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A SIMPLE APPROXIMATION FORMULA FOR MODAL CORRELATION COEFFICIENT IN CQC METHOD

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SUMMARY

Though CQC method is very effective, the formulas for the mode correlation coefficient playing the important role in this method are very complicated, and are different by the kinds of the response quantity, i.e. relative displacement, relative velocity and absolute acceleration. From the practical viewpoint, this paper proposed a simple approximation formula for the mode correlation coefficient, and verified its usefulness. It was concluded that the proposed approximation formula has enough accuracy and simplicity for practical use, and is more useful than other approximation formulas.

INTRODUCTION

The CQC method (Complete Quadratic Combination Method) [1] is representative as a response spectrum method considered the correlation between modes [2]. Though this method is very effective, the formulas for the mode correlation coefficient playing the important role in this method are very complicated. Not only the expressions themselves of the equations are very complicated, but also they are different by the kinds of the response quantity, i.e. relative displacement, relative velocity and absolute acceleration.

From the practical viewpoint, this paper proposes a simple approximation formula for the mode correlation coefficient for classically damped systems subjected to a white-noise input, and verifies its usefulness.

FORMULAS ALREADY PROPOSED

Exact formula

For convenience, the various response quantities (the relative displacement w, the relative velocity $\dot{w}$ and the absolute acceleration $\ddot{w}$) in a classically damped linear system (N degree of freedom) are represented by w. Its peak response $|w|_{\max}$ is given in the following equation according to the CQC method [1].

$$|w|_{\max}{}^2 = \sum_{s=1}^{N}\sum_{r=1}^{N}(\beta_s W_s \cdot S_s)\,\rho_{sr}(\beta_r W_r \cdot S_r) \tag{1}$$

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$$= \sum_{s=1}^{N} (\beta_s W_s \cdot S_s)^2 + \sum_{s=1}^{N} \sum_{\substack{r=1 \\ (r \neq s)}}^{N} (\beta_s W_s \cdot S_s)\, \rho_{sr} (\beta_r W_r \cdot S_r) \tag{2}$$

in which suffix s is the mode number, $\beta_s W_s$ is the participation vector corresponding to the response components, S_s is the response spectrum value of ground motions and ρ_{sr} is the mode correlation coefficient between the *s*-th mode and the *r*-th mode. The first term in Eq.(2) is a term of same modes ($r = s$) which is identical with the calculation formula of the SRSS method. The second term gives the effect of the mode correlation between different modes ($r \neq s$).

The mode correlation coefficient ρ_{sr} are different by the kinds of the response quantity, i.e. relative displacement, relative velocity and absolute acceleration. It is given by following equation, respectively [1,3,4]. These equations are induced under the assumption about the ground motion that the duration time is sufficiently longer than natural periods and the characteristics is a white-noise. In the following, these equations are called "texact formulas" for convenience.

$$\rho_{sr} = \frac{8\sqrt{h_s h_r}\,(h_s + \chi h_r)\chi^{\frac{3}{2}}}{(1-\chi^2)^2 + 4h_s h_r \chi(1+\chi^2) + 4(h_s{}^2 + h_r{}^2)\chi^2} \quad \text{[relative displacement]} \tag{3}$$

$$\rho_{sr} = \frac{8\sqrt{h_s h_r}\,(h_r + \chi_{sr} h_s)\chi_{sr}^{3/2}}{(1-\chi_{sr}^2)^2 + 4h_s h_r \chi_{sr}(1+\chi_{sr}^2) + 4(h_s^2 + h_r^2)\chi_{sr}^2} \quad \text{[relative velocity]} \tag{4}$$

$$\rho_{sr} = \frac{8\sqrt{h_s h_r}\left[h_r + \chi_{sr}^3 h_s + 4\chi_{sr} h_s h_r (h_r + \chi_{sr} h_s)\right]\sqrt{\chi_{sr}}}{\sqrt{(1+4h_s{}^2)(1+4h_r{}^2)}\left[(1-\chi_{sr}^2)^2 + 4h_s h_r \chi_{sr}(1+\chi_{sr}^2) + 4(h_s^2 + h_r^2)\chi_{sr}^2\right]} \quad \text{[absolute acceleration]} \tag{5}$$

in which h_s is a damping factor and χ_{sr} is the natural circular frequency ratio.

