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THE TIME-STEP NUMERICAL SIMULATION OF FREE FIELD MOTION OF LAYERED HALF-SPACE FOR INCLINED SEISMIC WAVES

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SUMMARY

In numerical simulation of wave motion scattering problem with obliquely incident seismic wave, besides the incident wave at bottom artificial boundary, the free field motion at incident side boundary, induced by the background layered half-space, is of input energy into the computational area. In order to replace complicated frequency domain method, a time-step numerical simulation method to calculate the free field motion is developed. The bases of this method are the decoupling motion equations of finite element and a interpolation formula of free field motion based on that the apparent horizontal velocity of free field motion is constant and can be calculated exactly.

INTRODUCTION

In numerical simulation of wave scattering problems by finite element method or finite difference method, the finite computational area must be truncated from the semi-infinite media and appropriate artificial boundary condition must be introduced (Fig.1). The artificial boundary condition can only simulate the outgoing wave propagating away from the computational area. The non-outgoing waves propagating along the boundary or toward the computational area should be eliminated from the total wave field when the artificial boundary condition is applied. For time and space

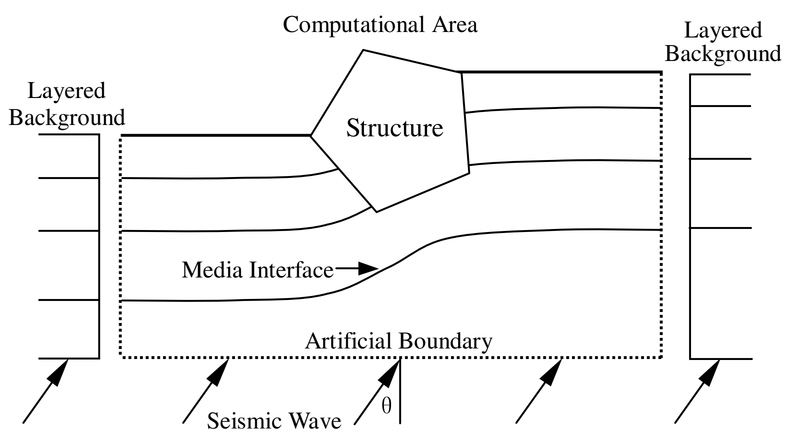

Fig.1 The mechanical model of finite element

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uncoupling artificial boundary condition, the non-outgoing wave of bottom boundary and the side boundaries can be defined respectively. For example, a common definition is took the incident seismic wave as the non-outgoing wave of bottom boundary and the free field of the background media as that of side boundaries (Liao, 2002). At present, the assumptions that the background media is layered half-space and the incident seismic wave is shearing or compressing wave propagating vertically are usually adopted, and the calculation of free field may be simplified to one-dimensional problem and performed in time domain conveniently. However, the seismic wave is not incidence vertically when the computational area is near the earthquake source. The study of Jin and Liao (1994), Sigaki (2000) shown that the average incident angle of the seismic wave is about 30° at the rock site based on the statistical analysis of the strong motion records. So in some important cases, it is necessary to consider the seismic wave incident obliquely. When the seismic wave is obliquely incident, besides the incident wave at bottom artificial boundary, the free field motion at incident side boundary, induced by the background layered half-space, is of the input energy into computational area, and must be calculated before the artificial boundary condition is applied. Now, if the background media is homogeneous half-space, the free field motion at the side boundary nodes can be expressed by simple analytical formula and calculated easily in the time domain. If the background media is general layered half-space, the free field motion at all the side boundary nodes and all time-steps can be calculated simultaneously by Haskell-Thomson method (Haskell, 1953, Thomson, 1950) and the Fourier transform, and then transmitted to the boundary processing of the wave scattering problem in the form of dimension or file. This procedure is not only complicated, but also needs more computer memory or hard-disk space. In order to simulate synchronously the input and the scattering wave field motion in time domain, a time-step numerical simulation method to calculate the free field motion is developed to replace the complicated frequency domain method. The bases of this method are the decoupling motion equations of finite element and a interpolation formula of free field motion based on that the apparent horizontal velocity of free field motion is constant and can be calculated exactly.

