-\frac{1}{2}}}{2}\right)
$$

$$
\mathbf{A}\left(\frac{\mathbf{J}_{k+1}-2\mathbf{J}_k+\mathbf{J}_{k-1}}{h^2}\right)+\frac{1}{2}\left(\left.\frac{\partial\varphi}{\partial\dot{\mathbf{J}}}\right|_{k+\frac{1}{2}}+\left.\frac{\partial\varphi}{\partial\dot{\mathbf{J}}}\right|_{k-\frac{1}{2}}\right)-\mathbf{B}^{\mathbf{T}}\left(\frac{\mathbf{u}_{k+1}-\mathbf{u}_{k-1}}{2h}\right)=\mathbf{0}
\tag{44}
$$

By using Discrete Variational Calculus ensures that the resulting time-integration scheme possesses energy and momentum conserving properties. A heuristic proof of this fact is given by Sivaselvan [10]. Kane [11] presents a discrete version of Noether's theorem (see for example, José [7]) by which it can be shown that any numerical integrator derived using the discrete calculus of variations approach inherits these conservation characteristics. Moreover, it is shown by Simo [12] that the midpoint rule inherits the contractivity or B-stability of the dissipative system, i.e., systems with neighboring initial conditions converge in the energy norm. This approach also provides a framework for consistently developing higher order methods and error estimation methods that preserve conservation.

TIME-STEP SOLUTION

In troducing the notation, $\mathbf{v}_n$ and $\mathbf{F}_n$ as the Central Difference approximations of the velocity and the internal force respectively, Eq.(44) then becomes:

$$\mathbf{M}\left(\frac{\mathbf{v}_{n+1}-\mathbf{v}_n}{h}\right)+\mathbf{C}\left(\frac{\mathbf{v}_{n+1}+\mathbf{v}_n}{2}\right)+\mathbf{B}\left(\frac{\mathbf{F}_{n+1}+\mathbf{F}_n}{2}\right)=\left(\frac{\mathbf{P}_{n+1}+\mathbf{P}_n}{2}\right) \tag{45}$$

$$\mathbf{A}\left(\frac{\mathbf{F}_{n+1}-\mathbf{F}_n}{h}\right)+\frac{1}{2}\left(\left.\frac{\partial\varphi}{\partial\mathbf{F}}\right|_{n+1}+\left.\frac{\partial\varphi}{\partial\mathbf{F}}\right|_n\right)-\mathbf{B}^{\mathbf{T}}\left(\frac{\mathbf{v}_{n+1}+\mathbf{v}_n}{2}\right)=0 \tag{46}$$

Where $n = k - \frac{1}{2}$, $\mathbf{v}_n=\left(\dfrac{\mathbf{u}_{n+\frac{1}{2}}-\mathbf{u}_{n-\frac{1}{2}}}{h}\right)$ and $\mathbf{F}_n=\left(\dfrac{\mathbf{J}_{n+\frac{1}{2}}-\mathbf{J}_{n-\frac{1}{2}}}{h}\right)$. It is common in modeling frame structures for

dynamic analyses to use a lumped mass matrix and to ignore rotational inertia. Hence the mass matrix could in general be singular. Similarly, the damping matrix could also be singular, for example when using mass proportional damping. Thus, consistent with the convexity assumptions and without loss of generality, equation (45) can be rearranged and partitioned as follows:

$$\frac{2}{h}\begin{bmatrix}\mathbf{M}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}\begin{Bmatrix}\mathbf{v}^1\\\mathbf{v}^2\\\mathbf{v}^3\\\mathbf{v}^4\end{Bmatrix}+\begin{bmatrix}\mathbf{C}_{11}&\mathbf{C}_{12}&\mathbf{0}&\mathbf{0}\\\mathbf{C}_{12}^{\mathbf{T}}&\mathbf{C}_{22}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}\begin{Bmatrix}\mathbf{v}^1\\\mathbf{v}^2\\\mathbf{v}^3\\\mathbf{v}^4\end{Bmatrix}+\begin{bmatrix}\mathbf{B}_1^{\mathbf{T}}\\\mathbf{B}_2^{\mathbf{T}}\\\mathbf{B}_3^{\mathbf{T}}\\\mathbf{B}_4^{\mathbf{T}}\end{bmatrix}^{\mathbf{T}}\mathbf{F}=\begin{Bmatrix}\mathbf{P}^1\\\mathbf{P}^2\\\mathbf{P}^3\\\mathbf{P}^4\end{Bmatrix}+\frac{2}{h}\begin{Bmatrix}\mathbf{Mv}_n^1\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{Bmatrix} \tag{47}$$

