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# A TWO-DIMENSIONAL INTEGRAL FORMULATION WITH FUNDAMENTAL SOLUTIONS FOR DYNAMIC POROELASTICITY IN TIME DOMAIN

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## SUMMARY

This paper presents a direct boundary element approach for solving two-dimensional problems of dynamic poroelasticity in the time domain. The derivation of the time-dependent integral equations is based on Biot's theory and reciprocal theorem. An analytical form of two-dimensional transient fundamental solutions for porous media saturated with incompressible fluid in the case of u-p formulation is obtained. After an analytical temporal integration of the fundamental solution kernels, a boundary element time-marching procedure is established. The comparison of different time interpolation functions shows that the mixed time interpolation gives more stable solutions. In addition, the linear θ method is used in order to improve the numerical stability of time-stepping procedure. Finally, two examples are presented to investigate the stability and the accuracy of this approach for wave propagation analysis.

## INTRODUCTION

The efficiency of Boundary Element Method (BEM) in dealing with the wave propagation phenomena in infinite or semi-infinite elastic media is well recognized. This method is actually widely applied in geomechanics, seismology and earthquake engineering. But the application of this method to dynamic poroelasticity has, until now, been limited. The first integral formulations for poroelastodynamics based on solid and fluid displacements have been established in Laplace transform by Manolis & Beskos [1]. Nevertheless, it can be shown that only the solid displacements and fluid pressure are independent (Bonnet [2]). Based on these four unknowns (three in 2D), integral formulations have been developed by Cheng *et al* [3], Dominguez [4] for frequency domain, and by Chen [5], Gatmiri & Kamalian [6] for Laplace domain. However, it is more natural to work in the real time domain and observe the phenomenon as it evolves. Wiebe & Ante [7] suggested a time domain formulation in terms of solid and fluid displacements and with the restriction of vanishing damping between solid and fluid particles. Another formulation of Chen & Dargush [8] is based on inverse transformation of Laplace domain fundamental

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solution. Recently, Schanz [9] proposed a time-stepping procedure, using only the Laplace transformed fundamental solution and convolution quadrature method.

A specific and important subject in BEM approach is to determine the fundamental solution, which is the response of the medium to unit point excitation. But a such solution is not available for porous elastic media although more half a century has passed since Biot [10] has developed the full dynamic poroelasticity theory. The first attempt to obtain fundamental solution for dynamic poroelasticity was made by Burridge & Vargas [11]. Later, Wiebe & Antes [7] found a time-domain solution by neglecting the viscous coupling, Kaynia & Banerjee [12] derived an approximation of transient short-time solution. The Burridge & Vargas' solution was obtained for three forces, which is not enough, while the solution of Wiebe & Antes and Kaynia & Banerjee was sought for six variables ($u_i$, $w_i$) and six forces, which is too much. Based on four unknowns *($u_i$, $p$)*, Chen [5] derived a complete fundamental solution in Laplace transform domain and, in the same work, proposed two approximations for corresponding transient solution. Gatmiri & Kamalian [6] suggested an adapted solution to the u-p formulation.

In this paper, a time domain integral formulation for dynamic poroelasticity is presented. In order to obtain time-dependent fundamental solutions, two simplifications are introduced. Firstly, the u-p formulation (Zienkiewicz *et al* [13]) is considered. Secondly, it is assumed that solid particles and fluid are incompressible. According to these assumptions, an analytical transient fundamental solution is derived (Gatmiri & Nguyen [14]), this formulation is more appropriate for the problems involving water saturated soils under earthquake solicitations. For the numerical implementation, three types of time interpolation functions are introduced and time integration is performed analytically. The numerical instability of BEM solution during the time-stepping procedure is observed, as in elastodynamics. In order to improve the numerical stability, the linear θ method (Yu *et al* [15]) is incorporated. In addition, the mixed time interpolation is recommended because it gives a more stable solution.

## GOVERNING EQUATIONS

By using the basic principles of continuum mechanics [13], the equations governing the transient response of poroelastic media can be expressed as follows:

$$\sigma_{ij,j} + F_i = \rho\, \ddot{u}_i + \rho_f \ddot{w}_i \tag{1}$$

$$\sigma_{ij} = \lambda u_{i,i} + \mu(u_{i,j} + u_{j,i}) - \alpha p \delta_{ij} \tag{2}$$

$$p = M(\zeta - \alpha u_{i,i}) \tag{3}$$

$$\zeta + w_{i,i} = Q \tag{4}$$

$$p_{,i} = -\frac{1}{\kappa}\dot{w}_i - \rho_f \ddot{u}_i - m\ddot{w}_i + f_i \tag{5}$$

$u_i$ is the displacement of the solid skeleton, $p$ denotes the fluid pressure, $w_i$ represents the average displacement of the fluid relative to the solid skeleton and $\zeta$ represents the increment of fluid content; $\lambda$ and $\mu$ are drained Lamé constants; $\rho_f$ is the fluid density, $\rho_s$ is the solid density, $\rho=(1-n)\,\rho_s + n\rho_f$ is the density of solid-fluid mixte and $m=\rho_f/n$ [13] is another mass parameter where $n$ is the porosity; $\kappa=k/\eta$ is the permeability coefficient, with $\eta$ and $k$ denoting the fluid dynamic viscosity and the intrinsic permeability of the solid skeleton respectively; $\alpha$ and $M$ are material parameters which describe the relative compressibility of the constituents and are defined as $\alpha=1-K_d/K_s$, $1/M=n/K_f+(\alpha-n)/K_s$ where $K_s$,$K_f$ are the bulk modulus of solid grains and the fluid while $K_d$ denote that of solid skeleton (drained bulk modulus); $F_i$ and $f_i$ are respectively the buck and fluid body force; $Q$ denotes the cumulative injected volume from a fluid source injection of strength γ.

