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DIMENSIONAL ANALYSIS OF YIELDING STRUCTURES

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SUMMARY

This paper summarizes recent work by the authors that revisits the way of presenting information on the dynamic response of inelastic structures. The nonlinear response of rigid-plastic, elastic-plastic and bilinear systems is presented in terms of the dimensionless Π-products that result from rigorous dimensional analysis. The main advantage of the analysis presented in this study is that it brings forward the concept of self-similarity—an invariance with respect to changes in scale or size—which is a decisive symmetry that shapes nonlinear behavior.

INTRODUCTION

Within the context of earthquake engineering the first systematic work on the response of an elastic-plastic single-degree-of-freedom system subjected to earthquake and pulse-type ground shaking was presented in the seminal papers by Veletsos and Newmark [1] and Veletsos et al. [2]. In these pioneering studies the response of the elastic-plastic system was normalized to the response of an elastic system having the same stiffness as the initial stiffness of the inelastic system. This approach was mainly motivated from: (a) the need to explain why the forces that develop in yielding structures are considerably smaller than the forces computed from elastic analysis; and (b) an idea that the energy input in the two systems should be comparable.

In this paper we first present an alternative way of presenting the nonlinear response of an elastic-plastic system which is derived from formal dimensional analysis (Langhaar [3], Housner and Hudson [4], Barenblatt [5]). The proposed dimensionless variables are liberated from the associated elastic system response and reveal remarkable order in the normalized response. It is most interesting, that the fundamental concepts upon which the proposed dimensional analysis is built have been put forward in the 1965 Newmark's Rankine lecture (Newmark [6]) and in the paper by Veletsos et al. [2].

Our interest in this study focusses on the response of yielding structures under strong earthquake shaking which is the strongest nearby the causative faults where, in most occasions the ground exhibits distinguish-

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able pulses. Accordingly, our investigation focusses on the response analysis of elastoplastic and bilinear single-degree-of-freedom oscillators subjected to pulse-type excitations. The ability of distinct pulses to generate structural response that resembles the earthquake induced response has been examined in past studies (Veletsos et al. [2], Yim et al. [7], Hall et al. [8], Makris and Chang [9], Makris and Roussos [10] among others).

Type-A (one-sine acceleration) pulse and Type-B (one cosine acceleration) pulse, shown in Figure 1, are of particular interest to this study as the existence of distinct pulse duration and pulse amplitude allows the introduction of dimensionless parameters that reveal the underlying physics of the response.

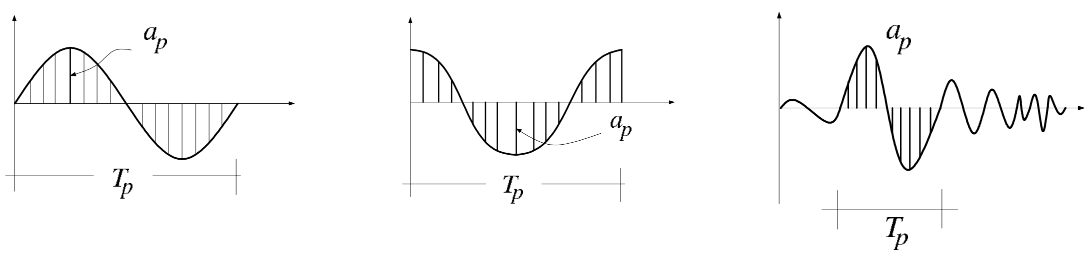

Figure 1. The one-sine Type-A pulse (left), and the one cosine Type-B pulse (center), which approximate strong ground motions with distinguishable pulses (right).

DIMENSIONAL ANALYSIS OF THE RIGID PLASTIC OSCILLATOR

A length scale of Ground Excitations Relevant to Structural Response

Within the context of earthquake engineering an early solution to the response of a rigid-plastic system (rigid mass sliding on a moving base—see Figure 2) subjected to a rectangular acceleration pulse has been presented by Newmark [6]. In this case, the strength of a rigid-plastic system is $Q = \mu mg$. Under a rectangular acceleration pulse with amplitude $a_p > \mu g$ and duration, T_p,

$$\ddot{u}_g(t) = a_p, \qquad 0 \le t \le T_p \tag{1}$$

the entire relative displacement of the mass on the moving surface is (Newmark [6])

$$u_{max} = \frac{a_p T_p^2}{2}\left(\frac{a_p}{\mu g} - 1\right) = \frac{a_p T_p^2}{2}\left(\frac{m a_p}{Q} - 1\right). \tag{2}$$

Equation (2) indicates that the plastic displacement is proportional to the intensity of the acceleration pulse, a_p, and the square of its duration, T_p^2. The product, $a_p T_p^2/2 \approx L_e$ is a characteristic length scale (in this case, the displacement of the base when the acceleration pulse expires) of the ground excitation and is a measure of the intensity of the excitation pulse. The rectangular acceleration pulse used by Newmark [6], which leads to an infinite base displacement, is probably the most well-suited example to introduce the finite length scale, $L_e = a_p T_p^2$, of the energetic pulse of the motion. Upon the expiration of the pulse, the base moves with a constant velocity and the inertia demand on the structure is zero. This situation is reminiscent of the minor seismic demands on structures subjected to selected near-source ground motions which, upon the expiration of the main pulse, induce incrementally very large ground displacements with minor inertia effects.