$$\chi_{sr} = \frac{\omega_r}{\omega_s} \tag{6}$$

When the damping factor is equal to each other ($h_s = h_r = h$), Eqs.(3)-(5) becomes an equation below, respectively.

$$\rho_{sr} = \frac{8h^2(1+\chi_{sr})\chi_{sr}^{3/2}}{(1-\chi_{sr}^2)^2 + 4h^2\chi_{sr}(1+\chi_{sr})^2} \quad \text{[relative displacement, relative velocity]} \tag{7}$$

$$\rho_{sr} = \frac{8h^2(1+\chi_{sr})\left[1-(1-4h^2)\chi_{sr} + \chi^2\right]\sqrt{\chi_{sr}}}{(1+4h^2)\left[(1-\chi_{sr}^2)^2 + 4h^2\chi_{sr}(1+\chi_{sr})^2\right]} \quad \text{[absolute acceleration]} \tag{8}$$

Basic property of the correlation coefficient

Fig.1 is a calculation example of the mode correlation coefficient ρ_{sr} for the each response quantity of relative displacement, relative velocity and absolute acceleration. Fig.1(a) is the cases in which the damping factor of both modes is equal to each other ($h_s = h_r = h$), and its value changes from 0.01 to 0.30 in 5 stages. Figs.1(b) and (c) is the case in which the damping factor of both modes differs to each other ($h_s \neq h_r$). Larger value of damping factors is 0.1, 0.2 respectively, and smaller value is 1/2 and 1/5 of them.

From Fig.(1), the difference among coefficients for 3 kinds of response quantities is smaller, as circular frequency ratio χ_{sr} is nearer to 1, and as damping factors are smaller. The range of χ_{sr} in which the difference of 3 coefficients is very small is 0.8-1.2, when the damping factors are about 0.2. However, this range becomes almost over the full range, when the damping factors are smaller than about 0.1.