## TIME DOMAIN BOUNDARY INTEGRAL EQUATIONS

### Reciprocal theorem

A reciprocal theorem generally provides a convenient starting point for a direct boundary integral equation. For the current coupled problem, Cheng & Predeleanu [16] have derived local reciprocal relation by using the constitutive equations (2), (3) and the symmetry of the elasticity tensor:

$$\sigma_{ij}\varepsilon_{ij}^{*} + p\zeta^{*} = \sigma_{ij}^{*}\varepsilon_{ij} + p^{*}\zeta \tag{6}$$

which relates two independent states at different spatial and time coordinates, denoted by a symbol * and without * superscript. In order to yields the reciprocal work theorem, equation (6) must be integrated over the solution domain and time. Then, after integrating by parts and considering the linear strain-displacement relation, the divergence theorem, the equilibrium equation (1) and the generalized Darcy's law (5), one obtains:

$$\begin{aligned}
0 = & \int_{\Gamma}\left[(t_i \times u_i^{*} - t_i^{*} \times u_i) - (p \times w^{*} - p^{*} \times w)\right]d\Gamma \\
& \int_{\Omega}\left[(F_i \times u_i^{*} - F_i^{*} \times u_i) + (f_i \times w_i^{*} - f_i^{*} \times w_i) - (Q \times p^{*} - Q^{*} \times p)\right]d\Omega \\
& \int_{\Omega}\Big[(\rho u_i + \rho_f w_i)v_{io}^{*} - (\rho u_{io} + \rho_f w_{io})\dot{u}_i^{*} + (\rho \dot{u}_i + \rho_f q_i)u_{io}^{*} - (\rho v_{io} + \rho_f q_{io})u_i^{*} \\
& + (\rho_f u_i + m w_i)q_{io}^{*} - (\rho_f u_{io} + m w_{io})q_i^{*} + (\rho_f \dot{u}_i + m q_i)w_{io}^{*} - (\rho_f v_{io} + m q_{io})w_i^{*} \\
& + \frac{1}{\kappa}(w_{io}w_i^{*} - w_i w_{io}^{*}) \ \Big]d\Omega
\end{aligned} \tag{7}$$

where symbol × indicates a Riemann convolution integral; subscript $o$ denotes the initial values; $t_i$ represents boundary traction vector, $w$ is normal component of relative fluid-solid displacement; Γ is the bounding surface of domain Ω, and $n_i$ the component of the unit outward normal to Γ.

### Integral equations

It appears from equation (7) that with the respective substitutions of delta functions (in space and in time) for the bulk body force $F_i$ and fluid dilatation $Q$, we can obtain Somigliana-type integral equations with solid displacements $u_i$ and fluid pressure $p$ as the left-hand side. However, these equations involving $w_i$ are not yet in suitable form for a BEM implementation since the physical conditions are hardly given in term of relative fluid-solid displacements. A formulation with displacements, tractions, pore pressure and fluid flux as primary quantities is therefore more suitable. For this purpose, it is better to consider the sources (point forces and fluid injection) which are delta functions in space and Heaviside step functions in time as:

$$F_{ij}^{*} = \delta_{ij}\delta(x-\chi)H(t-\tau), \quad f_i^{*} = 0, \quad \gamma^{*} = \delta(x-\chi)H(t-\tau) \tag{8}$$

Therefore, using these sources and admitting homogeneous initial conditions and absence of body forces and fluid injection lead to the following integral equations:

$$c_{ij}(\chi)u_j(\chi,t) = \int_{\Gamma}\left[(\dot{G}_{ij} \times t_i - \dot{H}_{ij} \times u_i) - (H_{3j} \times p - G_{3j} \times q\right]d\Gamma \tag{9}$$

$$c(\chi)p(\chi,t) = \int_{\Gamma}\left[(\ddot{G}_{i3} \times t_i - \ddot{H}_{i3} \times u_i) - (\dot{H}_{33} \times p - \dot{G}_{33} \times q\right]d\Gamma \tag{10}$$

or in a more convenient matrix notation as

$$\begin{bmatrix} c_{ij} & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} u_j \\ p \end{bmatrix} = \int_\Gamma \left( \begin{bmatrix} \dot{G}_{ij} & G_{3j} \\ \ddot{G}_{i3} & \dot{G}_{33} \end{bmatrix} \times \begin{bmatrix} t_i \\ q \end{bmatrix} - \begin{bmatrix} \dot{H}_{ij} & H_{3j} \\ \ddot{H}_{i3} & \dot{H}_{33} \end{bmatrix} \times \begin{bmatrix} u_i \\ p \end{bmatrix} \right) d\Gamma \tag{11}$$

In equations (9), (10), (11) we have introduced new tensor functions $G_{ij}$, $G_{3j}$, $H_{ij}$, $H_{3j}$ which are the displacement, pressure, traction and normal flux kernels due to a unit step point force in the *j*-direction along with $G_{i3}$, $G_{33}$, $H_{i3}$, $H_{33}$ which are similar kernels but due to a unit step point source of fluid injection. Notice that the traction and flux kernels $H_{ij}$, $H_{3j}$, $H_{i3}$, $H_{33}$ can be obviously derived from the displacement and pressure kernels through the constitutive relationship presented in the previous section. This can even be further compacted by generalizing the displacement and traction vectors to include pore pressure and flux, respectively, as an additional component. Then (11) becomes

$$c_{\alpha\beta}(\chi)\, u_\beta(\chi, t) = \int_\Gamma \left[ G_{\alpha\beta} \times t_\beta - H_{\alpha\beta} \times u_\beta \right] d\Gamma \tag{12}$$

$\alpha$, $\beta$ vary from 1 to 3; $c_{\alpha\beta}(\chi)$ is a matrix of constants and dependent only upon the local geometry of the boundary at $\chi$: $c_{\alpha\beta}=\delta_{\alpha\beta}/2$ for points $\chi$ on a smooth boundary and $c_{\alpha\beta}=\delta_{\alpha\beta}$ for points $\chi$ in the domain $\Omega$.