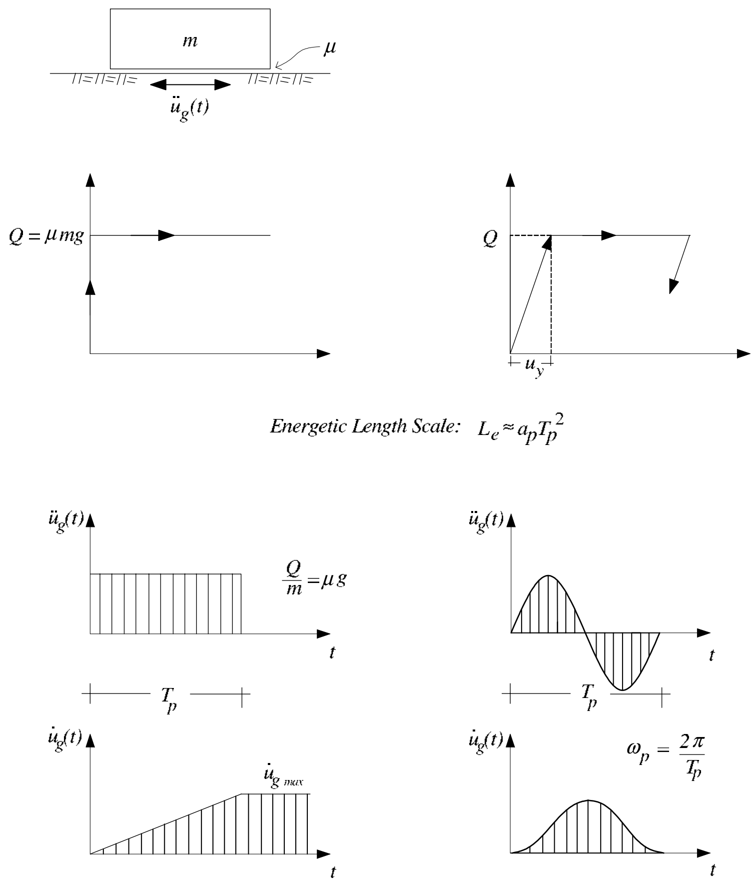

Figure 2. Rigid-plastic and elastic-plastic behavior (top); and the acceleration and velocity time histories of a rectangular and a one-sine acceleration pulse (bottom).

Following this discussion it is natural to normalize the relative structural response, u_{max}, to the length scale of the energetic excitation, L_e, and Equation (2), is re-written as

$$\frac{u_{max}}{a_p T_p^2} = \frac{1}{2}\left(\frac{m a_p}{Q} - 1\right). \tag{3}$$

Equation (3), which was obtained by solving the differential equation that governs the sliding response (Newmark [6] among others), relates the dimensionless displacement $\Pi_1 = u_{max}/a_p T_p^2$ to the dimensionless strength $\Pi_2 = Q/ma_p$. The relation, defined by Equation (3) is plotted with a heavy line in Figure 3 (left) in a logarithmic scale.

Review of Dimensional Analysis

Dimensional analysis is a mathematical tool that shapes the general form of relations that describe natural phenomena. The application of dimensional analysis to any particular physical phenomenon is based on the premise that the phenomenon can be described by a dimensionally homogeneous equation that relates the dependent variables, u, and the independent variables, $u_2, \ldots, u_k$:

$$u_1 = f(u_2, u_3, \ldots, u_k). \tag{4}$$

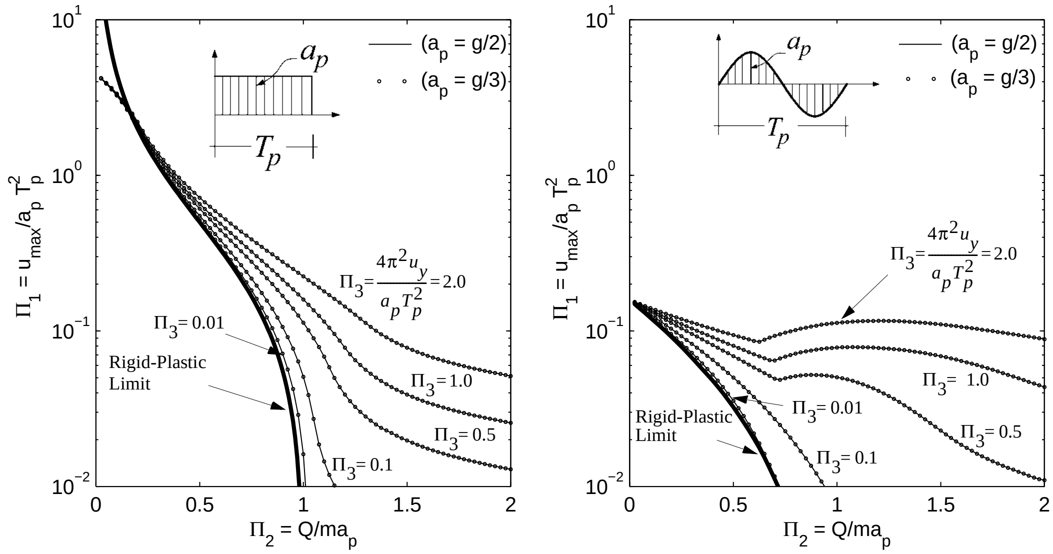

Figure 3. Normalized relative displacement curves of an elastic-plastic structure subjected to a rectangular and a one-sine acceleration pulses. The self-similar solutions approach the rigid-plastic limit as the normalized yield displacement tends to zero.

For instance, in the case of a rigid-plastic system subjected to an acceleration pulse with amplitude, a_p, and duration, T_p, it is expected that the maximum relative displacement, u_{max}, is a function of the specific strength of the system, $Q/m = \mu g$, and the characteristics of the pulse, a_p and T_p giving

$$u_{max} = f\left(\frac{Q}{m}, a_p, T_p\right). \tag{5}$$

In the case of Equation (5), the four variables $u_{max} \doteq [L]$, $Q/m \doteq [L][T]^{-2}$, $a_p \doteq [L][T]^{-2}$ and $T_p \doteq [T]$ involve only two independent reference dimensions ($r = 2$), that of length $[L]$ and time $[T]$. According to Buckingham's Π-theorem the number of independent dimensionless Π-products is equal to the number of physical variables appearing in Equation (5) (four variables) minus the number of reference dimensions (two). Therefore, for a rigid-plastic system subjected to an acceleration pulse, we have 4 - 2 = 2 Π-terms. Since we are focusing on the response to pulse-type motions, the obvious choice for the repeating variables is the acceleration amplitude of the pulse, a_p, and its duration, T_p, which gives $\Pi_1 = u_{max}/a_p T_p^2$ and $\Pi_2 = Q/ma_p$. With the two Π-terms established, Equation (5) reduces to

$$\frac{u_{max}}{a_p T_p^2} = \phi\left(\frac{Q}{ma_p}\right). \tag{6}$$

In the elementary case of a rectangular acceleration pulse the form of the function ϕ was obtained analytically by solving the differential equation (Newmark [6] among others) and is given by Equation (3). For trigonometric pulses such as the one sine pulse or the one cosine pulse introduced earlier, the response of the rigid-plastic system is also described by Equation (6) with the form of the function ϕ obtained numerically. Figure 3 (right) plots with a heavy line the response of the rigid-plastic system when subjected to a one-sine acceleration (Type-A) pulse. The response is plotted on a logarithmic scale next to the response from a rectangular pulse in order to illustrate the relative strength of a rectangular acceleration pulse and a forward displacement (one-sine acceleration pulse).

