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GENERAL H_∞ ENERGY CONTROL OF STRUCTURES AND ITS STABILITY ANALYSIS

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SUMMARY

In order to assign clear physical meanings to the H_∞ control in its applications to civil structures under seismic loads, a general energy control methodology is established in this study. With delicate definition of the regulated output vector, this H_∞ control can be interpreted as limiting the ratio for the sum of system energy and control energy with respect to the external excitation energy below a constant level. This limited value of output-to-input ratio is usually denoted in the H_∞ control literature by a conventional control parameter γ. Under this constraint, the balance between the system energy (including the elastic strain energy and kinetic energy) of structure and the control energy is then determined by the additional energy weighting parameters α and β. The performance of the controlled structure is consequently governed by the parameters appearing in the above two different phases. Based on the stability criterion of excluding pure imaginary or zero eigenvalues, the characteristic polynomial of the Hamiltonian matrix corresponding to the general H_∞ energy control is also explored in this paper to further derive an analytical expression for the greatest lower bound of γ that guarantees the system stability.

INTRODUCTION

Most of the early control theories, such as the conventional linear quadratic regulator (LQR) control, belong to the one-way control methodology where the external disturbance is ignored and the control force is solely manipulated to suppress the structural response. H_∞ control, on the other hand, is improved according to a more advanced two-way control philosophy where the control force and the external excitation are simultaneously considered to constrain the infinity norm of the transfer function between the excitation inputs and a set of regulated outputs. With this restriction, the goal of H_∞ control is to search for the least control force to withstand the worst external excitation. When first introduced in 1981 [1], the possibility of H_∞ control did not immediately gain much attention because of its corresponding mathematical difficulties in analysis. However, several methods attempted to simplify and analyze the optimal H_∞ control problem were explored by different researchers in the following few years [2-3]. The

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continuing works in late 1980s finally blossomed when Doyle et al. successfully solved this problem employing a unified framework of optimization in 1989 [4]. After 1990, more stringent and comprehensive mathematical basis was constructed for the H_∞ control theory [5] to reach a matured level and be gradually applied in various fields.

Even with well-established analytical techniques for H_∞ control, many practical concerns in its application to civil structures still require further study. For instance, design engineers would usually ask how to define the regulated outputs in H_∞ control such that desired physical quantities can be directly manipulated and what values of the control parameters should be adopted to attain the control goal at reasonable cost. In a recent study by the first author [6], an H_∞ energy control methodology was proposed to address the above issues. Via deliberately defining the regulated output vector in that study, the H_∞ control can be interpreted as limiting the ratio for the sum of system energy and control energy with respect to the external excitation energy below a constant level. This limited value of output-to-input ratio, usually denoted in the H_∞ control literature by a conventional control parameter γ, can be regarded as the first gate to regulate the control performance in H_∞ energy control. Either the strain or kinetic energy of structure was then individually adopted in that work to represent the system energy and the control performance was further determined with the additional use of another corresponding energy weighting parameter α or β. Accordingly, it led to two separate H_∞ energy control algorithms referred to as the elastic strain energy control and the kinetic energy control. In addition, with the introduction of γ, the stability of H_∞ controlled systems is no longer assured as in the conventional LQR control. Based on the two developed energy control formulations, another important phase of this study was to derive analytical expressions for γ that guarantee the system stability.

Although satisfactory results have been obtained from the above preliminary work [6] regarding H_∞ energy control, certain impractical simplifications were made to reduce the initial analytical difficulties. For example, the system energy usually can not be completely characterized with either the elastic strain energy or the kinetic energy of a system alone. It is consequently aimed in this article to further extend the H_∞ energy control theory such that a general control algorithm can be developed and its corresponding stability analysis can be effectively conducted for practical civil structures under seismic excitations.

$\mathbf{H}_\infty$ CONTROL WITH NONDIMENTIONALIZED CONTROL PARAMETER

Consider an actively controlled linear structure system represented by an n-DOF time-invariant discrete-parameter model. If this system is subjected to a $r\times 1$ external excitation vector $\mathbf{w}(t)$ and another $l\times 1$ control force vector $\mathbf{u}(t)$ is also applied, its equations of motion in state space can then be expressed as a first-order differential equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{E}\mathbf{w}(t) \tag{1}$$

where t denotes the time variable, $\mathbf{x}(t) = \langle \overline{\mathbf{x}}(t) \quad \dot{\overline{\mathbf{x}}}(t) \rangle^{\mathrm{T}}$ is a $2n\times 1$ state vector, and $\overline{\mathbf{x}}(t)$ represents the $n\times 1$ vector of structural displacements. In Equation (1), $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{E}$ symbolize the $2n\times 2n$ system matrix, $2n\times l$ control input matrix and $2n\times r$ disturbance input matrix, respectively, and take the form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\overline{\mathbf{M}}^{-1}\overline{\mathbf{K}} & -\overline{\mathbf{M}}^{-1}\overline{\mathbf{C}} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} \mathbf{0}_{n\times l} \\ -\overline{\mathbf{M}}^{-1}\overline{\mathbf{B}} \end{bmatrix}; \quad \mathbf{E} = \begin{bmatrix} \mathbf{0}_{n\times r} \\ -\overline{\mathbf{M}}^{-1}\overline{\mathbf{E}} \end{bmatrix} \tag{2}$$

where $\overline{\mathbf{M}}$ = $n\times n$ mass matrix of structure, $\overline{\mathbf{C}}$ = $n\times n$ viscous damping matrix of structure, $\overline{\mathbf{K}}$ = $n\times n$ stiffness matrix of structure, $\overline{\mathbf{B}}$ = $n\times l$ location matrix of control forces, and $\overline{\mathbf{E}}$ = $n\times r$ location matrix of external excitations; $\mathbf{0}$ signifies the zero matrix and $\mathbf{I}$ stands for the identity matrix, both with the corresponding dimension indicated in the subscript. Besides, the $p \times 1$ regulated output vector of the system can be generally defined as a linear combination of the system state and the control force vector:

$$\mathbf{z}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{3}$$

where the $p \times 2n$ matrix $\mathbf{C}$ and the $p \times l$ matrix $\mathbf{D}$ are usually called the output matrix and the direct transmission matrix, respectively.