## FUNDAMENTAL SOLUTION

### Laplace transformed fundamental solution

Taking the Laplace transform of equations (1)-(5), assuming that the initial conditions for $u_i$ *and* $w_i$ are zero and performing appropriate substitution, one obtains a set of coupled differential equations with solid displacement $\tilde{u}_i$ and fluid pressure $\tilde{p}$ as four independent variables to describe the behavior of the porous medium (Chen [5]) :

$$\mu \tilde{u}_{i,jj} + (\lambda + \mu)\tilde{u}_{j,ji} - \rho_1 s^2 \tilde{u}_i - \alpha^* \tilde{p}_{,i} + \tilde{f}_i = 0 \tag{13}$$

$$\zeta\, \tilde{p}_{,ii} - \frac{s}{M}\tilde{p} - \alpha^* \tilde{u}_{i,i} + \tilde{\gamma} = 0 \tag{14}$$

where *s* is the Laplace transform parameter and the tilde denotes the Laplace transformation,
$\zeta = 1/(ms + 1/\kappa)$, $\alpha^* = \alpha - \rho_f s\zeta$, $\rho_1 = \rho - \rho_f^2 s\zeta$ .

The explicit two-dimensional fundamental solution of equations (13), (14) was derived by Chen [5]. But the expressions of this transformed solution are too complex, so the analytical transient fundamental solution has long been thought to be extremely difficult, even considered impossible. In order to overcome this difficulty, we simplified the basic equations (1)-(5) so that the transformed solution of the modified equations can analytically be inversed. For this purpose, we adopted two assumptions. Firstly, the second time derivative of the relative fluid displacements is considered small and all terms involving $\ddot{w}_i$ are neglected. That corresponds to the well-known u-p formulation which is appropriate for the medium-speed phenomena. This approximation is valid for most problems of earthquake analysis and frequencies slower than this. Secondly, we assume that solid particles and fluid are incompressible. Indeed, the constant *M*, which represents the combined compressibility of the fluid and solid phases, tends to infinity and one can remove the term containing this coefficient. This hypothesis is valid for most soils. Moreover, such compressibility is now relatively unimportant as the wave motion in the fluid is essentially excluded by the omission of the fluid acceleration term. Thus the equations (13), (14) are modified as following:

$$\mu \tilde{u}_{i,jj} + (\lambda + \mu)\tilde{u}_{j,ji} - \rho s^2 \tilde{u}_i - \alpha \tilde{p}_{,i} + \tilde{f}_i = 0 \tag{15}$$

$$\kappa \tilde{p}_{,ii} - \alpha^* \tilde{u}_{i,i} + \tilde{\gamma} = 0 \tag{16}$$

By using the Kupradze method, the two-dimensional transformed solution is the following

$$\tilde{G}_{ij} = (A_{ij} H_{12} + B_{ij} H_{11}) \frac{\Lambda^2}{\rho s^3} - A_{ij} \frac{1}{r} \frac{\Lambda^2 - \lambda_1^2}{\lambda_1^2 \rho s^3} - (A_{ij} H_{22} + B_{ij} H_{21}) \frac{\lambda_2^2}{\rho s^3} + \frac{C_{ij}}{s} H_{21} \tag{17a}$$

$$\tilde{G}_{3j} = \frac{r_{,j}}{2\pi(\lambda + 2\mu)\kappa} \alpha^* \left( \frac{1}{\lambda_1^2 r} - H_{12} \right) \tag{17b}$$

$$\tilde{G}_{i3} = \frac{r_{,i}}{2\pi(\lambda + 2\mu)\kappa} \frac{\alpha}{s} \left( \frac{1}{\lambda_1^2 r} - H_{12} \right) \tag{17c}$$

$$\tilde{G}_{33} = \frac{1}{2\pi\kappa s \lambda_1^2} \left[ (\lambda_1^2 - \Lambda^2) H_{11} - \Lambda^2 \ln(r) \right] \tag{17d}$$

in which

$$\begin{array}{lll}
H_{k1}(r,s) = K_o(\lambda_k r) & H_{k2}(r,s) = K_1(\lambda_k r)/\lambda_1 & k=1,2 \\
A_{ij}(r) = (2r_{,i}r_{,j} - \delta_{ij})/2\pi r & B_{ij}(r) = r_{,i}r_{,j}/2\pi & C_{ij} = \delta_{ij}/2\pi\mu \\
\Lambda^2 = \rho s^2/(\lambda + 2\mu) & \lambda_1^2 = ((s+\beta)^2 - \beta^2)/v_1^2 & \lambda_2^2 = s^2/v_2^2 \\
v_1 = \sqrt{(\lambda + 2\mu)/(\rho - \alpha\rho_f)} & v_2 = \sqrt{\mu/\rho} & \beta = \alpha^2/(2\kappa(\rho - \alpha\rho_f))
\end{array}$$

$K_o(\lambda_k r)$ and $K_1(\lambda_k r)$ are the modified Bessel function of the second kind of zero and first order.