where the partitions 1 through 4 represent respectively (i) degrees of freedom with mass, (ii) those with damping but no mass, (iii) those with prescribed forces and (iv) those with prescribed displacements (or velocities). The symbols $\mathbf{F}$, $\mathbf{v}^i$ and $\mathbf{P}^i$ denote respectively $(\mathbf{F}_{n+1}+\mathbf{F}_n)/2$, $(\mathbf{v}_{n+1}^i+\mathbf{v}_n^i)/2$ and $(\mathbf{P}_{n+1}^i+\mathbf{P}_n^i)/2$. Eliminating for $\mathbf{v}^1$ and $\mathbf{v}^2$ using the first two parts of equation (47), we obtain the dynamic compatibility equation:

$$\overline{\mathbf{A}}\mathbf{F}_{n+1}+\frac{h}{2}\left(\left.\frac{\partial\varphi}{\partial\mathbf{F}}\right|_{n+1}\right)-\overline{\mathbf{b}}-h\mathbf{B}_3^{\mathbf{T}}\mathbf{v}^3=0 \tag{48}$$

where:

$$\overline{\mathbf{A}}=\mathbf{A}+\frac{h}{2}\mathbf{B}_2^{\mathbf{T}}\mathbf{C}_{22}^{-1}\mathbf{B}_2+\frac{h^2}{4}\overline{\mathbf{B}}_1^{\mathbf{T}}\overline{\mathbf{M}}^{-1}\overline{\mathbf{B}}_1 \tag{49}$$

$$\overline{\mathbf{b}}=\left[\left(\mathbf{A}-\frac{h}{2}\mathbf{B}_2^{\mathbf{T}}\mathbf{C}_{22}^{-1}\mathbf{B}_2-\frac{h^2}{4}\overline{\mathbf{B}}_1^{\mathbf{T}}\overline{\mathbf{M}}^{-1}\overline{\mathbf{B}}_1\right)\mathbf{F}_n+\frac{h^2}{2}\overline{\mathbf{B}}_1^{\mathbf{T}}\overline{\mathbf{M}}^{-1}\overline{\mathbf{P}}^1+h\mathbf{B}_2^{\mathbf{T}}\mathbf{C}_{22}^{-1}\mathbf{P}^2+h\overline{\mathbf{B}}_1^{\mathbf{T}}\overline{\mathbf{M}}^{-1}\mathbf{Mv}_n^1+\frac{h}{2}\left.\frac{\partial\varphi}{\partial\mathbf{F}}\right|_n\right] \tag{50}$$

$\overline{\mathbf{M}}=\mathbf{M}+\dfrac{h}{2}\overline{\mathbf{C}}_{11}$, $\overline{\mathbf{C}}_{11}=\mathbf{C}_{11}-\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{12}^{\mathbf{T}}$, $\overline{\mathbf{B}}_1=\mathbf{B}_1-\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{B}_2$ and $\overline{\mathbf{P}}^1=\mathbf{P}^1-\mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{P}^2$. Observe that the structure

of $\overline{\mathbf{A}}$, the equivalent dynamic flexibility matrix, is *dual* to that of the equivalent dynamic stiffness matrix of Newmark's method with $\gamma = \frac{1}{2}$. The roles of the flexibility and mass matrices are interchanged. Pre-multiplying equation (48) by $\delta\mathbf{F}_{n+1}$ and integrating gives:

$$\delta\left[\frac{1}{2}\mathbf{F}_{n+1}^{\mathbf{T}}\overline{\mathbf{A}}\mathbf{F}_{n+1}-\mathbf{F}_{n+1}^{\mathbf{T}}\overline{\mathbf{b}}+\frac{h}{2}\varphi(\mathbf{F}_{n+1})\right]=0 \tag{51}$$