The most commonly adopted norms in the control literature to measure a vector function $\mathbf{f}(t)$ include the H_2 norm:

$$\left\|\mathbf{f}(t)\right\|_2 = \left[\int_{-\infty}^{\infty} \mathbf{f}^{\mathrm{T}}(t)\mathbf{f}(t)\,\mathrm{d}t \right]^{\frac{1}{2}} \tag{4}$$

and the H_∞ norm:

$$\left\|\mathbf{f}(t)\right\|_\infty = \sup_t \left[\mathbf{f}^{\mathrm{T}}(t)\mathbf{f}(t) \right]^{\frac{1}{2}} \tag{5}$$

where sup indicates that the least upper bound is taken. As for measuring a system, it is most convenient to take the ratio of the output signal $\mathbf{z}(t)$ relative to the input excitation signal $\mathbf{w}(t)$ and this measurement depends on the norms adopted to measure the output and input signals. If both the output and input signals of a system S are measured using H_2 norm, the H_∞ norm of this system is defined as

$$\left\|S\right\|_\infty = \sup_{\mathbf{w}} \frac{\left\|\mathbf{z}(t)\right\|_2}{\left\|\mathbf{w}(t)\right\|_2} \tag{6}$$

Based on the definition in Equation (6), the concept of H_∞ control is to design the control force vector $\mathbf{u}(t)$ such that the output signal $\mathbf{z}(t)$ and the input excitation signal $\mathbf{w}(t)$ always satisfy the following condition under any circumstances:

$$\left\|S\right\|_\infty = \sup_{\mathbf{w}} \frac{\left\|\mathbf{z}(t)\right\|_2}{\left\|\mathbf{w}(t)\right\|_2} < \gamma \quad \text{or} \quad \int_0^{t_f} \mathbf{z}^{\mathrm{T}}(t)\ \mathbf{z}(t)\ \mathrm{d}t < \gamma^2 \int_0^{t_f} \mathbf{w}^{\mathrm{T}}(t)\ \mathbf{w}(t)\,\mathrm{d}t \tag{7}$$

where t_f represents the terminal time of the control process. In Equation (7), γ is a positive scalar parameter selected by the designer to guarantee that the ratio of output with respect to input is constrained under this specific value. A smaller designated value of γ means that more stringent performance of the controlled system is required. For covering different types of output, the formulation of Equation (7) is generally adopted in the H_∞ control literature and the dimension of γ totally depends on the dimension of $\mathbf{z}(t)$. Since the output term $\mathbf{z}^{\mathrm{T}}(t)\mathbf{z}(t)$ in Equation (7) should be established in corresponding to the system energy in H_∞ energy control, as will be discussed in the next section, Equation (7) can thus be deliberately redefined as

$$\left\|S\right\|_\infty = \sup_{\mathbf{w}} \frac{\left\|\mathbf{z}(t)\right\|_2}{\left\|\mathbf{w}(t)\right\|_2} < \frac{\gamma}{\sqrt{K_1}} \quad \text{or} \quad \int_0^{t_f} \mathbf{z}^{\mathrm{T}}(t)\ \mathbf{z}(t)\,\mathrm{d}t < \frac{\gamma^2}{K_1} \int_0^{t_f} \mathbf{w}^{\mathrm{T}}(t)\ \mathbf{w}(t)\,\mathrm{d}t \tag{8}$$

where K_1 can be chosen as any stiffness value, e.g., the effective stiffness of the first system vibration mode. With the definition of Equation (8), the parameter γ turns to be dimensionless and will simplify the presentation of many formulas in the following derivations, especially in the section of stability analysis. To mathematically analyze the H_∞ control problem, a performance index is usually defined as

$$J(\mathbf{u},\ \mathbf{w}) = \frac{1}{2} \int_0^{t_f} \left[\mathbf{z}^{\mathrm{T}}(t)\ \mathbf{z}(t)\ - \frac{\gamma^2}{K_1} \mathbf{w}^{\mathrm{T}}(t)\ \mathbf{w}(t) \right] \mathrm{d}t \tag{9}$$

Because the excitation input $\mathbf{w}(t)$ tends to increase the performance index J, but the control force input $\mathbf{u}(t)$ is designed to minimize J, Equation (8) can be further reformulated as

$$\min_{\mathbf{u}} \max_{\mathbf{w}} \frac{1}{2}\int_0^{t_f}\left[\mathbf{z}^{\mathrm{T}}(t)\ \mathbf{z}(t)-\frac{\gamma^2}{K_1}\mathbf{w}^{\mathrm{T}}(t)\ \mathbf{w}(t)\right]\mathrm{d}t=\min_{\mathbf{u}} \max_{\mathbf{w}} J<0 \tag{10}$$

If the type of state feedback is considered in designing the H_∞ control law, the whole problem is subsequently described as determining the control force in the form of

$$\mathbf{u}(t)=-\mathbf{G}\mathbf{x}(t) \tag{11}$$

such that the minmax problem of Equation (10) can be satisfied under the state motion constraint given by Equation (1). In Equation (11), **G** is conventionally called the feedback gain matrix. Calculus of variation has been applied in the literature [24] to solve this constrained optimization problem of H_∞ control and the resulting solution takes the form of the algebraic Riccati equation (ARE) as follows:

$$\begin{aligned}\mathbf{P}\left[\mathbf{A}-\mathbf{B}\ (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}\right]+\left[\mathbf{A}-\mathbf{B}\ (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}\right]^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left[\frac{K_1}{\gamma^2}\mathbf{E}\mathbf{E}^{\mathrm{T}}-\mathbf{B}\ (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{B}^{\mathrm{T}}\right]\mathbf{P}\\+\left[\mathbf{C}^{\mathrm{T}}\mathbf{C}-\mathbf{C}^{\mathrm{T}}\mathbf{D}\ (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{C}\right]=\mathbf{0}\end{aligned} \tag{12}$$

where **P** is the so-called Riccati matrix. It should be particularly indicated that the ARE of Equation (12) differs slightly from that in the literature by replacing γ^2 into γ^2/K_1 according to the modified definition for energy control as shown in Equation (8). In most practical applications, it is normally further assumed that

$$\mathbf{C}^{\mathrm{T}}\mathbf{D}=\mathbf{0}_{2n\times l}\quad \text{or}\quad \mathbf{D}^{\mathrm{T}}\mathbf{C}=\mathbf{0}_{l\times 2n} \tag{13}$$