DIMENSIONAL ANALYSIS OF THE ELASTIC-PLASTIC OSCILLATOR

The idealized rigid-plastic system analyzed in the preceding section exhibits zero yield displacement (infinite pre-yielding stiffness), and therefore infinite ductility. We now consider the response of an inelastic system that exhibits a finite yield displacement before sliding. With reference to Figure 2 (right) the response of an elastic-plastic system subjected to some acceleration pulse of amplitude a_p and duration T_p, should be a function of the specific strength, Q/m, the yield displacement, u_y, and the characteristics of the pulse, a_p and $\omega_p = 2\pi/T_p$. Accordingly,

$$u_{max} = f\left(\frac{Q}{m}, u_y, a_p, \omega_p\right) . \tag{7}$$

The five variables appearing in Equation (7) involve only two reference dimensions, that of length $[L]$ and time $[T]$. According to Buckingham's Π-theorem the number of independent dimensionless Π-products is now: (5 variables) - (2 reference dimensions) = 3 Π-terms.

$$\frac{u_{max}\omega_p^2}{a_p} = \phi\left(\frac{Q}{ma_p}, \frac{u_y\omega_p^2}{a_p}\right) . \tag{8}$$

Figure 3 illustrates how the response of the elastic-plastic system amplifies as the normalized yield displacement increases. The most notable observation from Figure 3 is that the normalized response is invariant with respect to the level of the acceleration amplitude, a_p; the solid and dotted lines have been computed for different levels of acceleration, a_p, yet the normalized response is identical. This scale invariance between the size of the maximum relative displacement, the size of the yield displacement, and the intensity of the acceleration pulse is known as self-similarity (Langhaar[3], Barenblatt[5]) — a special type of symmetry that has unique importance in understanding and ordering nonlinear response. Another interesting observation is that for values of normalized strength below one ($\Pi_2 = Q/ma_p \le 1$), the rectangular pulse induces much larger displacements than the one-sine acceleration pulse; whereas, as the value of the normalized strength, Q/ma_p increases the situation reverses.

DIMENSIONAL ANALYSIS OF THE LINEAR OSCILLATOR

Before considering the bilinear oscillator we first discuss the linear single-degree-of-freedom (SDOF) system. Dynamic equilibrium of a linear SDOF oscillator with mass *m*, stiffness K_o, and damping *C*, that is subjected to a ground excitation described either by a Type-A pulse or a Type-B pulse gives

$$\ddot{u}(t) + 2\xi\omega_o\dot{u}(t) + \omega_o^2 u(t) = -\ddot{u}_g(t) \tag{9}$$

where, $\xi = C/2m\omega_o$ is the viscous damping ratio, $\omega_o = \sqrt{K_o/m}$ is the undamped natural frequency and $\ddot{u}_g(t)$ is a cycloidal pulse ground excitation that is fully defined by the parameters a_p and ω_p. Accordingly, the maximum relative displacement of the linear SDOF oscillator, u_{max}, is a function of four variables

$$u_{max} = f(\omega_o, \xi, a_p, \omega_p) . \tag{10}$$

It is observed that the 5 variables appearing in Equation (10), $u_{max} \doteq [L]$, $\omega_o \doteq [T]^{-1}$, $\xi \doteq [1]$, $a_p \doteq [L][T]^{-2}$ and $\omega_p \doteq [T]^{-1}$ involve only two reference dimensions; that of length $[L]$ and time $[T]$. Therefore, for a linear SDOF oscillator we have 3Π-terms. Herein, as in the previous section we select as repeating variables the characteristics of the pulse excitation, a_p and ω_p. The three independent Π-terms are

$$\Pi_1 = \frac{u_{max}\omega_p^2}{a_p}, \quad \Pi_2 = \frac{\omega_o}{\omega_p}, \quad \Pi_3 = \xi. \tag{11}$$

With the three Π-terms given above, the relation of the five variables appearing in Equation (10) is reduced to a relation of three variables

$$\frac{u_{max}\omega_p^2}{a_p} = \phi\left(\frac{\omega_o}{\omega_p}, \xi\right). \tag{12}$$

Figure 4 plots the dimensionless maximum value of the solution of Equation (12), $\Pi_1 = u_{max}\omega_p^2/a_p$, as a function of $\Pi_2 = \omega_o/\omega_p$ for different fixed values of $\Pi_3 = \xi$. The two shock spectra for $\xi = 0\%$ and $\xi = 10\%$ show that for structures with $\omega_o/\omega_p \geq 1$, the peak relative displacement is nearly the same when they are subjected to a Type-A or a Type-B cycloidal pulses that exhibit the same peak pulse acceleration yet different pulse velocities.

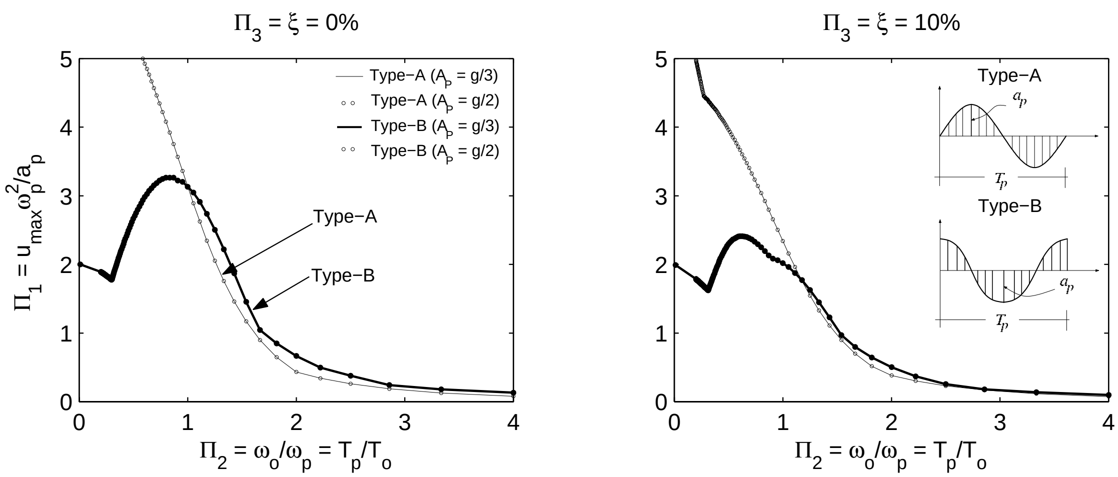

Figure 4. Normalized relative displacement spectra of a linear SDOF oscillator.