In obtaining equation (51), it has been noted that $\mathbf{B}_3\delta\mathbf{F}_{n+1}=\delta\mathbf{P}^3=\mathbf{0}$, since $\mathbf{P}^3$ is prescribed. Since $\mathbf{A}$, $\mathbf{C}_{22}$ and $\mathbf{M}$ are positive definite, from equation (49) we have $\overline{\mathbf{A}}$ is positive definite. Hence the quantity in brackets in equation (51) is minimized. If dissipation is limited to plasticity, then the function φ is the

regularized indicator function of the elastic domain. Hence, in the limit of rate-independent plasticity, the problem of obtaining $\mathbf{F}_{n+1}$ at each step may be stated as follows:

$$\text{Minimize } \Pi\left(\mathbf{F}_{n+1}\right) = \frac{1}{2}\mathbf{F}_{n+1}^{\mathbf{T}}\overline{\mathbf{A}}\mathbf{F}_{n+1} - \mathbf{F}_{n+1}^{\mathbf{T}}\overline{\mathbf{b}}$$

$$\text{Subject to (i) } \mathbf{B}_3\mathbf{F}_{n+1} = \mathbf{P}^3 \tag{52}$$

$$\text{and (ii) } \frac{h}{2}\phi_i\left(\mathbf{F}_{n+1}\right) \leq 0 \quad i = 1,2,\ldots,N_y$$

This is the **Principle of Minimum Incremental Complementary Potential Energy** which can be stated as: *Of all the* $\mathbf{F}_{n+1}$ *satisfying equilibrium with prescribed external forces at the un-damped quasi-static degrees of freedom and satisfying the yield conditions, the one that minimizes the incremental complementary potential energy* Π *is the one that satisfies equilibrium in the other degrees of freedom and compatibility*. It is to be noted that due to the nature of the velocity-dependent Lagrangian and dissipation functions, it was possible to eliminate the velocities, leading to a minimum principle in forces only. In general, however, the incremental potential would be a function of $\mathbf{F}_{n+1}$ and $\mathbf{v}_{n+1}$ and would result in a saddle-point problem at each time step. Equation (52) is similar to the rate variational principles of plasticity.

Constrained Minimization by the Augmented Lagrangian Method
An Augmented Lagrangian algorithm is used for the solution of the minimization problem (52). For a detailed treatment of the Augmented Lagrangian formulation, the reader is referred to Glowinski [15]. The problem (52) is reduced to a sequence of linearly constrained sub-problems using the Augmented Lagrangian regularization:

$$\Pi_{AL}\left(\mathbf{F}_{n+1},\boldsymbol{\lambda}\right) = \frac{1}{2}\mathbf{F}_{n+1}^{\mathbf{T}}\overline{\mathbf{A}}\mathbf{F}_{n+1} - \mathbf{F}_{n+1}^{\mathbf{T}}\overline{\mathbf{b}} + \frac{h}{2}\sum_{i=1}^{N_y}\left[\lambda_i\phi_i\left(\mathbf{F}_{n+1}\right) + \frac{\nu}{2}\left\langle\phi_i\left(\mathbf{F}_{n+1}\right)\right\rangle^2\right] \tag{53}$$

where $\boldsymbol{\lambda}=\{\lambda_1, \lambda_2,\ldots, \lambda_{Ny}\}^{\mathbf{T}}$ is the vector of plastic multipliers, ν is a penalty parameter and $\langle\rangle$ denotes the Mackaulay Bracket. The Augmented Lagrangian regularization is a combination of the usual Lagrangian term, $\lambda_i\phi_i(\mathbf{F}_{n+1})$ and the penalty function $\nu/2\langle\phi_i(\mathbf{F}_{n+1})\rangle^2$. The latter helps accelerate convergence while the former eliminates the need for the penalty parameter to be large, which leads to numerical ill-conditioning. Both terms vanish at a feasible point. A dense matrix is presented by Sivaselvan [10].