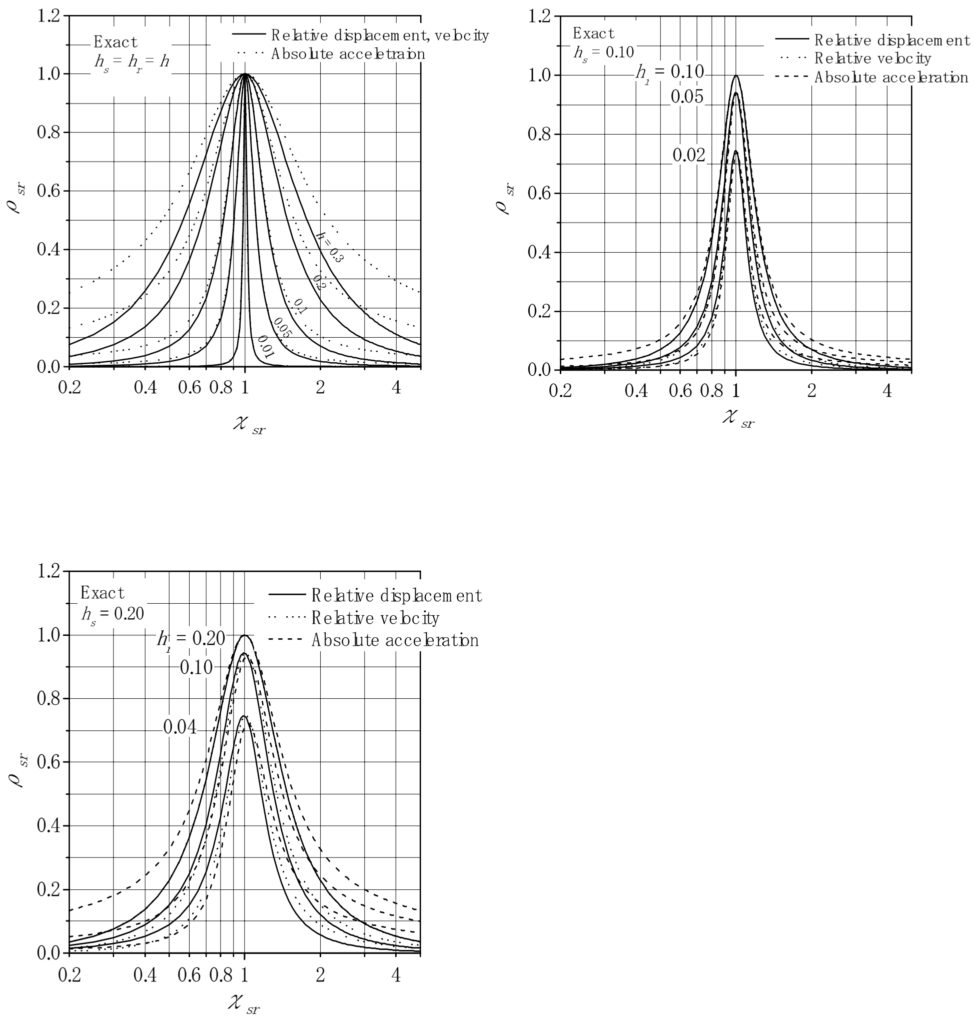


Fig.1 Mode correlation coefficient by exact formula.

In addition, it is possible to point out next points on the basic property of mode correlation coefficient ρ_{sr}, from Fig.1: The coefficient takes the value of 0-1. The correlation coefficient between the *r*-th mode and the *s*-mode and correlation coefficient between the *s*-mode and the *r*-th mode is equal to each other.

$$0 \le \rho_{sr} \le 1 \tag{9}$$

$$\rho_{sr} = \rho_{rs} \tag{10}$$

In the cases in which the damping factor of both modes is equal to each other (Fig.1(a), $h_s = h_r = h$), correlation coefficient ρ_{sr} becomes 1 ($\rho_{sr} = 1$) at natural circular frequency ratio $\chi_{sr} = 1$, regardless of the value of damping factors *h*. As the natural circular frequency ratio χ_{sr} of both modes separates from 1, the mode correlation coefficient ρ_{sr} rapidly decreases. This degree of discretion is larger, as the damping factors are small.

On the other hand, in the cases in which the damping factor of both modes is not equal to each other (Figs.1(b) and (c), $h_s \ne h_r$), correlation coefficient ρ_{sr} becomes smaller than 1 even at natural circular frequency ratio $\chi_{sr} = 1$. The degree of this decrease depends on the ratio of the damping factor of both modes and is larger as this ratio separates from 1.

