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NON-BOUC SMOOTHLY-VARYING HYSTERESIS DIFFERENTIAL EQUATION MODELS - DERIVATIONS

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SUMMARY

Non-linear dynamic analysis of a structure by the displacement-based approach requires a hysteresis model of the restoring force characteristics of the structural system. Thus far, it appears that the Bouc-Wen model, or variants thereof, is the only smoothly-varying differential equation model in the literature. This paper presents a methodology for the systematic derivation of smoothly-varying differential equation models that is applied to derive a set of alternative non-Bouc models. Like the Bouc-Wen model, these differential equations are expressed in terms of a number of undetermined parameters that are identified by reference to experimentally acquired hysteresis loop data for the structural system. As an indication of comparative performance, an arbitrarily selected set of experimental hysteresis loops is used as the basis of a parameter identification procedure that is applied to each model. The standard error for each model is thereby determined and comparisons made. Sample plots of experimental and calculated hysteresis loops for each model are also presented. The results indicate that the models presented may indeed be viable alternatives.

INTRODUCTION

The shift towards performance-based seismic design has promoted the approach to design by the displacement-based methods. Non-linear dynamic analysis of a structure by the displacement based approach requires a hysteresis model of the restoring force characteristics of the structural system. If the problem is considered as a stochastic process and the solution approach is via the powerful equivalent linearization, then it has been shown that the hysteresis model must be in a differential equation form if the wide-band feature of the response is to be preserved. Thus far, the most prevalent smoothly-varying differential equation model in the literature is the Bouc [1] model which was subsequently modified by Wen [2] and others. The model also results in elegant solutions to the deterministic non-linear dynamics problem, and can be applied to the modeling of supplemental damping and base isolation devices, and ferromagnetic materials in electrical engineering.

Cimampi *et al* [3] have shown that the Bouc-Wen model represents a form of endochronic plasticity theory. The resulting system of equations for the stochastic or deterministic problem is solved using

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numerical analysis algorithms. Given the complexity of the system, issues of modeling error, computational stability, and processing time can arise. It is therefore reasonable that alternative smoothly-varying differential equation models should be available to the analyst. This paper presents the derivation of a set of alternative non-Bouc models via a systematic methodology that can be applied to derive a large number of models of that classification. Like the Bouc-Wen model, these differential equations are expressed in terms of a number of undetermined parameters that are identified by reference to experimentally acquired hysteresis loop data for the structural system. As an indication of comparative performance, an arbitrarily selected set of experimental hysteresis loops is used as the basis of a parameter identification procedure which is applied to each model. The standard error for each model is thereby determined and comparisons made.

MODEL DERIVATION METHODOLOGY

The derivation methodology is based on the principles of endochronic plasticity theory originally developed by Valanis [4] primarily for application to metals, and subsequently refined and developed into a mature paradigm of material behaviour. The conceptual framework offered by endochronic plasticity theory facilitates the methodology, the key aspect of which has been described by Lubliner [5]. The methodology is as follows and being algorithmic, should be able to be automated by hi-level mathematical symbolic programming languages such as those offered by Mathematica, Maple, etc. The methodology can also be extended to multiple dimensions as well, but only the uniaxial case is presented.

1. Select a rheological model for the system (e.g. Maxwell, Kelvin, Burgers, etc., or combinations of these).
2. Select an internal time measure as a function of deformation.
3. Define an internal time scale as a function equivalent to one of the classical constitutive relations of linear visco-elasticity (i.e. the differential equation steady creep laws) but with the internal time - t', replacing real time, t.

   Some of the classical constitutive relations are:

   $d\varepsilon/dt = A\sigma^n$ (Norton-Bailey power law) (1)

   $d\varepsilon/dt = A\sinh(B\sigma)$ (Ludwik law) (2)

   $d\varepsilon/dt = A(e^{B\sigma} - 1)$ (Soderberg law) (3)
4. Given the functions defined in steps 2 and 3, apply the equilibrium and displacement compatibility rules to the rheological model of step 1 to eliminate the internal time variable, thereby resulting in the smoothly-varying differential equation hysteresis model.