The non-dimensional results presented in Figure 4 are independent of the intensity of the ground shaking showing that the non-dimensional response of a linear SDOF oscillator is self-similar—meaning that the normalized peak response values follow the same master curve for any value of the induced peak ground acceleration, a_p. While this result is well known to the literature for the linear SDOF oscillator, the dimensional analysis presented in this study shows that the same is true not only for the elastic-plastic oscillator but also the bilinear oscillator.

DIMENSIONAL ANALYSIS OF THE BILINEAR OSCILLATOR

In previous sections we examined the dimensional response analysis of rigid-plastic and elastic-plastic systems which are systems that exhibit a zero post-yielding stiffness. In reality, steel structures and other traditional structures exhibit a non-zero post-yielding stiffness and their response is better approximated with a bilinear idealization, shown in Figure 5, rather than with an elastic-plastic idealization. In addition to traditional structures, the bilinear idealization has been found to be most appropriate in the modelling of modern seismic protection devices.

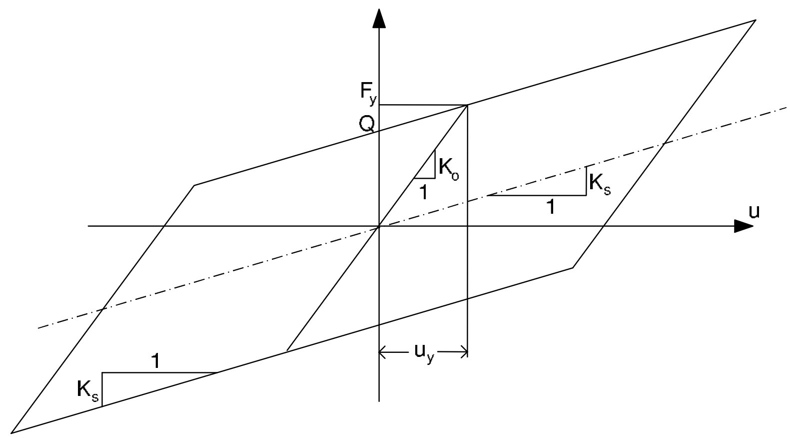

Figure 5. The bilinear mechanical model.

The main difference between the behavior of structures equipped with seismic protection devices and the behavior of traditional structures built either from concrete or steel is the intentional lower value of strength, Q, and the large value of available ductility, $\mu = u_{max}/u_y$. In the extreme case of a sliding spherical bearing the yield displacement of the teflon layer on the articulated spherical bearing is $u_y = 0.2mm$ (Mohka et al. [11]). Assuming for instance a design base displacement $u_{max} = 0.4m = 400mm$, the resulting ductility demand on a sliding bearing is $\mu = 2000$. In the case of lead rubber bearings, the yield displacement can be as high as $u_y = 2cm$ and in this case the ductility demand is $\mu = 0.4m/0.02m = 20$ (Kelly [12]). Similarly, for lead extrusion dampers $\mu = 15$ or even more (Skinner et al. [13]); whereas, for buckling-restrained braces $\mu = 10 \text{ to } 15$ (Black et al. [14]). While these aforementioned values of ductility are much larger than the values of ductility ($\mu \approx 4 \sim 5$) of well engineered traditional structures, the two orders-of-magnitude variability in the ductility capacities of seismic protection devices ($\mu = 20$ for lead rubber bearings; whereas $\mu = 2000$ for friction pendulum bearings) suggest that the ductility parameter should be immaterial in the response. Earlier parametric studies by Makris and Chang [9] concluded that when seismically isolated structures are excited by strong earthquakes, the value of the yield displacement has marginal effects. This conclusion, which applies in the case of relatively small values of strength ($Q/mg \le 0.2$) and a non-zero post-yielding stiffness, is confirmed in this study with rigorous dimensional analysis.

The relatively small strength and large values of ductility associated with the bilinear behavior shown in Figure 5 suggest that bilinear systems exhibit an oscillatory response that is characterized by the second stiffness $K_s = m\omega_s^2$. The mechanical model adopted herein is most appropriate for the modelling of seismic isolation systems (Kelly [12], Skinner et al. [13]) and has been used widely for the response analysis of seismically isolated bridges (Makris and Chang [9] and Chang et al. [15] among others), however, it can approximate the mechanical behavior of other yielding structures that exhibit a distinct yield force, F_y, while delivering appreciable ductility such as one-story frames equipped with buckling-restrained braces (Black et al. [14]).

The characterization of a structural element with bilinear behavior is fully described with any three of the five parameters shown in Figure 5—the strength, Q, the pre-yielding stiffness, K_o, the yield displacement, u_y, the yield force, F_y and the post-yield stiffness, K_s. Here, we define the force-displacement behavior of the bilinear model with the structural frequency $\omega_s = \sqrt{K_s/m} = 2\pi/T_s$, the characteristic strength, Q, and the yield displacement, u_y. Note that the structural frequency, ω_s, is the undamped frequency of the mass, m, of the SDOF structure supported on the elastic spring, $K_s = m\omega_s^2$, which is the second slope of the bilinear loop. This definition is consistent with the frequency of oscillation of isolated structures supported on friction pendulum bearings with radius of curvature, R, In this case

$$\omega_s = \sqrt{\frac{g}{R}} = \sqrt{\frac{K_s}{m}}. \tag{13}$$

With reference to Figure 5, the yield force, F_y, is

$$F_y = K_o u_y = Q + K_s u_y \tag{14}$$

from which

$$K_o = \frac{Q}{u_y} + K_s. \tag{15}$$

Dynamic equilibrium of the more general elasto-viscoplastic oscillator gives

$$\ddot{u}(t) + \frac{F(t)}{m} = -\ddot{u}_g(t) \tag{16}$$

where $u(t)$ is the relative-to-the-ground displacement history; $\ddot{u}_g(t)$ is the ground acceleration; and $F(t)$ is the internal force resulting from the restoring mechanism and damping (viscous or hysteretic). For the most general case, the internal force is expressed mathematically as

$$\frac{F(t)}{m} = 2\xi\omega_s\dot{u}(t) + \omega_s^2 u(t) + \frac{Q}{m}z(t) \tag{17}$$

where Q/m is the specific strength; $\omega_s = \sqrt{K_s/m}$ is undamped natural frequency; and $z(t)$ is a hysteretic dimensionless quantity with $|z(t)| \le 1$ that is governed by

$$u_y\dot{z} + \gamma|\dot{u}(t)|z|z|^{n-1} + \beta\dot{u}(t)|z|^n - \dot{u}(t) = 0\ . \tag{18}$$