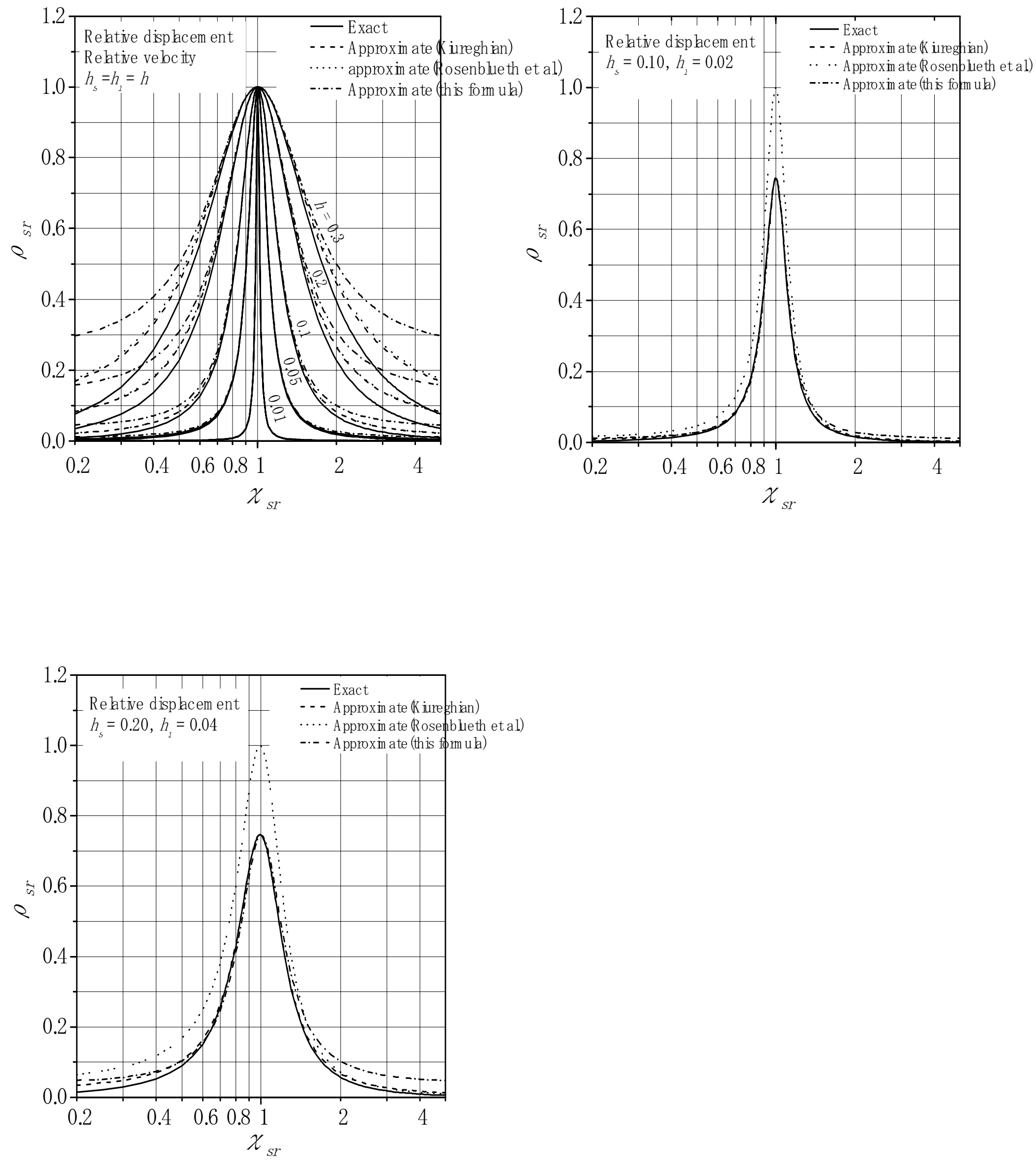


Fig.2 Comparison between approximation formulas and exact formula.

Approximation formulas

The approximation formulas [3] are proposed for the case in which the damping factor of both modes is small and the natural circular frequency is close each other.

$$\rho_{sr} = \frac{8\sqrt{h_s h_r}\left(h_s + \chi h_r\right)\chi^{\frac{3}{2}}}{\left(1-\chi^2\right)^2 + 4h_s h_r \chi\left(1+\chi^2\right) + 4\left(h_s^{\,2} + h_r^{\,2}\right)\chi^2} \quad \text{[relative displacement]} \qquad (11)$$

$$\rho_{sr} = \frac{2\sqrt{h_s h_r}\left[(1+\chi_{sr})^2 (h_s + h_r) - (1-\chi_{sr}^2)(h_s - h_r)\right]}{4(1-\chi_{sr})^2 + (h_s + h_r)^2 (1+\chi_{sr})^2} \quad \text{[relative velocity]} \tag{12}$$

Theses formulas are different as well as the exact formula by the kinds of the response quantity, i.e. relative displacement, relative velocity and absolute acceleration and the expression itself is not drastically simplified in comparison with the exact formula.