With this condition, Equation (12) can be conveniently simplified to:

$$\mathbf{P}\mathbf{A}+\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left[\frac{K_1}{\gamma^2}\mathbf{E}\mathbf{E}^{\mathrm{T}}-\mathbf{B}\ (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{B}^{\mathrm{T}}\right]\mathbf{P}+\mathbf{C}^{\mathrm{T}}\mathbf{C}=\mathbf{0} \tag{14}$$

After solving for **P**, the optimal control force is obtained as

$$\mathbf{u}(t)=-(\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) \tag{15}$$

GENERAL $\mathbf{H}_\infty$ ENERGY CONTROL

Various options, e.g. directly in terms of displacement or velocity, can be adopted to define the output vector $\mathbf{z}(t)$ for associating the H_∞ control with apparent physical meanings in its applications to structural systems. However, the choices with displacement or velocity will have to assign **D** as a zero matrix and consequently make the control algorithm infeasible because of Equation (15). The essential concept of H_∞ energy control, on the other hand, is aimed to correspond the whole output term $\mathbf{z}^{\mathrm{T}}(t)\mathbf{z}(t)$ in Equation (9) to a certain type of energy. To attain this goal, the $p\times 2n$ output matrix **C** and the $p\times l$ direct transmission matrix **D** should be deliberately specified such that Equation (13) can first be satisfied and then leads to

$$\mathbf{z}^{\mathrm{T}}(t)\mathbf{z}(t)=\mathbf{x}^{\mathrm{T}}(t)\mathbf{C}^{\mathrm{T}}\mathbf{C}\mathbf{x}(t)+\mathbf{u}^{\mathrm{T}}(t)\mathbf{D}^{\mathrm{T}}\mathbf{D}\mathbf{u}(t) \tag{16}$$

In Equation (16), the first term on the right-hand side is a function of the state vector and can be evidently used for corresponding to the system energy, whereas the second term relates to the control force and can be directly employed to characterize the control energy. As a result, $\mathbf{z}^{\mathrm{T}}(t)\mathbf{z}(t)$ represents the sum of these two energies and the H_∞ control expressed by Equation (8) or (10) can be subsequently interpreted as restricting this sum under a designated ratio of the energy produced by the external excitation.

If the term $\mathbf{x}^{\mathrm{T}}(t)\mathbf{C}^{\mathrm{T}}\mathbf{C}\mathbf{x}(t)$ in Equation (16) is intended to sum up the elastic strain energy and the kinetic energy as the total system energy and the term $\mathbf{u}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{D}\mathbf{u}$ is aimed to hold the dimension of energy, $\mathbf{z}(t)$ can be first defined as a $(2n+l)\times 1$ vector (i.e., $p = 2n+l$) such that **C** and **D** can then be designed as

$$\mathbf{C} = \begin{bmatrix} \sqrt{\alpha}\bar{\mathbf{K}}^{\frac{1}{2}} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \sqrt{\beta}\bar{\mathbf{M}}^{\frac{1}{2}} \\ \mathbf{0}_{l\times n} & \mathbf{0}_{l\times n} \end{bmatrix} \quad \text{and} \quad \mathrm{D} = \frac{1}{\sqrt{K_1}}\begin{bmatrix} 0_{n\times l} \\ 0_{n\times l} \\ \mathrm{I}_{l\times l} \end{bmatrix} \tag{17}$$

where α is a weighting parameter associated with the strain energy and β is another weighting parameter to regulate the kinetic energy, both nonnegative and dimensionless. With the form of Equation (17), it should be noted that Equation (13) is satisfied and Equation (16) can be reduced to

$$\mathbf{z}^{\mathrm{T}}(t)\mathbf{z}(t) = \alpha\,\bar{\mathbf{x}}^{\mathrm{T}}(t)\bar{\mathbf{K}}\bar{\mathbf{x}}(t) + \beta\,\dot{\bar{\mathbf{x}}}^{\mathrm{T}}(t)\bar{\mathbf{M}}\dot{\bar{\mathbf{x}}}(t) + \frac{1}{K_1}\mathbf{u}^{\mathrm{T}}(t)\mathbf{u}(t) \tag{18}$$

The terms $\alpha\,\bar{\mathbf{x}}^{\mathrm{T}}\bar{\mathbf{K}}\bar{\mathbf{x}}$ and $\beta\,\dot{\bar{\mathbf{x}}}^{\mathrm{T}}(t)\bar{\mathbf{M}}\dot{\bar{\mathbf{x}}}(t)$ in Equation (18) are proportional to the elastic strain energy and kinetic energy of structural system, respectively, and the other term $\mathbf{u}^{\mathrm{T}}\mathbf{u}/K_1$ corresponds to the control energy. Therefore, when the conventional control parameter γ is specified in this case, it implies that the sum of the strain energy, the kinetic energy, and the control energy should be limited to a level lower than γ with respect to the energy produced by the external excitation $\mathbf{w}^{\mathrm{T}}\mathbf{w}/K_1$. Under this constraint, the balance between the system energy of structure and the control energy is determined by the weighting parameters α and β. More specifically, increasing the values of α and β would lead to the requirement of larger control energy and smaller associated system energy.