The model given by Equations (17) and (18) is a special case of the Bouc-Wen model (Wen [16] [17]) enhanced with a viscous term. In Equation (18), β, γ and n are dimensionless quantities that control the shape of the hysteretic loop.

Substitution of Equation (17) into Equation (16) gives

$$\ddot{u}(t) + 2\xi\omega_s\dot{u}(t) + \omega_s^2 u(t) + \frac{Q}{m}z(t) = -\ddot{u}_g(t) \tag{19}$$

where $z(t)$ is governed by Equation (18). Equations (18) and (19) indicate that the SDOF with viscous and hysteretic behavior is characterized with parameters ω_s, ξ, Q/m and u_y.

Since viscous damping was included in the response analysis of the linear SDOF oscillator, in this section we focus our attention on a purely hysteretic bilinear oscillator that is described fully by the three parameters ω_s, Q/m and u_y, ($\xi = 0$), subjected to either a Type-A or Type-B cycloidal pulses that are fully described by the parameters a_p and ω_p. With these parameters the maximum relative displacement of the bilinear hysteretic SDOF oscillator is a function of five variables

$$u_{max} = f\left(\omega_s, \frac{Q}{m}, u_y, a_p, \omega_p\right). \tag{20}$$

The six variables appearing in Equation (20), $u_{max} \doteq [L]$, $\omega_o \doteq [T]^{-1}$, $Q/m \doteq [L][T]^{-2}$, $u_y \doteq [L]$, $a_p \doteq [L][T]^{-2}$ and $\omega_p \doteq [T]^{-1}$ involve only two reference dimensions; that of length $[L]$ and time $[T]$. Therefore, according to Buckingham's Π theorem, we have 6 - 2 = 4 Π-terms

$$\Pi_1 = \frac{u_{max}\omega_p^2}{a_p}, \quad \Pi_2 = \frac{\omega_s}{\omega_p}, \quad \Pi_3 = \frac{Q}{ma_p}, \quad \Pi_4 = \frac{u_y\omega_p^2}{a_p}. \tag{21}$$

Here again we select as repeating variables the characteristics of the pulse excitation, a_p and ω_p. With this selection the maximum displacement response of interest, u_{max}, is normalized with the length scale a_p/ω_p^2 that is representative of the inertia loading of the excitation. Had the maximum response been scaled with the yield displacement, u_y, $\Pi_1 = u_{max}/u_y = \mu$, the structural response would have been normalized with a length scale of the structural behavior—a poor choice which does not factor the inertial loading from the energetic pulse.

With the four Π-terms given above, the relation of the six variables appearing in Equation (20) is reduced to a relation of four variables

$$\frac{u_{max}\omega_p^2}{a_p} = \phi\left(\frac{\omega_s}{\omega_p}, \frac{Q}{ma_p}, \frac{u_y\omega_p^2}{a_p}\right). \tag{22}$$

Alternatively, if one describes the cycloidal pulse with the peak pulse velocity v_p and ω_p rather than a_p and ω_p, Equation (20) takes the form

$$\frac{u_{max}\omega_p}{v_p} = \phi\left(\frac{\omega_s}{\omega_p}, \frac{Q}{m\omega_p v_p}, \frac{u_y\omega_p}{v_p}\right). \tag{23}$$

This normalization and a comparison of the effect of normalizing the response with respect to velocity rather than acceleration is presented in Makris and Black [18].

Figure 6 plots the dimensionless value of the solution of Equation (22), $\Pi_1 = u_{max}\omega_p^2/a_p$, as a function of $\Pi_2 = \omega_s/\omega_p$ for two different fixed values of $\Pi_3 = Q/ma_p$ and $\Pi_4 = u_y\omega_p^2/a_p$. The two shock spectra in each of the plots show that for structures with $\omega_s/\omega_p \geq 1$, their peak relative displacement is nearly the same when they are subjected to Type-A or Type-B cycloidal pulses that exhibit the same peak pulse acceleration yet different peak pulse velocities (a result not seen if normalized by pulse velocity, see Makris and Black [18]).

Figure 6 reveals two additional important results that confirm the unique advantages of dimensional analysis. The first result is that for a fixed value of the Π_3 and Π_4 terms, the response of the bilinear oscillator follows the same master curve for all values of excitation levels (any value of peak ground acceleration, a_p) showing that the solutions for the dimensionless peak response-values are self-similar. This result confirms the important physical significance of the Π-terms given by Equation (21). The second important result is that the top row of plots is nearly identical to the bottom row—an observation that suggests that the response is insensitive to the value of $\Pi_4 = u_y\omega_p^2/a_p$. In fact the responses, Π_1, converges to a finite