Fig.2 compares value by this approximation formula (broken line) with value by exact formula (solid line). Figs.2(a) shows the relative displacement and relative velocity when the damping factor is equal to each other ($h = 0.01, 0.05, 0.1, 0.2, 0.3$) and Figs.2(b) and 2(c) show the relative displacement when the damping factor is different to each other ($h_s = 0.1,\ h_r = 0.02$ and $h_s = 0.2,\ h_r = 0.04$). The accuracy of this approximation formula is very high over the wide range of χ_{sr} when the damping factor of both modes is almost under 0.2, whether or not it is equal to each other.

Following approximation formula [5] is considerably simple compared to the approximation formula shown above.

$$\rho_{sr} = \frac{1}{1+\varepsilon_{sr}^2} \tag{13}$$

in which

$$\varepsilon_{sr} = \frac{\sqrt{1-h_s^2} - \chi_{sr}\sqrt{1-h_r^2}}{h_s + h_r \chi_{sr}} \quad \text{[relative displacement]} \tag{14}$$

The following equation is obtained, if the damping factors are small.

$$\varepsilon_{sr} = \frac{1-\chi_{sr}}{h_s + h_r \chi_{sr}} \tag{15}$$

$$\varepsilon_{sr} = \frac{\left(\dfrac{1-\chi_{sr}}{1+\chi_{sr}}\right)}{h} \tag{16}$$

In addition, the following formula is obtained, if the damping factors of both modes are equal to each other ($h_s = h_r = h$).

The dotted line in Fig.2 is calculated value by Eqs. (13) and (14). When the damping factors of both modes are equal to each other ($h_s = h_r = h$), it is almost equal to the calculated value by the above-mentioned approximation formulas, and therefore, the accuracy is very high. However, the accuracy becomes very low on this, when the attenuation constant of both modes is contrastively different to each other.

DERIVATION OF APPROXIMATION FORMULA

Assumption

In the induction of the approximation formula, next two assumptions for practical use are adopted; the damping ratios were sufficiently small, and that natural frequencies of both modes were close to each other.

$$h_s, h_r \ll 1 \tag{17}$$

$$|1-\chi_{sr}| \ll 1 \tag{18}$$

Results

By applying the assumptions Eqs.(17) and (18), the exact formulas Eqs.(3)-(5) finally become a next identical approximation formula.

$$\rho_{sr} = \rho_{0sr} \cdot \eta_{sr} \tag{19}$$

in which

$$\rho_{0sr} = \frac{1}{1 + \Lambda_{sr}{}^2} \tag{20}$$

$$\Lambda_{sr} = \frac{\lambda_{sr}}{2h_{0sr}} \tag{21}$$

$$\lambda_{sr} = \frac{\left|\omega_r{}^2 - \omega_s{}^2\right|}{\omega_r{}^2 + \omega_s{}^2} = \frac{\left|1 - \chi_{sr}{}^2\right|}{1 + \chi_{sr}{}^2} \tag{22),(23}$$

$$h_{0sr} = \frac{h_r + h_s}{2} \tag{24}$$

and

$$\eta_{sr} = \frac{\sqrt{h_s h_r}}{h_{0sr}} = \frac{2\sqrt{\gamma_{sr}}}{1 + \gamma_{sr}} \tag{25),(26}$$

$$\gamma_{sr} = \frac{h_r}{h_s} \tag{27}$$

Since $\eta_{sr} = 1,\ h_{0sr} = h$ when the damping factor of both modes is equal to each other ($h_s = h_r = h$), the approximation formula becomes

$$\rho_{sr} = \rho_{0sr} = \frac{1}{1 + \Lambda_{sr}{}^2} \tag{28}$$

in which

$$\gamma = \frac{h_r}{h_s} \tag{29}$$

From Eqs.(19) and (20), the condition for the mode correlation coefficient ρ_{sr} becoming over specific value ρ'_{sr} is

$$\Lambda_{sr} \le \sqrt{\left(\frac{\eta_{sr}}{\rho'_{sr}}\right) - 1} \tag{30}$$

DISCUSSION

Accuracy of proposed formula

Fig.3 compares the calculated value by this approximation formula with calculated value by exact formula. The broken lines are the approximate values. The solid lines (the relative displacement) and the dotted lines (the absolute acceleration) are the exact values.