With the form of **C** and **D** in Equation (17), the state feedback control law of Equation (15) can be reduced to

$$\mathbf{u}(t) = -K_1\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{x}(t) \tag{19}$$

Substitution of Equation (19) into the system state Equation (1) results in

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} - K_1\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{P}\right)\mathbf{x}(t) + \mathbf{E}\mathbf{w}(t) = \mathbf{A}_1\,\mathbf{x}(t) + \mathbf{E}\mathbf{w}(t) \tag{20}$$

where $\mathbf{A}_1$ represents the controlled system matrix. Equation (20) is a first-order linear differential equation which can be conveniently solved using convolution integral. For direct applications to practical cases of structural control, Equation (19) and the solution of Equation (20) need to be further discretized. In addition, the conventional control parameter γ and the additional energy weighting parameters α and β have to be selected by the design engineer before applying the H_∞ energy control. With the given control parameters and available system parameters, the ARE of Equation (14) can then be solved to obtain the Riccati matrix **P**. Following the above off-line work, the on-line analysis algorithm for H_∞ energy control has been summarized in Reference 26 and will not be repeated in this paper.

STABILITY ANALYSIS

It has been shown in the literature [7] that there exists a unique positive definite solution (Riccati matrix) for the ARE corresponding to the conventional LQR control and the stability of the controlled system can be consequently guaranteed. With the introduction of γ, however, the positive definiteness of the Riccati matrix and the control stability is no longer assured for H_∞ control unless γ is restricted to be greater than a specific value. Except for certain special cases, there are no general analytical formulas in the literature for determining this greatest lower bound (g.l.b.) of γ (denoted as $\gamma_{\text{g.l.b.}}$ in this paper) and numerical procedures such as the bi-section method have to be utilized. Based on the general H_∞ energy control proposed in this study, it will be demonstrated in this section that an analytical expression for $\gamma_{\text{g.l.b.}}$ can be mathematically derived to provide a more efficient way in determining the feasible range of γ.

Stability Criterion

With the definitions of **C** and **D** for the general H_∞ energy control shown in Equation (17), $\mathbf{D}^{\mathrm{T}}\mathbf{D}=\mathbf{I}_{l\times l}/K_1$ and the ARE of Equation (14) can be written as

$$\mathbf{PA}+\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left[K_1\left(\frac{1}{\gamma^2}\mathbf{E}\mathbf{E}^{\mathrm{T}}-\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)\right]\mathbf{P}+\mathbf{C}^{\mathrm{T}}\mathbf{C}=\mathbf{0} \tag{21}$$

To solve the $2n\times 2n$ Riccati matrix **P** from this nonlinear matrix equation, it is most convenient to transform Equation (21) into a linear eigenvalue problem. Let **Ω** denote the diagonal matrix constituted by all the eigenvalues of $\mathbf{A}+\left[K_1\left(\mathbf{E}\mathbf{E}^{\mathrm{T}}/\gamma^2-\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)\right]\mathbf{P}$ and **Φ** represent the matrix formed by all the corresponding eigenvectors. Assuming

$$\mathbf{P}=\bar{\mathbf{P}}\,\mathbf{\Phi}^{-1} \tag{22}$$

and following the detailed derivation described in the literature [8], it will lead to an equivalent form of Equation (21):

$$\begin{bmatrix}\mathbf{A} & K_1\left(\frac{1}{\gamma^2}\mathbf{E}\mathbf{E}^{\mathrm{T}}-\mathbf{B}\,\mathbf{B}^{\mathrm{T}}\right)\\ -\mathbf{C}^{\mathrm{T}}\mathbf{C} & -\mathbf{A}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}\\ \bar{\mathbf{P}}\end{bmatrix}=\begin{bmatrix}\mathbf{\Phi}\\ \bar{\mathbf{P}}\end{bmatrix}\mathbf{\Omega} \tag{23}$$

where

$$\mathbf{H}=\begin{bmatrix}\mathbf{A} & K_1\left(\frac{1}{\gamma^2}\mathbf{E}\mathbf{E}^{\mathrm{T}}-\mathbf{B}\,\mathbf{B}^{\mathrm{T}}\right)\\ -\mathbf{C}^{\mathrm{T}}\mathbf{C} & -\mathbf{A}^{\mathrm{T}}\end{bmatrix} \tag{24}$$

is usually referred to as the Hamiltonian matrix. After the eigenvalue problem of Equation (23) is solved, **P** can be obtained from Equation (22).

It should be noted that **H** is a $4n\times 4n$ matrix with a total of $4n$ eigenvalues and $4n$ corresponding eigenvectors. However, Equation (23) merely contains $4n\times 2n$ simultaneous equations and the Riccati solution **P** for the ARE of Equation (21) is a $2n\times 2n$ matrix. Therefore, only $2n$ eigenvalues and their $2n$ corresponding eigenvectors of **H** are needed for Equation (22) in solving **P**. The subsequent question is then: which $2n$ eigenvalues and eigenvectors should be adopted among $4n$ of them? It has been shown in the literature [8] that the $2n$ eigenvalues with negative real parts have to be selected from all the eigenvalues of **H** in the determination of **P** such that the system stability can be guaranteed. In addition, it has also been proved that the eigenvalues of the Hamiltonian matrix **H** appear in pairs of $\pm\lambda_1,\ \pm\lambda_2,\ \cdots,\ \pm\lambda_{2n}$ [9]. With this feature, at most $2n$ eigenvalues with negative real parts can be obtained for **H**. A stability criterion for the H_∞ controlled system can consequently be established by prohibiting any eigenvalues of **H** locating on the imaginary axis of the complex plane. Based on this stability criterion, it will be shown in the following subsection that the theoretical formula for $\gamma_{\text{g.l.b.}}$ can be analytically established for a general single-degree-of-freedom (SDOF) system.