Since many combinations of the spring and dashpot elements of the rheological model required in step 1 are possible, a large number of hysteresis models can be derived and that number increased further by extension to the multi-dimensional cases.

MODELS DERIVATION

Model 1

Step 1:

Consider the simple Maxwell rheological model for a SDOF oscillator shown in Fig.1 which is composed of an elastic spring in series with a viscous element, where U refers to the displacement, and F, the force.

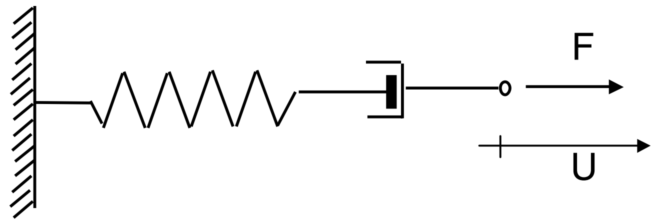

Figure 1. Maxwell Rheological Model

Step 2:
The simplest definition of internal time is stated by Cimampi *et al* [3], based on the definition by Valanis (4), as

$$dt' = |dU| \tag{4}$$

Step 3:
If equation (1) above is selected, this leads to a Bouc-type model. For an alternative, select equation (2),

$$\frac{dU}{dt'} = A\sinh(BF) \tag{5}$$

where t' is internal time, and A,B are constants.

Step 4:
Equation (5) can be partitioned as,

$$\frac{dU}{dt'} = \left[\frac{1}{\beta+\gamma}(\beta\, A\sinh(BF) + \gamma\, A\sinh(BF))\right] \tag{6},$$

where β, γ are constants. As with the Bouc model, this allows for the possibility of different loading and unloading shapes.

Considering the consistency of the signs of the terms of equation (6) corresponding to when the displacement is positive or negative, and noting equation (4), then equation (6) can be re-written as,

$$\frac{dU}{dt'} = \frac{A}{\beta+\gamma}\left[\beta\frac{F}{|F|}|\sinh(BF)| + \gamma\frac{dU}{|dU|}|\sinh(BF)|\right] \tag{7}$$

Returning to the rehological model of Fig.1, by compatibility considerations we get,

$$\frac{dU}{dt'} = \frac{dU_e}{dt'} + \frac{dU_p}{dt'} \tag{8}$$

where subscripts "e" and "p" refer to "elastic" and "endochronic plastic" respectively.
Hence,

$$\frac{dU}{dt'} = \frac{1}{K}\frac{dF}{dt'} + \frac{dU_p}{dt'} \tag{9}$$

where K is the spring stiffness. Therefore,

$$\frac{dF}{dt'} = K\left(\frac{dU}{dt'} - \frac{dU_p}{dt'}\right) \tag{10}$$

Hence from equation (7),

$$\frac{dF}{dt'} = K\left(\frac{dU}{dt'} - \frac{A}{\beta+\gamma}\left[\beta\frac{F}{|F|}|\sinh(BF)| + \gamma\frac{dU}{|dU|}|\sinh(BF)|\right]\right) \tag{11}$$

However,

$$\frac{dF}{|dU|} = \frac{dF}{dU}\,\frac{dU}{|dU|} \quad \text{and,} \quad \left(\frac{dU}{|dU|}\right)^2 = 1$$

Therefore, dividing equation (11) by $\frac{dU}{|dU|}$, and noting equation (4) we get,

$$\frac{dF}{dU} = K\left(1 - \frac{A}{\beta+\gamma}\left[\beta\ \frac{dU}{|dU|}\frac{F}{|F|}|\sinh(BF)| + \gamma|\sinh(BF)|\right]\right) \tag{12}$$

Normalizing the force and displacement by the force and displacement at the ultimate point of the primary curve, F_p and U_p respectively (i.e. $Z=F/F_p$ and $X=U/U_p$), and since, $\frac{dU}{|dU|}$ is the same as the signum function, sgn(), we get,

$$\frac{dZ}{dX} = \left(\frac{KU_p}{F_p}\right)\left(1 - \frac{A}{\beta+\gamma}\left[\beta\quad \operatorname{sgn}(X)\frac{Z}{|Z|}\ |\sinh(BF_pZ)| + \gamma|\sinh(BF_pZ)|\right]\right) \tag{13}$$

where Z, X are the normalized force and displacement respectively.