Its accuracy is very high at 0.8-1.2 range of natural frequency ratio of both modes, if the damping factors are smaller than 0.2. If the damping factors are smaller than 0.1, this range becomes almost over the full range.

The dot-dashed lines in Fig.2 are calculated values according to this approximation formula. The accuracy of this formula is lower a little than approximation formulas already proposed. However, this approximation formula has the sufficient accuracy in the practical use, as mentioned above. Moreover, the

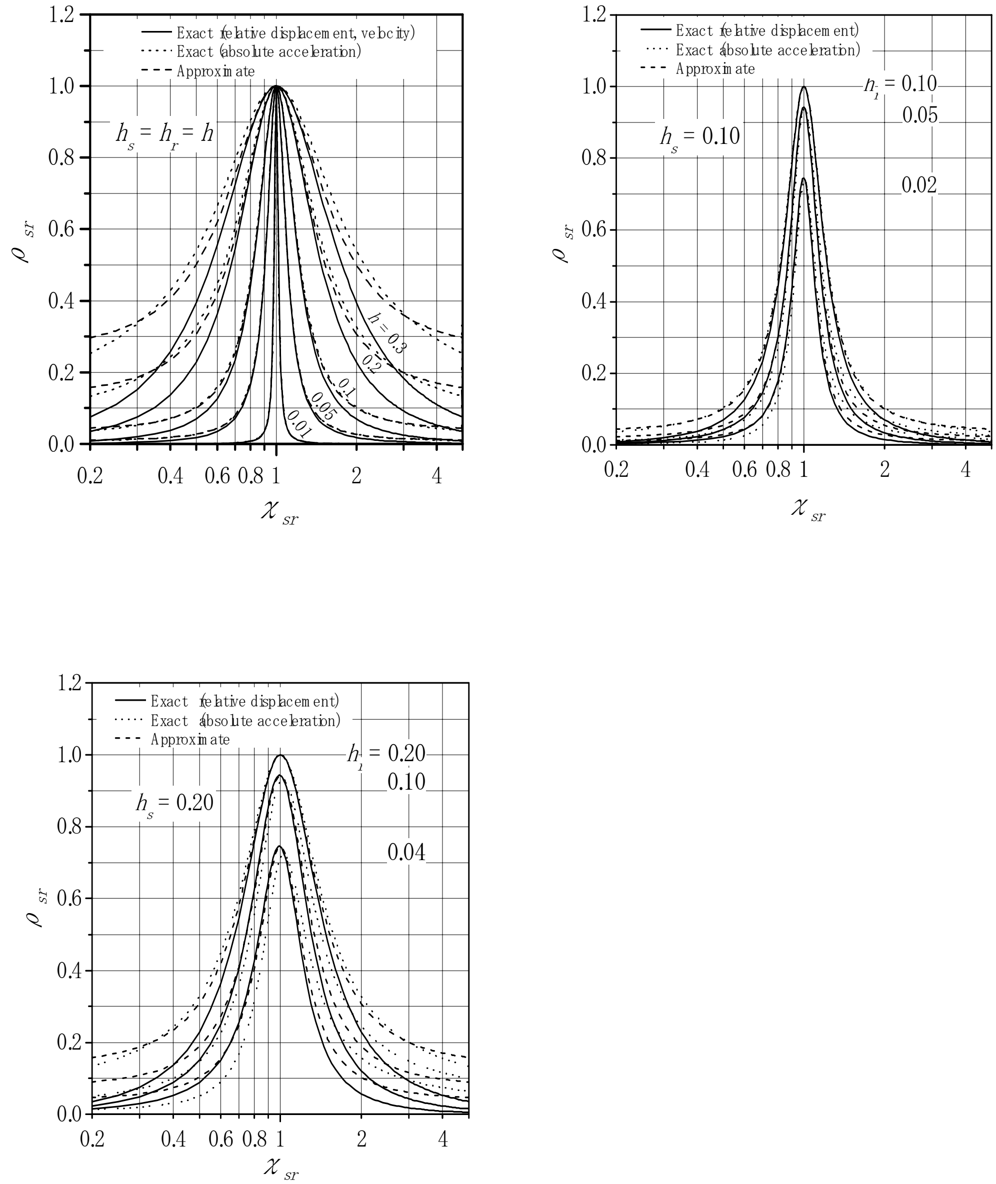


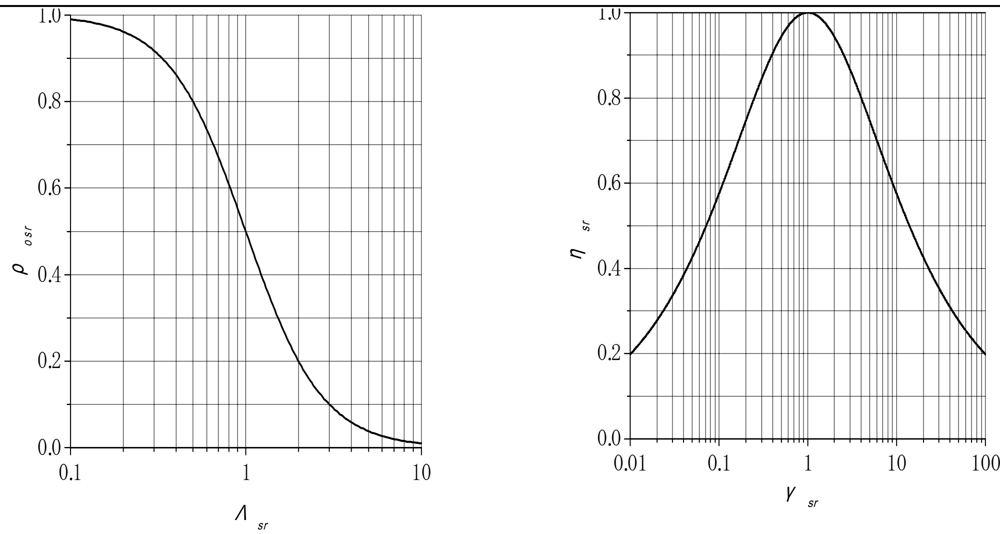
Fig.3 Comparison between proposed approximation formula and exact formula.

accuracy of this approximation formula is high for the absolute acceleration that approximation formulas were not yet proposed.

Simplicity in form of proposed formula