Analytical stability formula for SDOF systems

For an SDOF system with mass $\bar{\mathbf{M}} = m$, stiffness $\bar{\mathbf{K}} = K_1 = k$, and damping $\bar{\mathbf{C}} = c$, its natural frequency and damping ratio can be express as $\omega = \sqrt{(k/m)}$ and $\zeta = c/2m\omega$. If the general H_∞ energy control is applied, $\bar{\mathbf{B}} = 1$ and $\bar{\mathbf{E}} = 1$ in this case and its 4×4 Hamiltonian matrix $\mathbf{H}$ is in the form of

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & -2\zeta\omega & 0 & \dfrac{\omega^2}{m}\left(\dfrac{1}{\gamma^2}-1\right) \\ -\alpha m\omega^2 & 0 & 0 & \omega^2 \\ 0 & -\beta m & -1 & 2\zeta\omega \end{bmatrix} \tag{25}$$

To solve the eigenvalue problem of $\mathbf{H}$, its characteristic determinant should be expanded into the corresponding fourth-order characteristic polynomial:

$$\begin{aligned} \det\left[\mathbf{H} - \lambda \mathbf{I}_{4x4}\right] &= \det \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ -\omega^2 & (-2\zeta\omega - \lambda) & 0 & \dfrac{\omega^2}{m}\left(\dfrac{1}{\gamma^2}-1\right) \\ -\alpha\ m\omega^2 & 0 & -\lambda & \omega^2 \\ 0 & -\beta m & -1 & (2\zeta\omega - \lambda) \end{bmatrix} \\ &= \lambda^4 + \omega^2\left[2 - 4\zeta^2 + \beta\left(\frac{1}{\gamma^2}-1\right)\right]\lambda^2 + \omega^4\left[1 - \alpha\left(\frac{1}{\gamma^2}-1\right)\right] \\ &= \lambda^4 + a_2\lambda^2 + a_0 \end{aligned} \tag{26}$$

where

$$a_2 = \omega^2\left[2 - 4\zeta^2 + \beta\left(\frac{1}{\gamma^2}-1\right)\right] \quad \text{and} \quad a_0 = \omega^4\left[1 - \alpha\left(\frac{1}{\gamma^2}-1\right)\right] \tag{27}$$

It is noteworthy that this polynomial contains only the terms in even orders. Consequently, its corresponding eigenvalues must appear in pairs of opposite signs, as indicated in the previous subsection.

Solving the characteristic polynomial shown in Equation (26) directly leads to

$$\lambda^2 = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_0}}{2} \tag{28}$$

To satisfy the stability criterion that excludes pure imaginary or zero eigenvalues, λ^2 is not allowed holding negative or zeroing real values. In other words, the two roots for λ^2 illustrated in Equation (28) can only be a pair of complex conjugates or two positive real numbers. For the former case to be true, it is necessary that

$$a_2^2 - 4a_0 = \omega^4\left[\beta^2\left(\frac{1}{\gamma^2}-1\right)^2 + 4\left(\alpha + \beta - 2\beta\zeta^2\right)\left(\frac{1}{\gamma^2}-1\right) + 16\zeta^2\left(\zeta^2 - 1\right)\right] < 0 \tag{29}$$

and the set containing all the feasible values of γ to satisfy Equation (29) can be expressed as

$$S_1 = \left\{\gamma \;\middle|\; \beta^2\left(\frac{1}{\gamma^2}-1\right)^2 + 4\left(\alpha + \beta - 2\beta\zeta^2\right)\left(\frac{1}{\gamma^2}-1\right) + 16\zeta^2\left(\zeta^2 - 1\right) < 0\right\} \tag{30}$$

On the other hand, for the validity of the latter case, three conditions are simultaneously required to have

$$a_2^2 - 4a_0 \geq 0 \tag{31}$$

leading to

$$\overline{S}_1=\left\{\gamma \;\middle|\; \beta^2\left(\frac{1}{\gamma^2}-1\right)^2+4\left(\alpha+\beta-2\beta\zeta^2\right)\left(\frac{1}{\gamma^2}-1\right)+16\zeta^2\left(\zeta^2-1\right)\geq 0\right\} \tag{32}$$

$$a_2=\omega^2\left[2-4\zeta^2+\beta\left(\frac{1}{\gamma^2}-1\right)\right]<0 \tag{33}$$

leading to

$$S_2=\left\{\ \gamma \;\middle|\; \left[\beta+2\left(2\zeta^2-1\right)\right]\gamma^2>\beta\right\} \tag{34}$$

and

$$a_0=\omega^4\left[1-\alpha\left(\frac{1}{\gamma^2}-1\right)\right]>0 \tag{35}$$

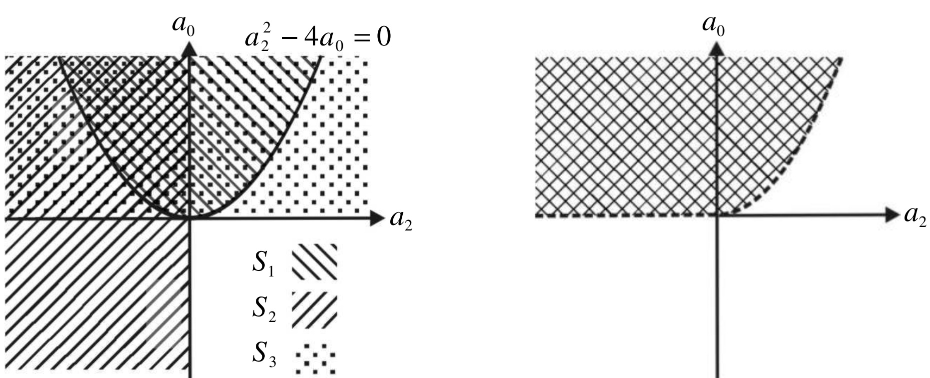
leading to

$$S_3=\left\{\gamma \;\middle|\; \gamma^2>\frac{\alpha}{\alpha+1}\right\} \tag{36}$$

In Equation (32), $\overline{S}_1$ signifies the complementary set of S_1. Illustrations for the three sets S_1, S_2, and S_3 in the a_2 - a_0 plane are shown in Figure 1(a), clearly indicating that S_1 belongs to S_3 (i.e., $S_1 \subset S_3$). Finally, the set containing all the possible values of γ to maintain the system stability can be written and arranged as

$$\begin{aligned} S_1\cup\left(\overline{S}_1\cap S_2\cap S_3\right)&=\left(S_1\cup\overline{S}_1\right)\cap\left(S_1\cup S_2\right)\cap\left(S_1\cup S_3\right)\\ &=\left(S_1\cup S_2\right)\cap S_3 \end{aligned} \tag{37}$$

where $\cup$ and $\cap$ denote the union and intersection set operation, respectively. With Equation (37), the analytical formula of $\gamma_{\text{g.l.b.}}$ can be systematically derived by separately investigating the feasible ranges of γ from the three sets S_1, S_2, and S_3 first and then performing the required union and intersection operations.