Equation (13) defines the basic hysteresis loop shape. If A is set to unity then the equation has 3 undetermined parameters whereas the Bouc model has 4.

From this point onwards, the capabilities of the model can be enhanced, as done for the Bouc model, by the use of augmentation functions. This may be needed to, for example, model the phenomena of loop pinching, or strength degradation, or stiffness degradation, or asymmetrical loop shape in the positive and negative directions, or combinations of these effects. Baber and Noori [6], among others, developed such augmentation functions for the Bouc-Wen model and these can be applied to the alternative models as well.

One can consider different behaviour in the opposite displacement directions, hence asymmetrical loops, for the most general case by supplying two sets of constants. If only the possibility of different peak resistances is considered then to achieve this, the "A" parameter would need to take on different values depending on whether the loop is in the positive or negative direction. To retain applicability to the equivalent linearisation method of non-linear stochastic structural dynamics then A must be a continuous and differentiable "switching" function such as that used by Dobson *et al* [7]. If applicability only to deterministic non-linear structural dynamic problems is required, then "A" can be supplied as discrete values. Therefore,

$$\frac{dZ}{dX} = \left(\frac{KU_p}{F_p}\right)\left(1 - \frac{A}{p_3+p_5}\left[p_3\operatorname{sgn}(X)\frac{Z}{|Z|}|\sinh(p_4F_pZ)| + p_5|\sinh(p_4F_pZ)|\right]\right) \tag{14}$$

where $A = p_1$ when $Z>0$, and $A = p_2$ when $Z<0$.

For stiffness and strength degradation equation (13) becomes,

$$\frac{dZ}{dX} = \left(\frac{KU_p}{F_p}\right)\left(1 - \eta(\varepsilon)\upsilon(\varepsilon)D(X,Z)\right) \tag{15}$$

where,

$$D(X,Z)=\frac{A}{p_3+p_5}\left[p_3\,\mathrm{sgn}(X)\frac{Z}{|Z|}|\sinh(p_4F_pZ)|+p_5|\sinh(p_4F_pZ)|\right] \tag{16}$$

$$\upsilon(\varepsilon)=1+\upsilon_0\,\varepsilon \tag{17}$$

$$\eta(\varepsilon)=1+\eta_0\,\varepsilon \tag{18}$$

$$\varepsilon=\int_{X_0}^{X_f} Z.dX \tag{19}$$

ε is the cumulative dissipated hysteretic energy (i.e. the sum of the loops areas), ν(ε) is the stiffness degradation function, η(ε) is the strength degradation function, ν_0 and η_0 are constants, and X_f, X_0 refer to the normalised displacements at any instant, and at the start of the excitation respectively.

For the consideration of pinching only, the Baber and Noori (6) function can be used. The resulting model is therefore given by:

$$\frac{dZ}{dX}=h(\varepsilon)\left[\left(\frac{KU_p}{F_p}\right)\left(1-D(X,Z)\right)\right] \tag{20}$$

where h(ε) is the pinching function and is given by,

$$h(\varepsilon)=1-g_1\,e^{-(z\,\mathrm{sgn}(X))^2/g_2^2} \tag{21}$$

$$g_1=p_6\left(1-e^{-p_7\varepsilon}\right) \tag{22}$$

$$g_2=\left(p_8+p_9\,\varepsilon\right)\left(p_{10}+g_1\right) \tag{23}$$

D(X,Z) and ε are previously given by equations (16) and (19). This model is referred to as the Da model. Application of the model to test data from 20 walls is the subject of another paper by Clarke (8).