**Transient fundamental solution**

After an inverse Laplace transformation, we have the corresponding analytical form of time-dependent fundamental solution as following

$$\begin{aligned} G_{ij}(r,t) = {} & e_1 \int_{r/v_1}^{t} h_{12}(r,\tau) d\tau + e_2 \int_{r/v_1}^{t} h_{11}(r,\tau) d\tau + e_3 t^2/2 + e_4 t + e_5 (1 - e^{-2\beta t}) \\ & + \left[ e_8 \ln(t + \sqrt{t^2 - r^2/v_2^2}) + e_6 v_2^2 t \sqrt{t^2 - r^2/v_2^2}/2r - e_8 \ln(r/v_2) \right] H(t - r/v_2) \end{aligned} \tag{18a}$$

$$G_{3j}(r,t) = r_{,j} \left[ d_1 h_{12}(r,t) + d_2 \dot{h}_{12}(r,t) \right] + r_{,j} (d_3 + d_4 e^{-2\beta t})/r \tag{18b}$$

$$G_{i3}(r,t) = r_{,i} f_1 \int_{r/v_1}^{t} h_{12}(r,\tau) d\tau + r_{,i} (f_2 t + f_3 (1 - e^{-2\beta t}))/r \tag{18c}$$

$$G_{33}(r,t) = \int_{r/v_1}^{t} (g_1 + g_2 e^{-2\beta(t-\tau)}) h_{11}(r,\tau) d\tau + g_2 \ln(r) e^{-2\beta t} \tag{18d}$$

The expressions of the functions $h_{kl}(r,t)$ as well as the coefficients *d, e, f, g* are given in the appendix.

Since the analytical expressions of the fundamental solutions are extremely complicated, it is better to investigate graphically. The new time-domain fundamental solution are compared to the Chen's complete solution [5] obtained by numerical Laplace transform inversion (figures 1-4). The soil material properties are: $\lambda$ = 12,5 MPa; $\mu$ = 8,33 MPa; $K_s = 10^5$ MPa; $K_f = 0{,}22.10^4$ MPa; $\alpha$ = 1; $\rho_s$ = 2600 kg/m$^3$; $\rho_f$ = 1000 kg/m$^3$; $m$ = 5533,33 kg/m$^3$; $\kappa$ = 3,55.10$^{-2}$ m/s. The applied force point (or fluid source point) is located at (0;0), the receiver is chosen at the  coordinate (0,1;0,2).

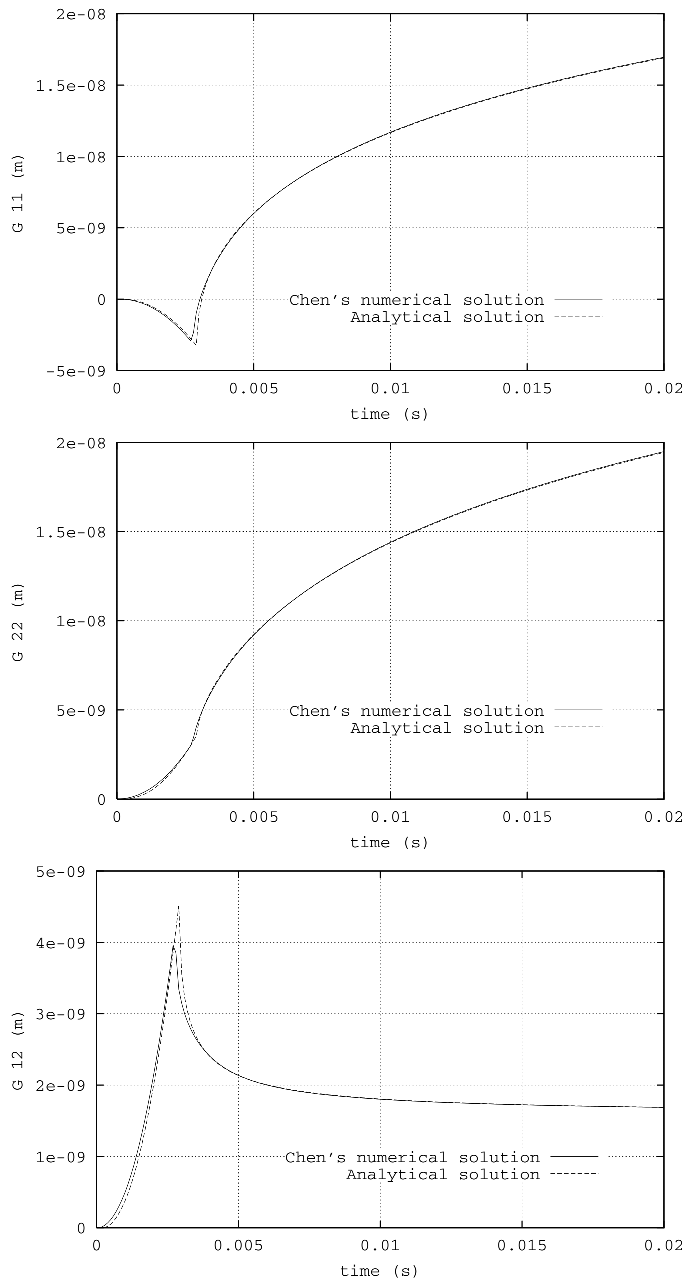



**Figure 1: Two-dimensional displacement time history at (0,1;0,2) due to point force at (0;0)**

From this numerical result, several features of the new fundamental solution are deduced as following

- This new solution agrees very well with the complete solution at the long time. As the time increases, both of these solutions approach to the corresponding quasi-static state. Nevertheless, they differentiate one with the other at the short time, especially for the components of pressure $G_{3j}$ and $G_{33}$.