(a) S_1, S_2, and S_3 (b) $\left(S_1\cup S_2\right)\cap S_3$

Figure 1. Illustrations of S_1, S_2, S_3, and $\left(S_1\cup S_2\right)\cap S_3$

Starting with S_1 to discuss the stable range of γ, it is found that Equation (30) is composed of a quadratic inequality of $\left(1/\gamma^2\right)-1$, which corresponds to the following three possible solutions:

$$S_1=\left\{\frac{\beta^2}{d_2}>\gamma^2>\frac{\beta^2}{d_1}\right\},\ \text{when}\ d_1\geq d_2>0 \tag{38a}$$

$$S_1 = \left\{ \gamma^2 > \frac{\beta^2}{d_1} \right\}, \text{ when } d_1 > 0 \ge d_2 \tag{38b}$$

$$S_1 = \text{empty set, otherwise} \tag{38c}$$

where

$$d_1 = \beta^2 - 2\left(\alpha + \beta - 2\beta\zeta^2\right) + 2\sqrt{\left(\alpha+\beta\right)^2 - 4\alpha\beta\zeta^2} \tag{39}$$

and

$$d_2 = \beta^2 - 2\left(\alpha + \beta - 2\beta\zeta^2\right) - 2\sqrt{\left(\alpha+\beta\right)^2 - 4\alpha\beta\zeta^2} < d_1 \tag{40}$$

On the other hand, the linear inequality of $\left(1/\gamma^2\right) - 1$ in Equation (34) can be easily arranged to obtain two possible solutions of S_2:

$$S_2 = \left\{ \gamma^2 > \frac{\beta^2}{d_3} \right\}, \text{ when } \beta + 2\left(2\zeta^2 - 1\right) > 0 \tag{41a}$$

$$S_2 = \text{empty set, when } \beta + 2\left(2\zeta^2 - 1\right) \le 0 \tag{41b}$$

where

$$d_3 = \beta\left[\beta + 2\left(2\zeta^2 - 1\right)\right] = d_2 + 2\alpha + 2\sqrt{\left(\alpha+\beta\right)^2 - 4\alpha\beta\zeta^2} \ge d_2 \tag{42}$$

It should be also noted that ζ^2 must be smaller than 1/2 when $\beta + 2\left(2\zeta^2 - 1\right) < 0$.

After solving S_1 and S_2 individually, Equation (37) indicates that the union of these two sets has to be subsequently taken. Nevertheless, there exist three potential solution sets for S_1 depending on the values of d_1 and d_2 and two possible solution sets for S_2 depending on the value of d_3, as expressed in Equations (38) and (41). For the convenience of systematically analyzing $S_1 \cup S_2$, the arrangement of d_1, d_2, and d_3 in the real axis needs to be discussed in advance. With this reason, a discriminant

$$D = \beta + 2\alpha\left(1 - 2\zeta^2\right) \tag{43}$$

is introduced in this study. According to the derivation in Appendix I, it can be proved that

$$d_3 \ge d_1, \text{ when } D \le 0 \tag{44a}$$

$$d_3 < d_1, \text{ when } D > 0 \tag{44b}$$

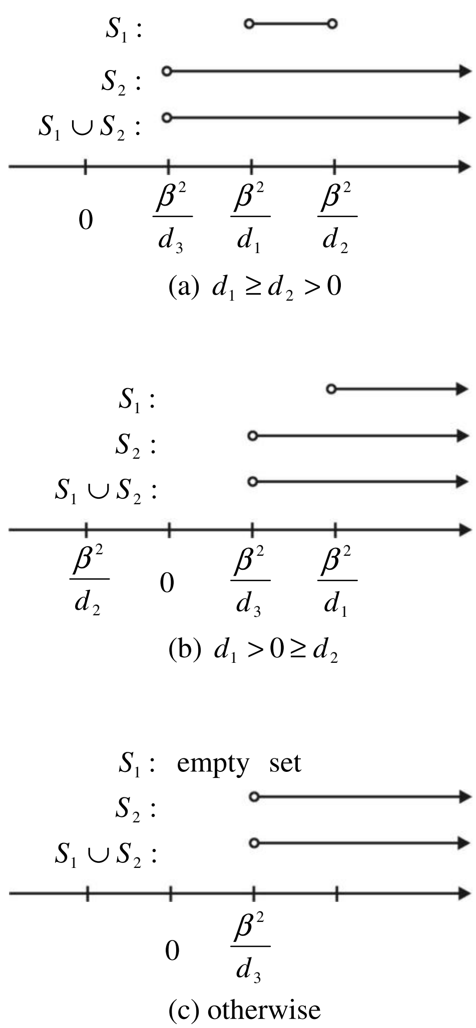
The case of $D = \beta + 2\alpha\left(1 - 2\zeta^2\right) \le 0$ is first examined. Since ζ^2 has to be greater than 1/2 in this case, it will naturally lead to $\beta + 2\left(2\zeta^2 - 1\right) > 0$ and the solution of S_2 is consequently in the form of Equation (41a). Consider next the three feasible solutions of S_1 shown in Equations (38a), (38b), and (38c). Union operation is then taken for Equation (41a) and each of these three sets to result in three possible cases of $S_1 \cup S_2$ when $D \le 0$. Parts (a), (b), and (c) of Figure 2 illustrate these three cases, respectively, and it can be obviously concluded that

$$S_1 \cup S_2 = \left\{ \gamma^2 > \frac{\beta^2}{d_3} \right\}, \text{ when } \beta + 2\alpha\left(1 - 2\zeta^2\right) \le 0 \tag{45}$$