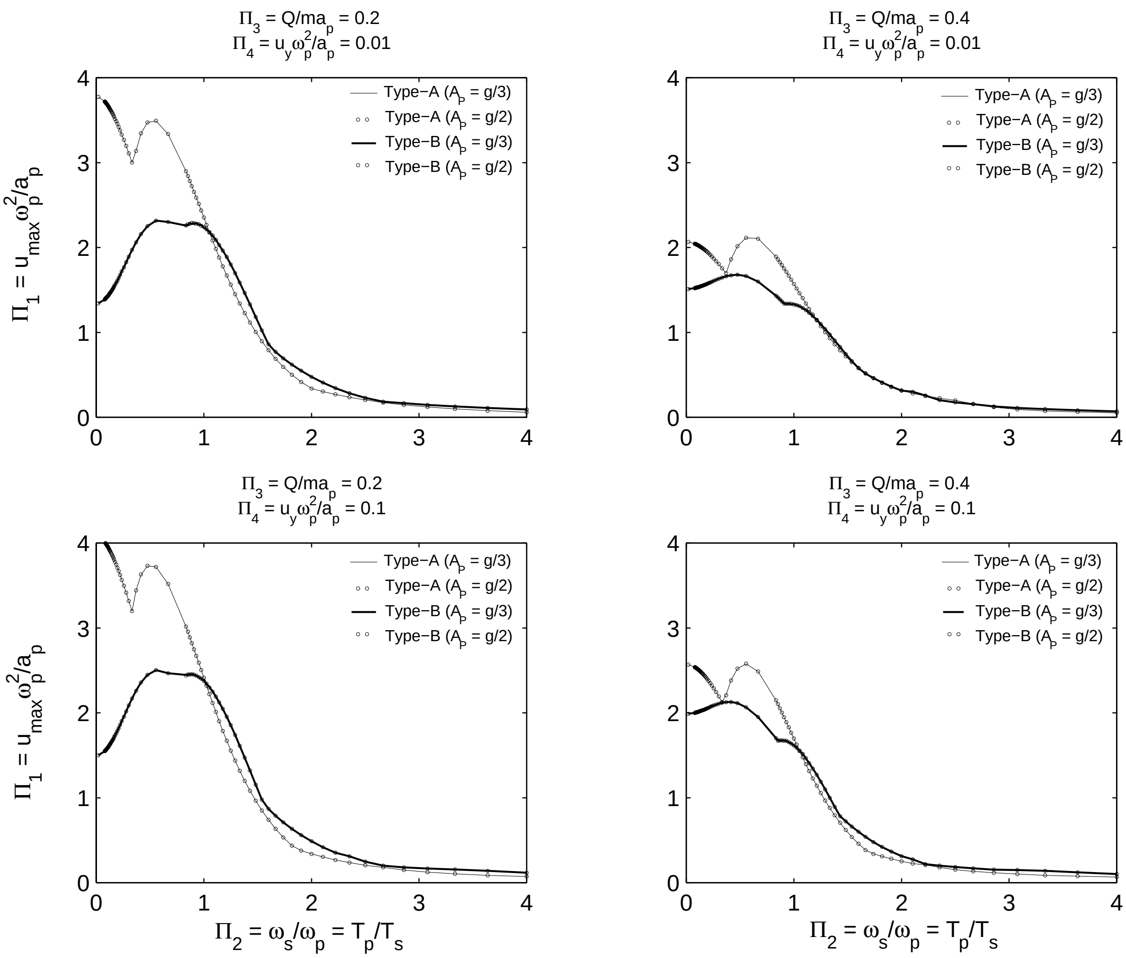

Figure 6. Normalized relative displacement spectra of a bilinear oscillator. The response is seen to be self-similar (independent of excitation intensity) and insensitive to yield displacement.

limit as Π_4 tends to zero; and according to the theory of dimensional analysis, one can simply replace Equation (22) or (23) by its limiting expression in which $\Pi_4 = 0$ (Barenblatt 1996)

$$\frac{u_{max}\omega_p^2}{a_p} \approx \phi\left(\frac{\omega_s}{\omega_p}, \frac{Q}{ma_p}, 0\right). \tag{24}$$

The existence of a finite limit for Π_1 as Π_4 tends to zero indicates that the normalized relative displacement of a bilinear oscillator, Π_1, exhibits a complete similarity, or similarity of the first kind (Barenblatt [5]), in the yield displacement, Π_4, and thus, the exact value of the yield displacement, u_y, is immaterial in the response of a bilinear SDOF oscillator that exhibits large values of ductility. Furthermore, with this limiting operation the number of arguments in the function $\phi(\cdot)$ appearing in Equations (22) and (23) reduce by one and therefore, the number of analysis in a parametric study reduces appreciably. For instance, Equation (22), by way of Equation (24), becomes

$$\frac{u_{max}\omega_p^2}{a_p} \approx \phi\left(\frac{\omega_s}{\omega_p}, \frac{Q}{ma_p}\right). \tag{25}$$

The plots shown in Figure 6 are computed for Π_4 =0.01 and 0.1 which are values much smaller than most values of the dimensionless parameter Π_1. The other limiting case is where a low frequency pulse excites a structure with a protective device that exhibits a very small yield displacement (friction pendulum bearings). In this case the dimensionless yield displacement can be as low as $\Pi_4 \approx 0.001$.

Figure 7 plots the variation of $\Pi_1 = u_{max}\omega_p^2/a_p$ when subjected to a Type-A pulse (left) and a Type-B pulse (right) for different values of the $\Pi_4 = u_y\omega_p^2/a_p$ parameter and three values of the $\Pi_3 = Q/ma_p$ parameter. Figure 7 confirms the very small variation of the response for two orders of magnitude of the parameter Π_4—a result that further demonstrates that the value of the yield displacement, u_y, is immaterial in the response of a bilinear SDOF oscillator or, in mathematical terms, that the response of the bilinear oscillator exhibits complete similarity in the parameter Π_4.