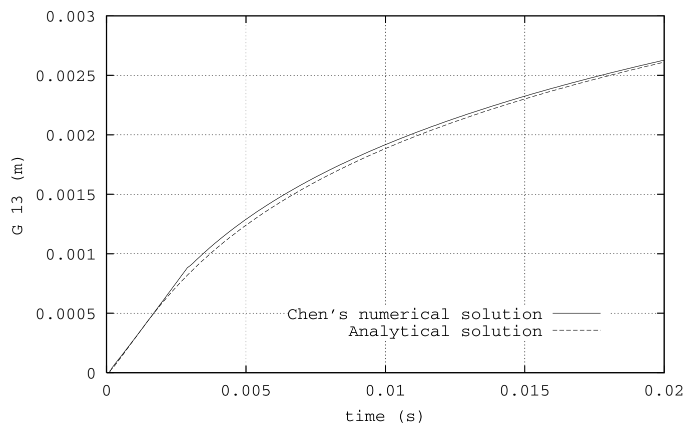



**Figure 2: Two-dimensional displacement time history at (0,1;0,2) due to fluid injection at (0;0)**

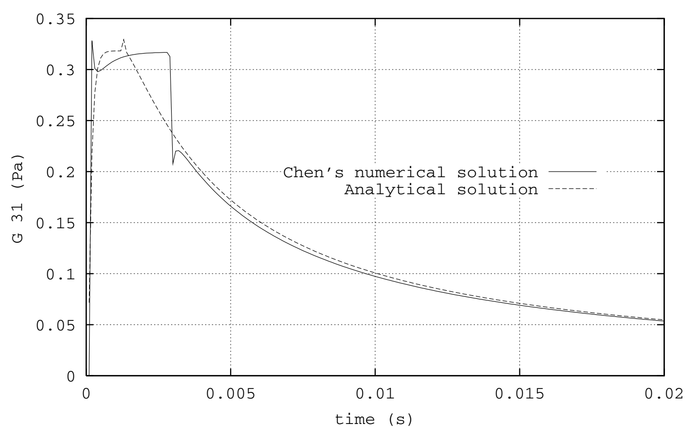



**Figure 3: Two-dimensional pressure time history at (0,1;0,2) due to point force at (0;0)**

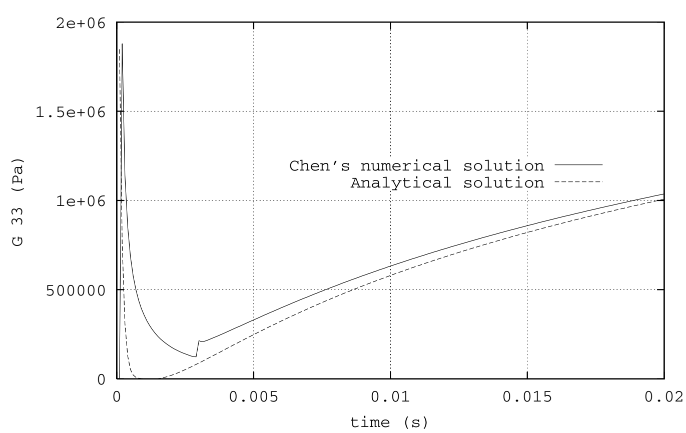



**Figure 4: Two-dimensional pressure time history at (0,1;0,2) due to fluid injection at (0;0)**

- In the expressions of the new solution, there are only two waves: pressure wave $P_1$ and shear wave $S$ associated with the coefficients $\lambda_1$ and $\lambda_2$, respectively. The arrival time of these two waves can be detected on figures 1-4 by sudden changes appearing in displacements or by pulse appearing in excessive pore fluid pressure, while the pulse associated to the coming of low compression wave $P_2$ does not appears in the curves of the new solutions. Indeed, once the assumption of the incompressibility of the components is adopted, there is only one dilation in phase of the ensemble solid-fluid. Thus, the slow-velocity wave $P_2$ is completely replaced by a process of consolidation. Consequently, the initial values of the pressures $G_{3j}$ and $G_{33}$ differ from zero, and the diffusion begins early before the arrival of the waves.

- The velocity of pressure wave $P_1$ calculated by analytical solution is rather small to that obtained by the complete solution.

## NUMERICAL IMPLEMENTATION

Equation (12) can be considered as generalized Somigliana-type integral equations for dynamic poroelasticity. This equation is formally equivalent to the one obtained for elastodynamics, but with a different range of the indices. Therefore, the numerical treatment of equation (12) is achieved following the usual point collocation procedure as in elastodynamics. We have a system of algebraic equations containing the generalized displacements and tractions at all collocation points at time $t=N\Delta t$

$$\overline{\mathbf{H}}^1\mathbf{u}^N = \mathbf{G}^1\mathbf{t}^N + \sum_{1}^{N-1}\left(\mathbf{G}^{N-m+1}\mathbf{t}^m - \mathbf{H}^{N-m+1}\mathbf{u}^m\right) \tag{19}$$

or finally, after applying the specified boundary conditions

$$\mathbf{A}\mathbf{X}^N = \mathbf{B}\mathbf{Y}^N + \sum_{1}^{N-1}\left(\mathbf{G}^{N-m+1}\mathbf{t}^m - \mathbf{H}^{N-m+1}\mathbf{u}^m\right) \tag{20}$$

where $\mathbf{u}^m$, $\mathbf{t}^m$ are nodal generalized displacements and tractions at the moment $t_m$; $\mathbf{G}^{N-m+1}$, $\mathbf{H}^{N-m+1}$ are kernel matrices, with $c_{\alpha\beta}$ included in the diagonal blocks of $\overline{\mathbf{H}}^1$; $\mathbf{X}^N$, $\mathbf{Y}^N$ are respectively the unknown and known components of $\mathbf{u}^N$, $\mathbf{t}^N$ and $\mathbf{A}$, $\mathbf{B}$ are the associated coefficients matrices.