As for the case of $D > 0$, it leads to $d_1 > d_3 \ge d_2$ from Equations (42) and (44b) and can be demonstrated that $d_1 > 0$ (see Appendix II). When $d_2 > 0$, the solutions of S_1 and S_2 have to be in the forms of Equation (38a) and (41a), respectively, and $S_1 \cup S_2$ in this case is illustrated in Figure 3(a). If $d_2 \le 0$, on the other hand, the solution of S_1 is in the form of Equation (38b) and the solution of S_2 can be either in

the form of equation (41a) or (41b). Illustrations for $S_1 \cup S_2$ in both of these two cases are shown in Figures 3(b) and 3(c). From the three parts of Figure 3, it can be summarized that
at

$$S_1 \cup S_2 = \left\{ \gamma^2 > \frac{\beta^2}{d_1} \right\}, \text{ when } \beta + 2\alpha\left(1 - 2\zeta^2\right) > 0 \tag{46}$$



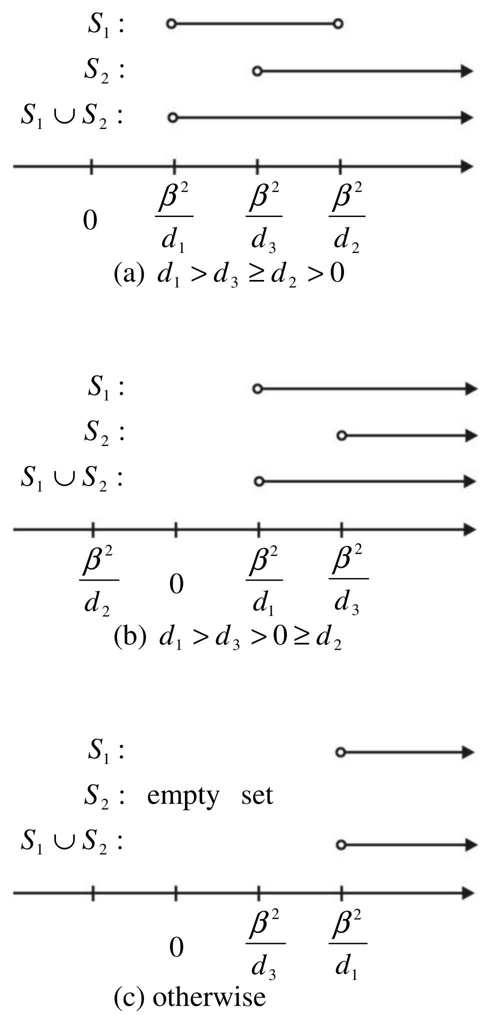


Figure 2. Illustrations of three possible cases for $S_1 \cup S_2$ when $D \leq 0$

Figure 3. Illustrations of three possible cases for $S_1 \cup S_2$ when $D > 0$

According to Equation (37), intersection operation should be finally taken for the solution set of S_3 expressed in Equation (36) and each of the two possible forms of $S_1 \cup S_2$ shown in Equations (45) and (46) to complete the stability analysis. Since the derivation in Appendix III can further prove that

$$\frac{\alpha}{\alpha+1} \geq \frac{\beta}{\beta + 2(2\zeta^2 - 1)}, \text{ when } \beta + 2\alpha\left(1 - 2\zeta^2\right) \leq 0 \tag{47}$$

and

$$\frac{\beta^2}{\beta^2-2\left(\alpha+\beta-2\beta\zeta^2\right)+2\sqrt{\left(\alpha+\beta\right)^2-4\alpha\beta\zeta^2}}\geq\frac{\alpha}{\alpha+1} \tag{48}$$

under any circumstances, the results from the above intersection process can be concisely simplified to a stable range of $\gamma > \gamma_{\text{g.l.b.}}$ with the greatest lower bound

$$\gamma_{\text{g.l.b.}}=\sqrt{\frac{\alpha}{\alpha+1}}\text{, when }\beta+2\alpha\left(1-2\zeta^2\right)\leq 0 \tag{49a}$$

and

$$\gamma_{\text{g.l.b.}}=\sqrt{\frac{\beta^2}{\beta^2-2\left(\alpha+\beta-2\beta\zeta^2\right)+2\sqrt{\left(\alpha+\beta\right)^2-4\alpha\beta\zeta^2}}}\text{, when }\beta+2\alpha\left(1-2\zeta^2\right)>0 \tag{49b}$$

With Equations (49a) and (49b), the stable limit $\gamma_{\text{g.l.b.}}$ can be analytically determined in a very convenient way as long as the three parameters α, β, and ζ are given for a general SDOF system under the general H_∞ energy control proposed in this study. On another note, it should be noted that the expression in Equation (49b) will result in zero values both in the numerator and denominator in the special case of $\beta=0$ and $0\leq\zeta<\sqrt{2}/2$. To deal with this problem, L'Hospital's rule can be applied to take the limit of Equation (49b) to obtain

$$\gamma_{\text{g.l.b.}}=\sqrt{\frac{\alpha}{\alpha+4\zeta^2(1-\zeta^2)}} \tag{50}$$