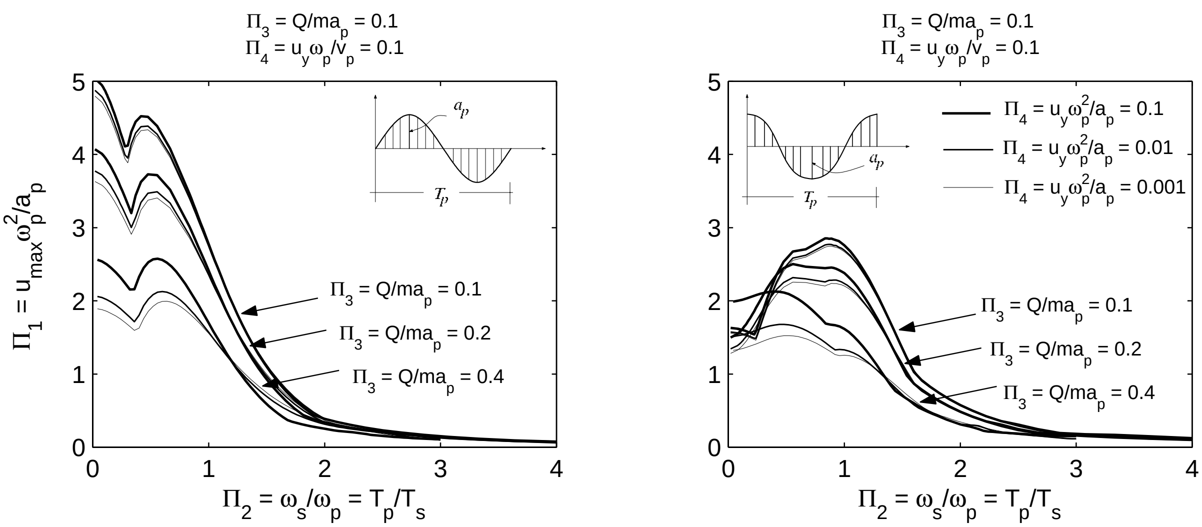

Figure 7. Non-dimensional relative displacement spectra of a bilinear oscillator subjected to a forward pulse (left) and a forward-and-back pulse (right). For a given non-dimensional strength, Π_3, the response is virtually independent to two orders-of-magnitude variation of the non-dimensional yield displacement, Π_4.

Another response quantity of interest is the total acceleration of the SDOF oscillator, $a(t) = \ddot{u}(t) + \ddot{u}_g(t)$, that is directly related with the base shear $V(t) = ma(t)$. The maximum total acceleration $a_{max}(t)$ is also a function of the five variables:

$$a_{max}(t) = f\left(\omega_s, \frac{Q}{m}, u_y, a_p, \omega_p\right) \qquad (26)$$

and according to the previous discussion there are four Π-terms. In this case

$$\Pi_1 = \frac{a_{max}}{a_p} \qquad (27)$$

while Π_2, Π_3 and Π_4 are given by Equation (21). With four Π-terms being established the relation of the six variables appearing in Equation (26) is reduced to a relation of four variables

$$\frac{a_{max}}{a_p} = \phi\left(\frac{\omega_s}{\omega_p}, \frac{Q}{ma_p}, \frac{u_y\omega_p^2}{a_p}\right). \tag{28}$$

Figure 8 plots the dimensionless value of the solution of Equation (28), $\Pi_1 = a_{max}/a_p$ as a function of $\Pi_2 = \omega_s/\omega_p$ for the two different fixed values of $\Pi_3 = Q/ma_p$ and $\Pi_4 = u_y\omega_p^2/a_p$. The two shock spectra appearing in each of the four top plots show that for most structures, their peak total acceleration (base shear) is nearly the same when subjected to Type-A or Type-B cycloidal pulses that exhibit the same peak pulse acceleration yet different peak pulse velocities (a result not seen if normalized with velocity).

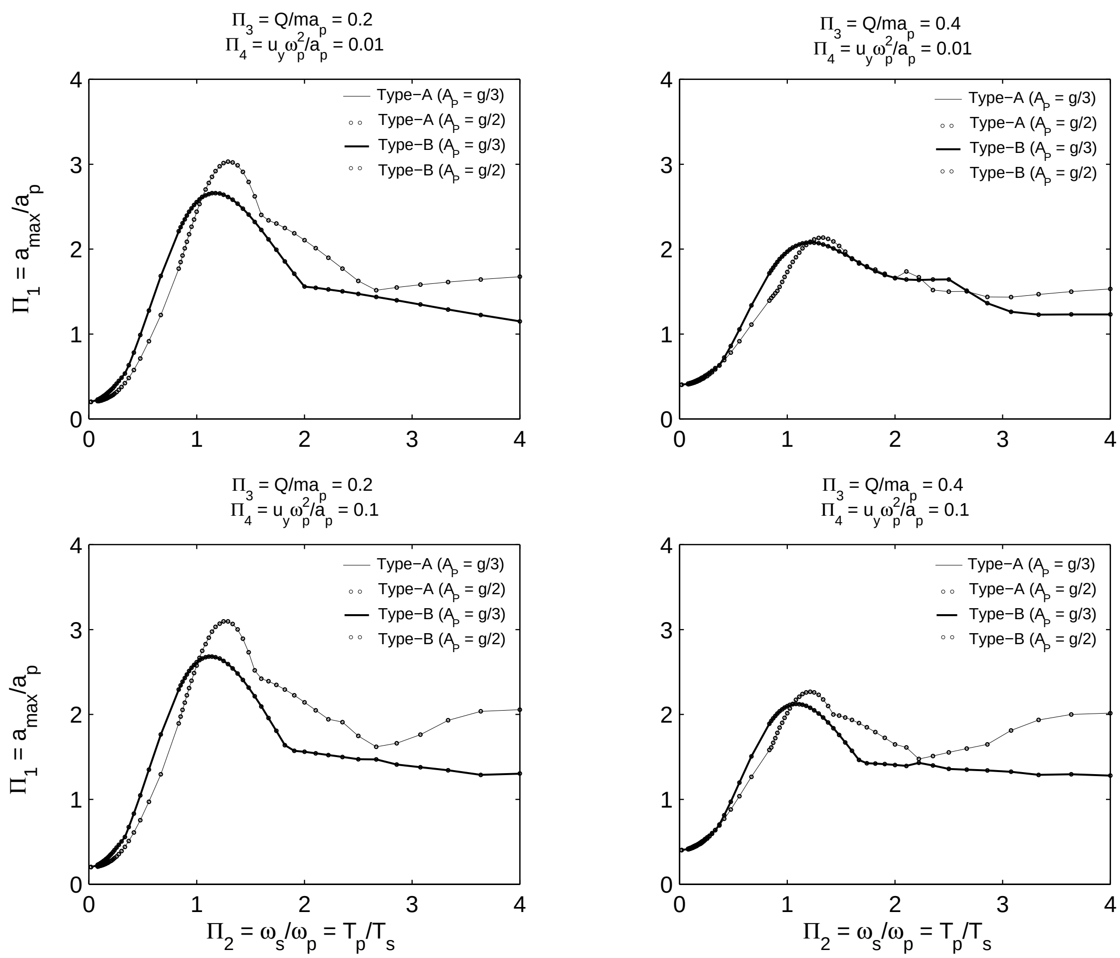

Figure 8. Normalized total acceleration spectra of a bilinear oscillator. The response is seen to be self-similar (independent of excitation intensity) and insensitive to yield displacement.