Equation (20) can be solved for the unknown boundary values using any standard solution procedure. The spatial integrals are performed numerically by invoking self-adaptive schemes to ensure both accuracy and efficiency. Strongly singular blocks in $\overline{\mathbf{H}}^1$ are evaluated indirectly with a generalization of the rigid body technique (Chen [5]). This technique is valid only for a closed boundary, thus for half-space problems; the domain of interest must be enclosed with a fictitious boundary (Ahmad & Banerjee [17]). The temporal integration is achieved analytically according to the choice of time interpolation functions: constant, linear or mixed. During a time step, the field variables (generalized displacements and tractions) are assumed to remain constant in the case of constant interpolation functions, while in the case of linear interpolation functions, they vary linearly. However, we noted that the displacements $u_i$ and the fluid flow $q$ are always continuous, it is thus logical to use the piecewise linear temporal interpolation function for these quantities. On the other hand, the tractions $t_i$ and the pore pressure $p$ can be discontinuous in time in certain cases: the sudden application of the boundary external forces or the reflection of wave at a fixed boundary. In these cases, the use of a mixed kind of variation seems more correct. In this formulation, the tractions and pressure are assumed to remain constant during two consecutive time-steps while the displacements and fluid flux are taken to be linear during a time step, then the generalized displacement and traction kernels take the form

$$G_{\alpha\beta}^k(x) = \left[\begin{array}{c:c} \frac{1}{2}G_{ij}\Big|_{(k-2)\Delta t}^{k\Delta t} & \frac{1}{\Delta t}\left(G_{i3}\Big|_{(k-1)\Delta t}^{k\Delta t} - G_{i3}\Big|_{(k-2)\Delta t}^{(k-1)\Delta t}\right) \\ \hdashline \frac{1}{2}\overline{G}_{3j}\Big|_{(k-2)\Delta t}^{k\Delta t} & \frac{1}{\Delta t}\left(\overline{G}_{33}\Big|_{(k-1)\Delta t}^{k\Delta t} - \overline{G}_{33}\Big|_{(k-2)\Delta t}^{(k-1)\Delta t}\right) \end{array}\right] \tag{21a}$$

$$H^{k}_{\alpha\beta}(x)=\left[\begin{array}{c:c} \frac{1}{\Delta t}\left(\overline{H}_{ij}\Big|^{k\Delta t}_{(k-1)\Delta t}-\overline{H}_{ij}\Big|^{(k-1)\Delta t}_{(k-2)\Delta t}\right) & \frac{1}{2}\dot{H}_{i3}\Big|^{k\Delta t}_{(k-2)\Delta t} \\ \hdashline \frac{1}{\Delta t}\left(\overline{\overline{H}}_{3j}\Big|^{k\Delta t}_{(k-1)\Delta t}-\overline{\overline{H}}_{3j}\Big|^{(k-1)\Delta t}_{(k-2)\Delta t}\right) & \frac{1}{2}H_{33}\Big|^{k\Delta t}_{(k-2)\Delta t} \end{array}\right] \tag{21b}$$

where $\overline{G}_{\alpha\beta}(x,t)=\int_0^t G_{\alpha\beta}(x,\tau)d\tau$ and $\overline{\overline{G}}_{\alpha\beta}(x,t)=\int\int_0^t G_{\alpha\beta}(x,\tau)d\tau$

$\overline{H}_{\alpha\beta}(x,t)=\int_0^t H_{\alpha\beta}(x,\tau)d\tau$ and $\overline{\overline{H}}_{\alpha\beta}(x,t)=\int\int_0^t H_{\alpha\beta}(x,\tau)d\tau$

Numerical results have shown that numerical instability can occur in the temporal scheme of dynamic BEM formulation (19). In order to improve the stability, we have applied in this work the linear θ method, proposed by Yu *et al* [15] in elastodynamics, because of its simplicity and effectiveness. According to this method, generalized tractions and displacements are supposed vary linearly from time $t$-$\Delta t$ to time $t+\theta\Delta t$ ($\theta\geq 1$), the response is first evaluated at time $t = (N-1+\theta)\Delta t$, and subsequently the solution at time $t=N\Delta t$ is obtained. Hence, the formulation (19) is modified like following:

$$\overline{\mathbf{H}}^1(\theta)\mathbf{u}^N=\mathbf{G}^1(\theta)\mathbf{t}^N+\frac{\theta-1}{\theta}\left[\mathbf{G}^1(\theta)\mathbf{t}^{N-1}-\overline{\mathbf{H}}^1\mathbf{u}^{N-1}\right]+\frac{1}{\theta}\sum_{1}^{N-1}\left(\mathbf{G}(\theta)^{N-m+1}\mathbf{t}^m-\mathbf{H}(\theta)^{N-m+1}\mathbf{u}^m\right) \tag{22}$$

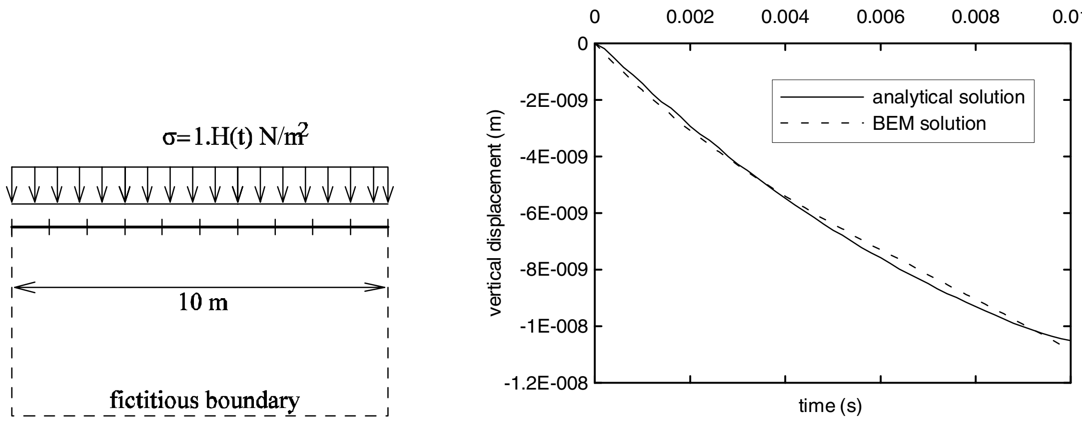
Note that if θ=1, equation (22) becomes equation (19). The kernels $G^k_{\alpha\beta}(\mathbf{x},\theta)$, $H^k_{\alpha\beta}(\mathbf{x},\theta)$ are given exactly like $G^k_{\alpha\beta}(\mathbf{x})$ and $H^k_{\alpha\beta}(\mathbf{x})$ except that $k$ is replaced by $k$-1+θ.