This formula Eqs.(19)-(27) is identical for all kinds of response quantities and is drastically simplified in comparison with the exact formulas Eqs.(3)-(5). Moreover, this formula is considerably simplified in comparison with Eqs.(11) and (12) which are one of approximation formulas already proposed.

However, the expression of this formula resembles Eqs.(13) and (16) which is the other approximation formula already proposed. The difference is that the proposed approximation formula is expressed by χ_{sr}^2, while this approximation formula is expressed by χ_{sr}. On this point, this approximation formula is rather convenient to induce the mode correlation coefficient by the amount of the system in the explicit form expression [6].



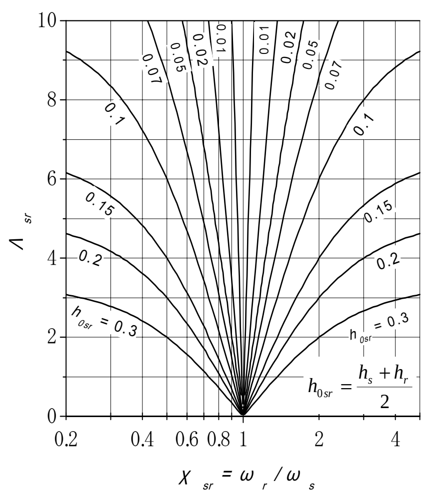


Fig.4 Coefficients constituting proposed approximation formula ($\rho_{sr} = \rho_{0sr} \cdot \eta_{sr}$)

Fig.5 Calculation of index Λ_{sr}.