Model 2

To develop Model 2, select equation (3) of the derivation methodology and proceed as per model 1. The resulting equations are the same as for Model 1, but equation (16) becomes,

$$D(X,Z)=\mathrm{sgn}(X)\left[\frac{A}{p_3+p_5}\left(p_3e^{p_4ZF_p}+p_5e^{p_4ZF_p}\right)-2\right] \tag{24}$$

MODELS COMPARATIVE PERFORMANCE

The following is the Bouc-Wen model augmented with the pinching function given by equations (19) and (21) to (23), and the asymmetry constants A of equation (14):

F = αKU + (1-α)KC where, (25)

$$\frac{dC}{dX}=h(\varepsilon)\left[\left(\frac{U_p}{F_p}\right)\left(p_6-A\left(p_3\,\mathrm{sgn}(X)|C|^{p_4-1}C+p_5|C|^{p_4}\right)\right)\right] \tag{26}$$

α, the ratio of post-yield to elastic stiffness, is set to 0.05. In each model, F_p and U_p is set to unity. Hence, Model 1 (i.e. Da model) and Model 2 each have 10 parameters, and the Bouc-Wen model has 11 parameters.

To compare the models' performance, one set of experimental hysteresis loop data from the testing of a shear wall is arbitrarily selected. This data comprises of 5 cycles of loading and the initial stiffness, K, is 25 kN/mm. Therefore, the only difference between the 3 models is due to their core loop shape function.

A system identification procedure is applied to each model to determine the parameters and at convergence, the error is noted. The procedure uses the Nelder and Mead Simplex algorithm and Runge-Kutta numerical integration and is described in the paper by Clarke [8].

RESULTS

The results of the system identification in terms of comparative performance are shown in Table 1.

Table 1. Comparative Performance of Hysteresis Models

	Model 1 (Da)	Model 2	Bouc-Wen
Error (%)	20.19	33.29	22.43
No. of iterations	176	208	461
Time (min)	11	12	24

Figures 2 to 5 show the experimental hysteresis loops, and the calculated hysteresis loops of Model 1 (Da), Model 2, and the Bouc-Wen model, respectively.