NUMERICAL EXAMPLE

The example structure is shown in Figure 3. It is a portal frame consisting of three element. The connections are assumed rigid. The stress-strain curve of the material is assumed bilinear with the following properties: E = 199955 kN/mm^2 and σ_y = 248.2 kN/mm^2. In order to verify the results obtained the program DRAIN-2DX (Allahabadi [16]) is used here. Two analyses, one with and one without PΔ effect, are performed.

Properties I-section
Columns
d = 203.2 mm
t_w = 7.2 mm
b_f = 203.1 mm
t_f = 11.1 mm

Beam
d = 303.3 mm
t_w = 7.5 mm
b_f = 203.3 mm
t_f = 13.1 mm

Figure 3. Example portal frame

Analysis without PΔ effect
First, a dynamic analysis is performed without external axial load on the columns, hence no significant geometric nonlinearity and PΔ effects. Figure 4(a) shows the horizontal displacement history of node 2 (Fig. 3.4). The permanent displacements resulting from plastic deformation have close agreement when calculated using the Lagrangian Approach or DRAIN-2DX. Figure 4(b) shows that there is significant difference between the two analyses in predicting the vertical displacements. While the plastic material
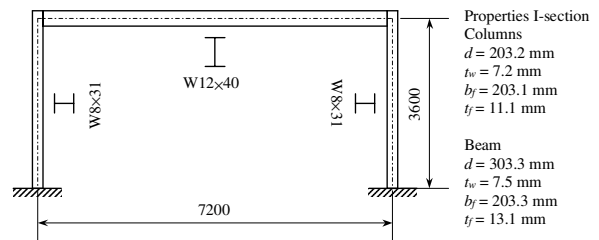
model in DRAIN-2DX accounts for the reduction of bending moment capacity resulting from the axial force interaction; it does not consider the development of centroidal axial plastic strain from plastification caused by bending according to the normality rule. It is important to consider this effect when relying on tension stiffening in beams for collapse prevention. Figure 4(c) shows the time history of the rotation of node 2. The curve is flat when plastic rotations develop in the columns at constant joint rotations. Other differences between the Lagrangian approach and DRAIN-2DX come from the additional joint rotations caused by differential settlements of the columns resulting from permanent axial deformation. Figure 4(d) shows a plot of the horizontal reaction at node 1 versus the horizontal displacement at node 2, showing good agreement between the two approaches.

Analysis with PΔ effect
A dynamic analysis is performed with an axial force of 731.05 kN on each column, corresponding 50% of the yield force. In this case there is significant geometric nonlinearity. Figure 5(a) and Figure 5(b) show that the horizontal and vertical displacements continue to grow. The point marked "collapse" in Figure 5(d) is the point beyond which an external horizontal force is required to pull the structure back to keep displacements from growing autonomously under the vertical loads acting on it. During a dynamic analysis, when this point is crossed, displacements continue to grow without reversal even when the input reverses; the analysis is terminated at this point. It is also noticed that under load reversal, the yield force in the opposite direction is higher than the original yield force since the moments resulting from PΔ effects need to be overcome in addition.