## VALIDATION

### Transient load on surface of half-space

The first problem addressed concerns plane strain step loading applied uniformly to the surface of a soil half-space. That is equivalent to the problem of one-dimensional wave propagation in a column of soil. The loading boundary conditions for this test problem include a step total stress $\sigma = 1.H(t)$ N/m$^2$, and free fluid flow, $p$=0, at the top surface of the half-space. The following material properties are specified: $E$ = 254,4 MPa; $M$ = 5210 MPa; ν = 0,3; α = 0.981; ρ = 1884 kg/m$^3$; $\rho_f$ = 1000 kg/m$^3$; κ = 3,55.10$^{-2}$ m/s. The half-space is modeled by using 10 quadratic boundary elements of equal length $\Delta h$=1 m, enclosing elements are used to regularize singular elementary integrals (figure 5a). Time step $\Delta t$=2.10$^{-4}$ s is chosen, and the mixed interpolation function is used. The variation of vertical displacement at the surface versus time is shown in figure 5b. A good agreement between BEM and analytical solution (Schanz & Cheng [18]) is observed.

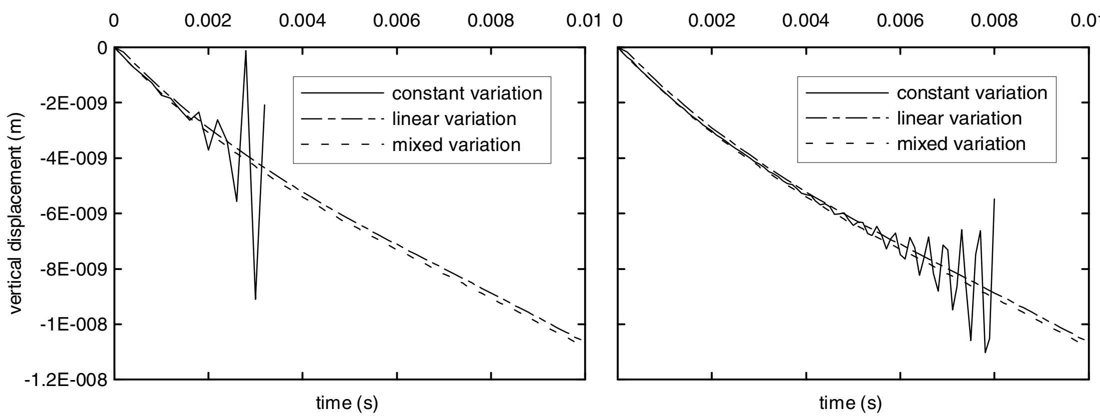
In order to study the numerical stability of the time-marching procedure, the effect at different time step values and the different time interpolation functions are examined (figure 6). It is shown that the instability is more important for the larger time step and the linear or mixed interpolation functions give more stable results than constant interpolation function. By using the θ method, the stability of time domain BEM formulation is improved considerably (figure 7).




a) Geometry and discretization

b) Vertical displacement at the surface

**Figure 5: Half-space under uniform load**




a) $\Delta t = 2.10^{-4}$ (s)

b) $\Delta t = 1.10^{-4}$ (s)

**Figure 6: Influence of time step size and choice of time interpolation functions**

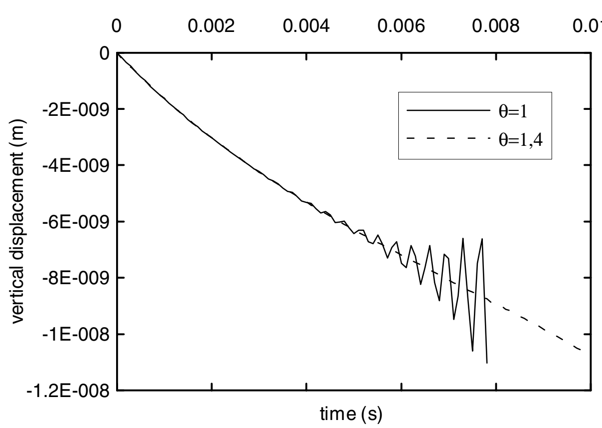
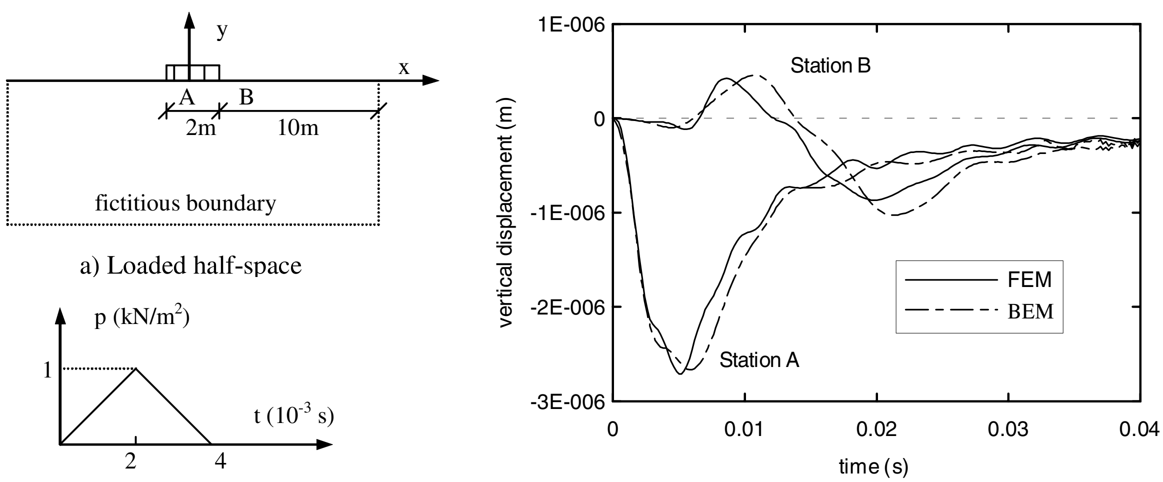