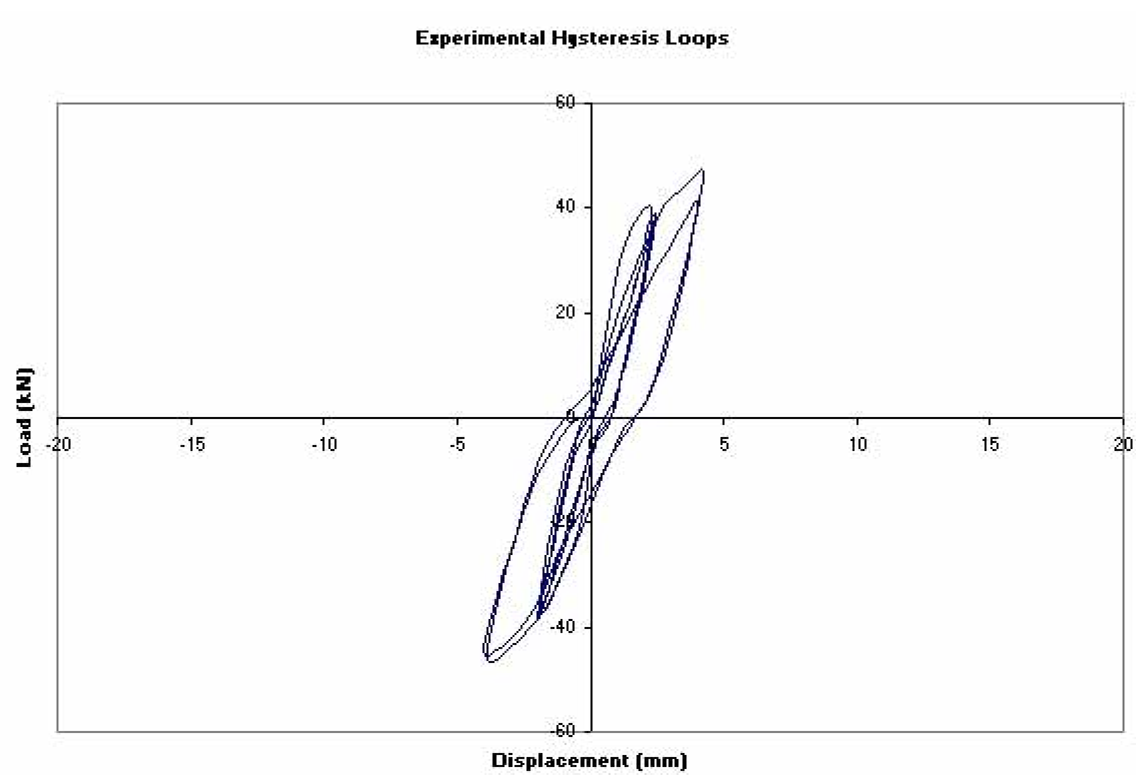

Figure 2. Experimental Hysteresis Loops

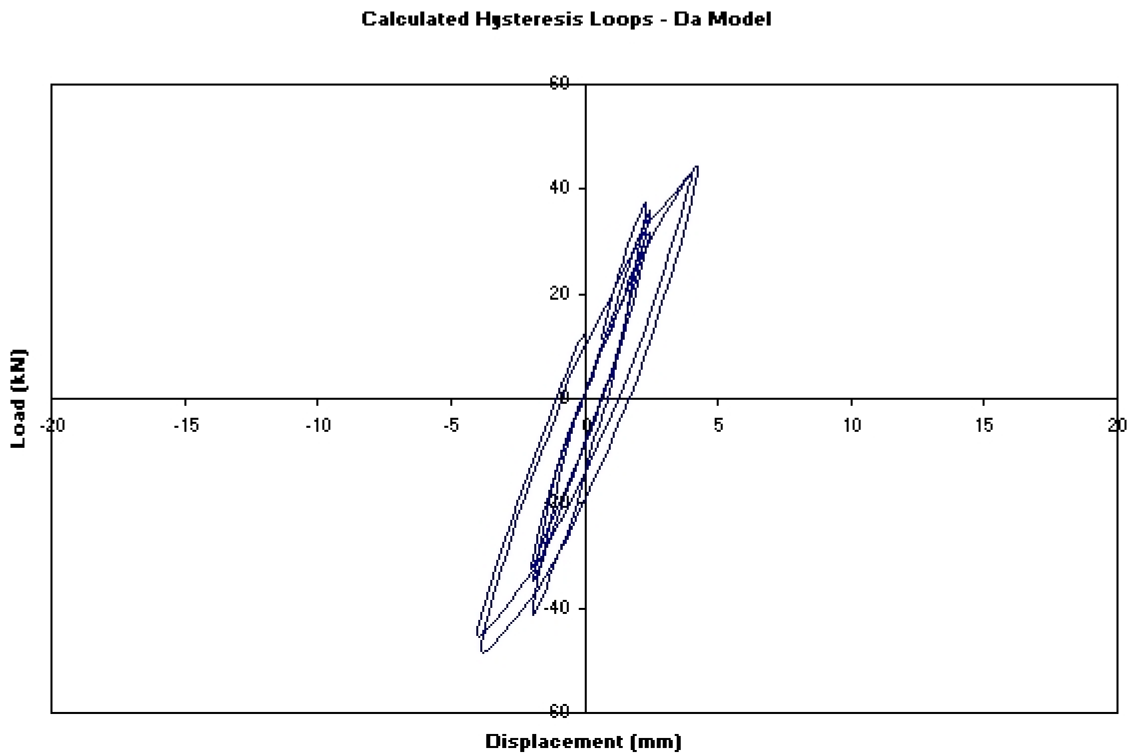

Figure 3. Calculated Hysteresis Loops – Model 1 (Da)