SUMMARY AND CONCLUSIONS

The evolution of the elastic-plastic structural state in time is provided a weak formulation using Hamilton's principle. It is shown that a certain class of structures called reciprocal structures has a mixed weak formulation in time involving Lagrangian and dissipation functions. The new form of the Lagrangian developed in this work involves not only displacements and velocities but also internal forces and their impulses leading to the concept of the generalized momentum for framed structures. This Lagrangian has been shown to extend to continua. The derivative of the compatibility operator with respect to displacements possesses a symmetry that renders the Lagrangian invariant under finite displacements. The formulation can therefore be used in geometric nonlinear analysis. A discrete variational integrator has been derived starting from a weak formulation.
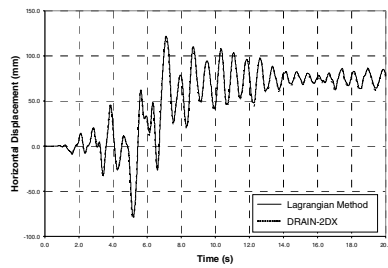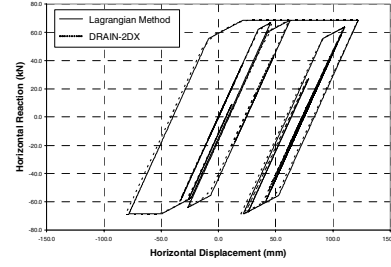

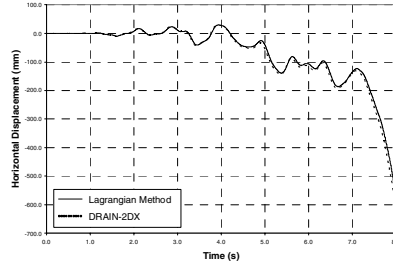(a) Horizontal displacement   (b) Vertical displacement
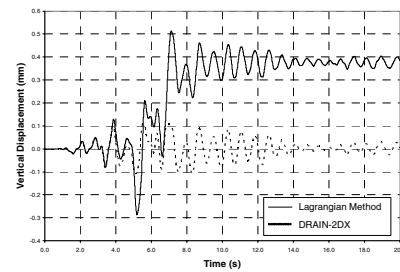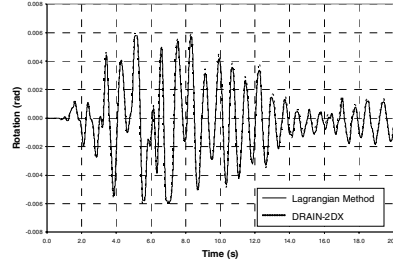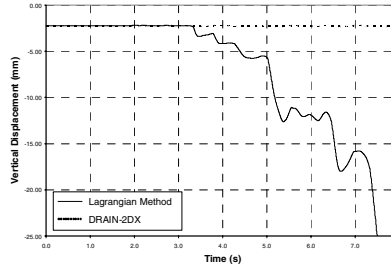
(c) Joint rotation   (d) Column shear force-displacement
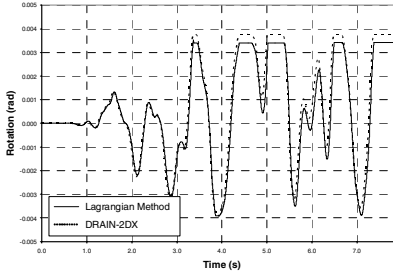
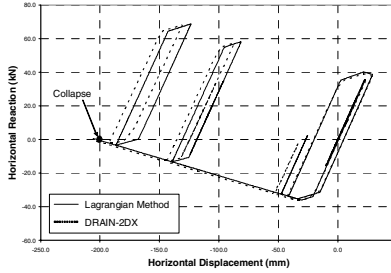Figure 4. Results with no external axial force

(a) Horizontal displacement   (b) Vertical displacement

(c) Joint rotation   (d) Column shear force-displacement

Figure 5. Results with external axial force

This integrator inherits the energy and momentum conservation characteristics for Lagrangian systems and the contractivity in the energy norm of dissipative systems. The integration of each step has been shown to be a constrained minimization problem – the principle of incremental minimum complementary potential energy. An Augmented Lagrangian algorithm and a dense matrix implementation have been derived for the solution of this problem. Since the matrix $\overline{\mathbf{A}}$ of the minimization problem (52) is positive definite, the solution is globally convergent, allowing for larger time steps for computation. This is however not the case in the conventional incremental iterative approach where the tangent matrix may not be positive definite and the Newton iterations may not be globally convergent, limiting the time step. In the continuum case, the minimization problem (52) would be over the function space of stresses rather than over the vector space of internal forces as shown here.

ACKNOWLEDGEMENTS

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