The self-similarity in the response observed in Figure 6 for the relative displacement is also observed in Figure 8 for the total accelerations. For a fixed value of the Π_3 and Π_4 terms, the response of the bilinear oscillator follows the same master curve for all values of excitation levels (any value of the peak pulse acceleration, a_p). The complete-similarity in the Π_4 parameter is again seen for this dimensionless response quantity ($\Pi_1 = a_{max}/a_p$), as the response is insensitive to an order of magnitude difference in Π_4.

CONCLUSION

In this study the nonlinear response of inelastic structures was investigated using the theory of dimensional analysis. Starting from the rigid-plastic and elastoplastic systems, the dimensional analysis presented in this study shows that what is important matters when ordering inelastic response is not the yield displacement u_y, alone, but its normalized value to the energetic length scale of the excitation, $L_e \approx a_p / \omega_p^2$.

Table 1 presents a summary of the systems considered and the associated dimensionless Π-terms along with the pulse approximations. When the response of the nonlinear systems studied here is presented in terms of the dimensionless Π-terms the response curves are self-similar and follow a single master curve. This remarkable order in the response is invariant with respect to changes in scale or size.

It was shown that for a given value of dimensionless strength and dimensionless yield displacement the dimensionless response (normalized relative displacement or base shear) is independent of the intensity level of the pulse shaking.

In addition, it was shown that the value of the normalized yield displacement, $\Pi_4 = u_y \omega_p^2 / a_p$, is immaterial in the response of bilinear oscillators that exhibit large values of ductilities. This finding shows that the response (peak relative displacements and base shears) exhibits a complete similarity (similarity of the first kind) in the dimensionless parameter Π_4 that is associated with the yield displacement.

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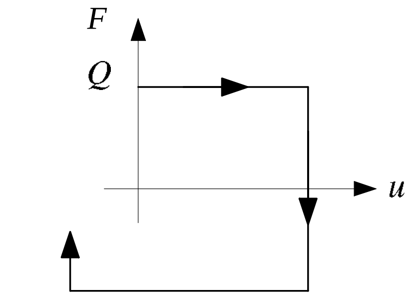
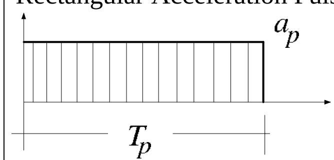
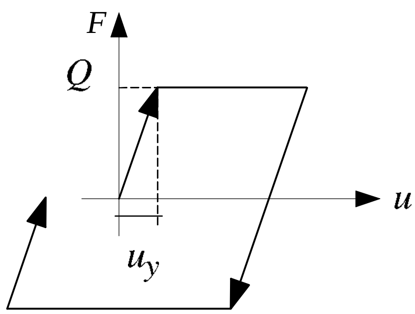
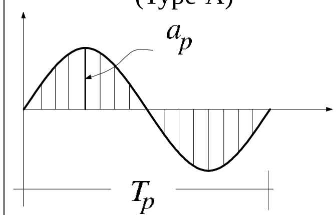
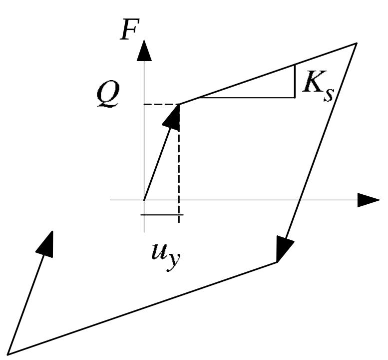
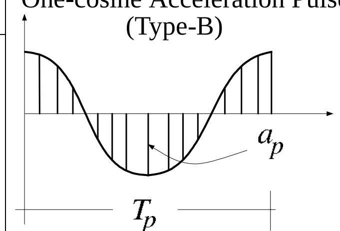
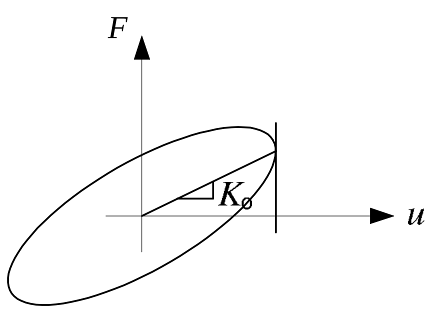
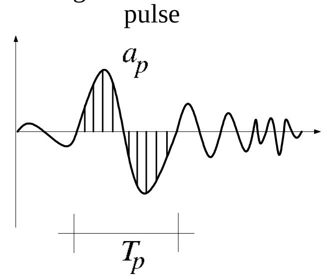
Table 1. Practical idealizations of structural behavior (left); the dimensionless response functions $\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_n)$ together with the associated dimensionless Π-products (center); and simple acceleration pulses that approximate the strong earthquake shaking (right).

Structural Idealization	Response Functions and dimensionless Π-products	Pulse-type Excitations
	$\Pi_1 = \phi(\Pi_2)$ $\Pi_1 = \frac{u_{max}\omega_p^2}{a_p}$ $\Pi_2 = \frac{Q}{ma_p}$	Rectangular Acceleration Pulse 
	$\Pi_1 = \phi(\Pi_2, \Pi_3)$ $\Pi_1 = \frac{u_{max}\omega_p^2}{a_p}$ $\Pi_2 = \frac{Q}{ma_p}$ $\quad \Pi_3 = \frac{u_y\omega_p^2}{a_p}$	One-sine Acceleration Pulse (Type-A) 
	$\Pi_1 = \phi(\Pi_2, \Pi_3, \Pi_4)$ $\approx \phi(\Pi_2, \Pi_3, 0)$ $\Pi_1 = \frac{u_{max}\omega_p^2}{a_p}$ $\Pi_2 = \frac{\omega_s}{\omega_p}, \quad \omega_s = \sqrt{\frac{K_s}{m}}$ $\Pi_3 = \frac{Q}{ma_p}$ $\quad \Pi_4 = \frac{u_y\omega_p^2}{a_p}$	One-cosine Acceleration Pulse (Type-B) 
	$\Pi_1 = \phi(\Pi_2, \Pi_3)$ $\Pi_1 = \frac{u_{max}\omega_p^2}{a_p}$ $\Pi_2 = \frac{\omega_o}{\omega_p}, \quad \omega_o = \sqrt{\frac{K_o}{m}}$ $\Pi_3 = \xi$	Earthquake record with distinguishable acceleration pulse 

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