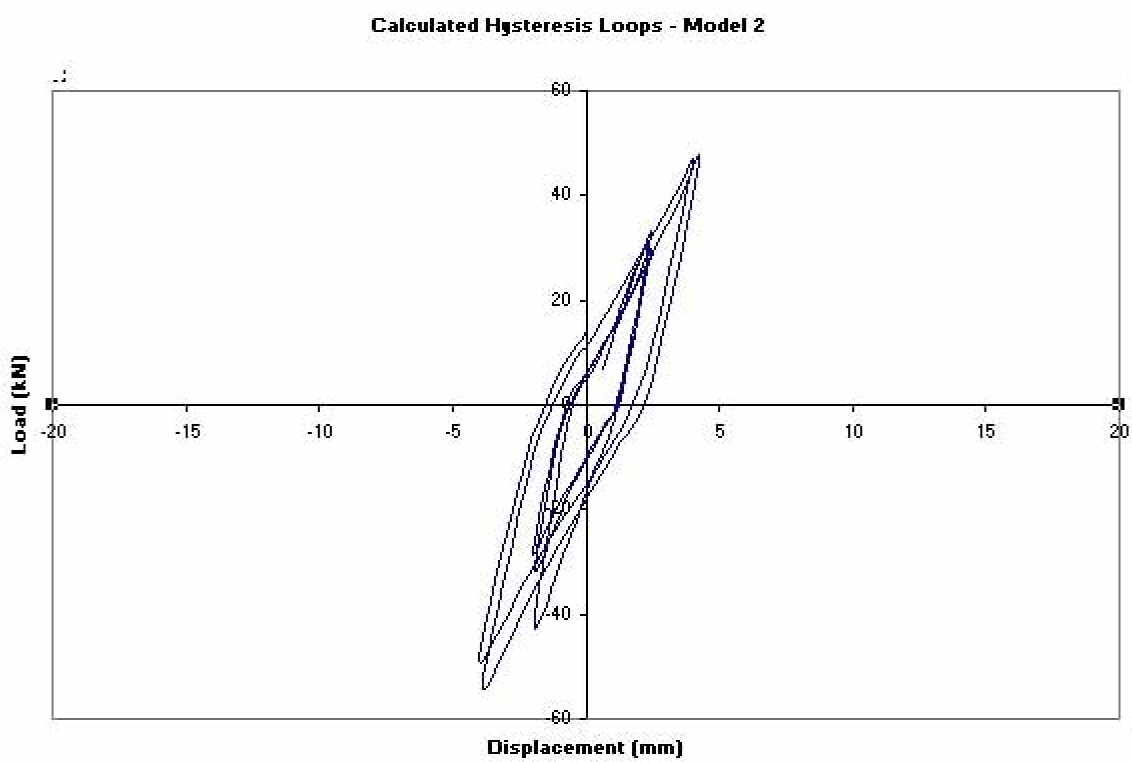

Figure 4. Calculated Hysteresis Loops – Model 2

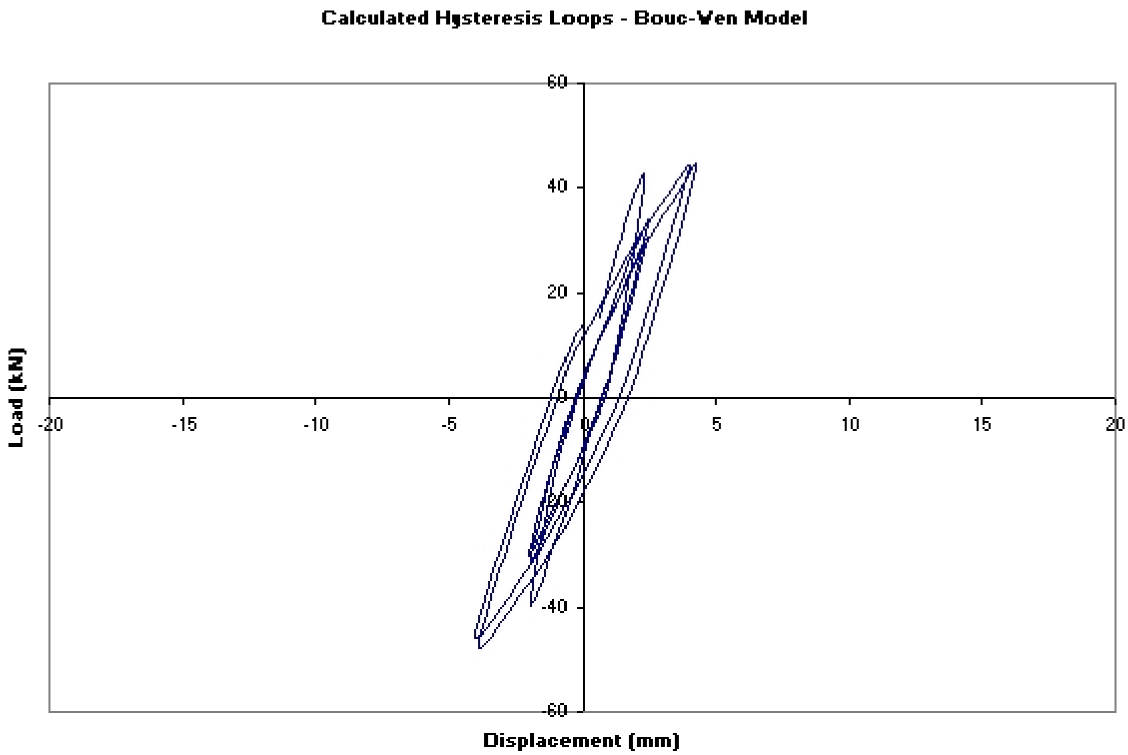

Figure 5. Calculated Hysteresis Loops – Bouc-Wen

DISCUSSION AND CONCLUSIONS

Table 1 indicates that the alternative smoothly-varying differential equation hysteresis Model 1 (Da Model) is more accurate than the Bouc-Wen model. It is also to be noted that the model requires one less parameter. The run time of the system identification computer program for Model 1 was less than half that of the Bouc-Wen model, and the number of iterations to convergence also significantly smaller. Given the endochronic plasticity framework it appears that the higher accuracy and efficiency is due to the core function sinh() compared to the power function, σ^n. In the development of the Ludwik Law of creep it was observed that better agreement with experimental data is obtained. The power function is more sensitive to changes in "n" compared to the sinh() and since the expansion of the sinh() is a sum of σ^n terms, the effect of using sinh is to smooth out the error over increments in the displacement. Comparing the calculated plots for Model 1 and the Bouc-Wen model, with the experimental plot, the former is also visibly in better agreement with the experimental plot. Though the difference in the accuracy is not large, the degree of visible dissimilarity seems to increase rapidly with increasing error. Since the system identification procedure does not start with the same initial estimates for each model, the observed faster convergence for Model 1 comapred with the Bouc-Wen model can be due to those estimates being closer to the final values. However, it is reasonable that the smaller number of parameters of Model 1 will result in faster execution.

Comparison of the results for Model 2 against those for the Bouc-Wen model indicates that the Bouc-Wen model is significantly more accurate though the former is faster and requires less parameters. This is not unexpected since the exponential function does not offer a means of smoothing out the values of the function as displacement increases, unlike the Bouc-Wen model in which case "n" has that effect.

The aforesaid conclusions regarding comparative performance are constrained by the fact that only one set of experimental data was investigated, and for a particular selection of model augmentation functions. Also, the issue within the context of the inelastic equations of motion, and the consideration of computational stability criteria remains to be explored. Nevertheless, a methodology for the systematic generation of alternative smoothly-varying differential equation hysteresis models has been presented and the results seem compelling.

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