From the analytical formulas in Equations (49a) and (49b), it can be further verified that $\gamma_{\text{g.l.b.}}$ is always less than one and the details are described in Appendix IV. Moreover, $\gamma_{\text{g.l.b.}}$ is merely related to the strain energy weighting parameter α if $\beta+2\alpha\left(1-2\zeta^2\right)\leq 0$. When $\beta+2\alpha\left(1-2\zeta^2\right)>0$, however, $\gamma_{\text{g.l.b.}}$ turns to depend on α, β, and ζ. Figure 4 illustrates how the value of $\gamma_{\text{g.l.b.}}$ varies with ζ for different values of α and β. It is first observed that $\gamma_{\text{g.l.b.}}$ generally decreases with the increasing value of ζ and stays at a constant level if $\beta+2\alpha\left(1-2\zeta^2\right)\leq 0$ (i.e., $\zeta\geq\sqrt{(2\alpha+\beta)/(4\alpha)}$). Besides, $\gamma_{\text{g.l.b.}}$ is also found to increase with the increasing value of $\alpha+\beta$. But, under the condition of keeping a constant value for $\alpha+\beta$, the balance between α and β has no crucial effects on the values of $\gamma_{\text{g.l.b.}}$, especially in the cases with smaller values of ζ. As for the cases with larger values of ζ, the greater value of α generally induce a trend to increase the value of $\gamma_{\text{g.l.b.}}$. For most of the civil structures whose damping ratios are small (usually less than 10%), their corresponding values of $\gamma_{\text{g.l.b.}}$ will always follow the analytical formula of Equation (49b). No matter what values of α and β are adopted, it can either observe from the portion of $\zeta\leq 0.1$ in Figure 4 or take the limit of $\zeta\to 0$ for Equation (49b) to conclude that the value of $\gamma_{\text{g.l.b.}}$ is very close to one. Therefore, $\gamma_{\text{g.l.b.}}=1$ can be used as a rule of thumb for designing the parameter γ if the general H_∞ energy control is applied in practical cases.

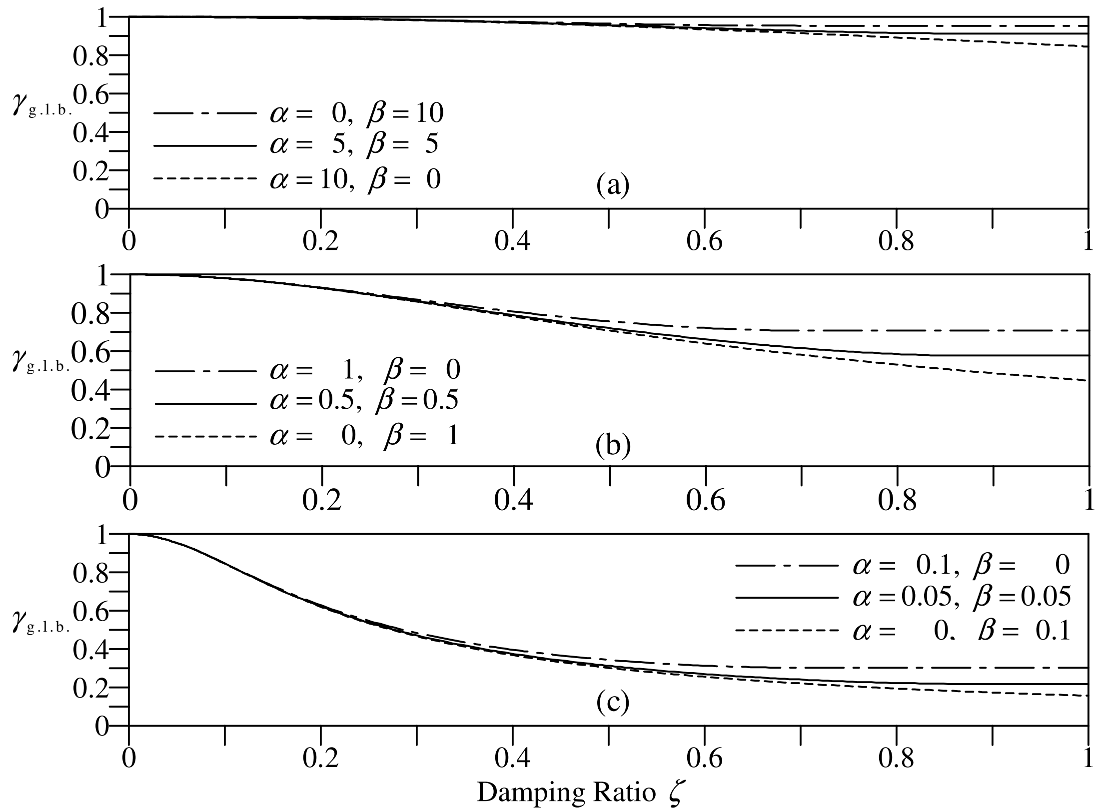


Figure 4. Greatest lower bound of SDOF structural system under H_∞ energy control

CONCLUSIONS

To associate H_∞ control with lucid physical meanings in its applications to civil structures under seismic loads, a general H_∞ energy control formulation is established in this study. In this new energy control algorithm, the conventional parameter γ is specified to limit the ratio for the sum of system energy and control energy with respect to the external excitation energy below this constant level. Under this constraint, the balance between the system energy (including the elastic strain energy and kinetic energy) of structure and the control energy is then determined by the additional energy weighting parameters α and β. The performance of the controlled structure is consequently governed by the parameters appearing in the above two different phases. In other words, more significant reduction in the structural response can be attained by either adopting smaller value of γ or larger value of $\alpha+\beta$, at the expense of requiring larger control force.

Based on the stability criterion of excluding pure imaginary or zero eigenvalues, the characteristic polynomial of the Hamiltonian matrix corresponding to the general H_∞ energy control is also explored in this paper for a general SDOF structural system to further derive an analytical expression for the greatest lower bound of γ. From this formula, it can be easily concluded that $\gamma_{g.l.b.}$ is always less than one, generally increases with increasing value of $\alpha+\beta$, and decrease with increasing value of ζ. For most of the civil structures, usually with light damping, it is also found that the value of $\gamma_{g.l.b.}$ is very close to one and $\gamma_{g.l.b.} = 1$ can be used as a rule of thumb for applying the energy control in practical cases.

For simplicity, the stability analysis and parametric study in this paper is focused on the SDOF structural system even though the H_∞ energy control algorithm is generally developed for any multi-degree-of-freedom (MDOF) system. Generalization of the analytical methodology in stability analysis to determine $\gamma_{\mathrm{g.l.b.}}$ for an MDOF system may further require performing the procedures of reduction-of-order and diagonalization to simplify the characteristic determinant of the higher-order Hamiltonian matrices.

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