From the above discussion, it was confirmed that the proposed approximation formula has both accuracy and simplicity, and is excellent on this point in comparison with other approximation formulas.

Substantially explicitness of proposed formula

In addition, the expression of the formula reflects directly and explicitly fundamental properties of the mode correlation which was mentioned previously.

From Eq.(19), proposed approximation formula is expressed by the product of two coefficients ρ_{0sr} and η_{sr}. Coefficient ρ_{0sr} is the mode correlation coefficient in assuming that the damping factor of both modes was equal to each other and that therefore it is together the average h_{0sr} of them, as is clear from the Eq.(28). The other coefficient η_{sr} shows effect of decreasing by actual difference of both damping factors for the mode correlation coefficient mentioned above.

From Eq.(20), mode correlation coefficient ρ_{0sr} is a function only of coefficient Λ_{sr}. This is shown in Fig.4(a). It is possible to show all curves in Fig.1(a) only by this curve. From Fig.4(a), the correlation coefficient ρ_{0sr} monotonously decreases with the increase of coefficient Λ_{sr}. Therefore, it is possible that coefficient Λ_{sr} is considered to be the index which shows the weakness of the mode correlation, namely the independence between modes. From Eq.(21), this coefficient Λ_{sr} is defined as a ratio of coefficient λ_{sr} to the sum of the damping factors of both modes ($2h_{0sr}$). From Eqs.(22) and (23), coefficient λ_{sr} shows the degree of separate between natural circular frequencies, and is a function only of natural circular frequency ratio χ_{sr}.

From the above, when natural circular frequency ratio χ_{sr} is 1, the reason why correlation coefficient ρ_{0sr} becomes 1, regardless of the value of damping factors *h*, is because the coefficient λ_{sr} showing the degree of separate between natural circular frequencies becomes 0, and therefore, coefficient Λ_{sr} becomes 0. As the natural circular frequency ratio χ_{sr} of both modes separates from 1, the reason why the mode correlation coefficient ρ_{sr} rapidly decreases is because the independence between modes coefficient λ_{sr} increases, and therefore, the coefficient Λ_{sr} increases. Moreover, as the damping factors are small, the reason why the degree in which the mode correlation coefficient ρ_{sr} decreases is larger is because the coefficient Λ_{sr} increases.

On the other hand, from Eq.(26), the coefficient η_{sr} is a function only of damping factor ratio γ_{sr} of both modes. This is shown in Fig.4(b). All curves in Figs.1(b) and (c) are possible to be drawn by multiplying the curve of Fig.4(a) by this coefficient. Therefore, as the damping factor ratio γ_{sr} of both modes separates from 1, the reason why the degree in which the mode correlation coefficient ρ_{sr} decreases is larger is because the coefficient η_{sr} decreases.

Fig.5 is prepared for convenient for calculating coefficient Λ_{sr}, which is the index showing the independent degree of the modes, from natural circular frequency ratio χ_{sr} and the average h_{0sr} of damping factors.

CONCLUSIONS

From the practical viewpoint, this paper proposed a simple approximation formula for the mode correlation coefficient playing the important role in CQC method and verifies its usefulness.

The proposed approximation formula (Eqs.(19)-(27)) has both enough accuracy and simplicity for practical use, and is more useful than other approximation formulas. Its accuracy is very high if both of the damping constants of two modes are less than 0.1. Compared with the exact solutions, this formula is greatly simplified. For example, this is common to all kinds of response values and has a simple expression directly reflecting the fundamental properties of the modal correlation.

The proposed approximation formula is applicable for the wide problem even if it is a classically damped linear system with small damping and it can simply confirm the degree of the effect of the mode correlation. The convenient figures (Figs.4 and 5) were also prepared for the purpose.

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