**Figure 7: Result with θ method ($\Delta t = 1.10^{-4}$ s, constant interpolation function)**

The numerical instability during the BEM time-stepping scheme, observed in this example, is an important remark. In elastodynamics, a similar problem involving a half-space does not practically pose any problem of instability, thus a broad choice of $\Delta t$ is possible. But in poroelastodynamics, there is a new source of error concerning the condition number of the matrix $\mathbf{A}$. In fact, this matrix is obtained by a columns permutation of matrices $\mathbf{G}^1$ and $\overline{\mathbf{H}}^1$, and the order of the components of these two matrices are so different that the matrix $\mathbf{A}$ is usually ill-conditioned.

**Generation of Rayleigh wave**

Objective of this example is to model the generation of the Rayleight wave on the top surface of a fluid-saturated soil half-space. The study of this wave is interesting because of its serious consequence in earthquake. A part of the free half-space of width 2m is excited by a triangular impulse as a vertical stress field (figure 8b). The discretization is extended up to a distance of 10m from loading area (figure 8a). This distance is considered sufficiently faraway so that the response to the interested points A (0,0) and B (3,0) is not compromised by the truncation. A uniform mesh of 22 quadratic elements of length $\Delta h$=1 m is built, enclosing elements are introduced to regularize the singular integrals. The same material as preceding example is considered. The mixed temporal interpolation functions with linear θ method (θ=1,4) are used with time step $\Delta t$=2.10$^{-4}$ s. Figure 8c shows the vertical displacement at points A, B. In comparison with the result obtained by finite element method, a good agreement can be observed.




a) Loaded half-space

b) Loading history

c) Vertical displacement at the surface

**Figure 8: Generation of Rayleigh wave**

## CONCLUSION

In the present work, a boundary element formulation and fundamental solutions are presented for the two-dimensional simplified u-p formulation of dynamic poroelasticity. The analytical time-dependent fundamental solutions for saturated porous media of incompressible components are derived. It seems that the proposed formulation is more appropriate for the saturated soils under earthquake solicitations. Numerical studies show that the results of the proposed BEM formulation can be unstable during time-marching procedure and the choice of time step size becomes significant. The numerical stability can be improved by using a linear θ method. In addition, the mixed time interpolation is recommended for having a more stable solution.

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## APPENDIX

$$e_1=\frac{A_{ij}}{\lambda+2\mu}\quad e_2=\frac{B_{ij}}{\lambda+2\mu}\quad e_3=\frac{D_{ij}}{\rho}\quad e_4=-\frac{D_{ij}}{2\beta(\rho-\alpha\rho_f)}\quad e_5=-e_4/2\beta\quad e_6=-\frac{A_{ij}}{\mu}\quad e_7=-\frac{B_{ij}}{\mu}+C_{ij}$$

$$e_8=e_7-e_6 r/2\quad A_{ij}=\frac{1}{2\pi r}\left(2r_{,i}r_{,j}-\delta_{ij}\right)\quad B_{ij}=\frac{r_{,i}r_{,j}}{2\pi}\quad C_{ij}=\frac{\delta_{ij}}{2\pi\mu}\quad \beta=\frac{\alpha^2}{2\kappa(\rho-\alpha\rho_f)}$$

$$d_1=-\frac{\alpha}{2\pi\kappa(\lambda+2\mu)}\quad d_2=\frac{\rho_f}{2\pi(\lambda+2\mu)}\quad d_3=\frac{1}{2\pi\alpha}\quad d_4=-\frac{\rho}{2\pi\alpha(\rho-\alpha\rho_f)}$$

$$f_1 = d_1 \quad f_2 = d_3 \quad f_3 = -\frac{\kappa(\rho - \alpha\rho_f)}{2\pi\alpha^3} \quad g_1 = \frac{1}{2\pi\kappa} \quad g_2 = -\frac{\rho}{2\pi\kappa(\rho - \alpha\rho_f)} \quad g_3 = g_1 + g_2$$

$$h_{11}(r,t) = \frac{e^{-\beta t}\cosh(\beta\sqrt{t^2 - r^2/v_1^2})}{\sqrt{t^2 - r^2/v_1^2}} H(t - r/v_1) \quad \left( v_1 = \sqrt{\frac{\lambda + 2\mu}{\rho - \alpha\rho_f}} \quad v_2 = \sqrt{\frac{\mu}{\rho}} \right)$$

$$h_{12}(r,t) = \frac{v_1^2 e^{-\beta t}\sinh(\beta\sqrt{t^2 - r^2/v_1^2})}{\beta r} H(t - r/v_1)$$

$$\dot{h}_{12}(r,t) = \left[ \frac{\cosh(\beta\sqrt{t^2 - r^2/v_1^2})t}{\sqrt{t^2 - r^2/v_1^2}} - \sinh(\beta\sqrt{t^2 - r^2/v_1^2}) \right] \frac{v_1^2 e^{-\beta t}}{r} H(t - r/v_1)$$