CHARACTERISTICS OF FREE FIELD MOTION OF LAYERED HALF-SPACE

When plane P-wave or SV-wave are obliquely incident on the interface of the horizontal layered media at an angle α_1 or β_1, the reflected P-Wave, SV-wave, and transmitted P-wave, SV-wave are induced (Fig. 2). Based on the Snell's law, the reflection angles of the P-wave or the SV-wave are equal to the corresponding incident angles, and the relationship between the incident angle and transmission angle can be written as

$$\frac{V_{P1}}{\sin\alpha_1} = \frac{V_{S1}}{\sin\beta_1} = \frac{V_{P2}}{\sin\alpha_2} = \frac{V_{S2}}{\sin\beta_2} \qquad (1)$$

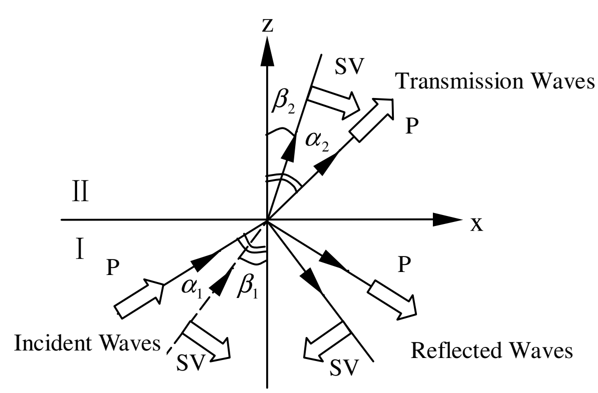

Fig.2 Reflection and transmission of P and SV waves

Where V_p and V_s are the propagating velocities of P and SV wave in two media.

Considering the definition of apparent horizontal velocity $V = \frac{V}{\sin\theta}$ (V is plane wave propagating velocity, θ is incident angle), the Snell's law can be illustrated as the whole wave system composed of the incident wave, reflected and transmitted plane waves induced by the media interface, has the same apparent horizontal velocity. Extending to the general situation of layered half-space subjected to obliquely incident plane wave, the free field wave system has the same apparent horizontal velocity.

The apparent horizontal velocity of P wave is $V_x = \frac{V_{PN}}{\sin\alpha}$ and that of S wave is $V_x = \frac{V_{SN}}{\sin\beta}$. Where V_{PN} and V_{SN} are the velocities of P-wave and the S-wave in the incident half space respectively, α and β are their incident angles respectively.

BASIC IDEA OF THE EXPLICIT FINITE ELEMENT METHOD

The nodes in computational area can be divided into two groups: one is interior nodes whose motion can be calculated by explicit finite element method, another is boundary nodes whose motion can be calculated by artificial boundary condition.

Displacement calculation of the interior nodes

Based on that the wave velocity is finite, Prof. Liao (1984) presented a finite-element decoupling method, where the dynamic equations of the interior nodes are transformed into a decoupling explicit step-by-step integration by adopting the lumped mass and the approximate concept of difference. In time aspect, the recursion is explicit because the displacement field at third time step can be recurred from the displacement field at the former two time steps. In Space aspect, the displacement of any nodes at (p+1)Δt can be calculated from the displacement of the adjacent nodes at pΔt and (p-1)Δt, which improve the calculating efficiency greatly, because no stiffness, mass and damping matrix are assembled and no large coupling equation group is solved. Where p is a positive integer and Δt is the time step.

Taking the elastic and undamped media as example, the dynamic equation of any node (i, j) at any time pΔt can be written as:

$$M_{i,j}\ddot{\mathbf{u}}_{i,j}^{p} + \sum_{m}\sum_{n} K_{ij,mn}\mathbf{u}_{m,n}^{p} = \mathbf{F}_{i,j}^{p} \tag{2}$$

Where, $M_{i,j}$ is the lumped mass of the node (i, j); $K_{ij,mn}$ is the stiffness coefficient, representing the elastic restoring force between node (i, j) and the adjacent node (m, n); $\ddot{\mathbf{u}}_{i,j}^{p}$ is the acceleration vector of node (i, j) at $t = p\Delta t$; $\mathbf{u}_{m,n}^{p}$ is the displacement vector of node (m, n) at $t = p\Delta t$; $\mathbf{F}_{i,j}^{p}$ is the external load of node (i, j) at $t = p\Delta t$, which is zero when not considering the volume force. Using central differences, $\ddot{\mathbf{u}}$ can be approximated as

$$\ddot{\mathbf{u}}_{i,j}^{p} = \frac{1}{\Delta t^2}(\mathbf{u}_{i,j}^{p+1} - 2\mathbf{u}_{i,j}^{p} + \mathbf{u}_{i,j}^{p-1}) \tag{3}$$

Substitution of Eq.(3) into Eq.(2) leads to

$$\mathbf{u}_{i,j}^{p+1} = 2\mathbf{u}_{i,j}^{p} - \mathbf{u}_{i,j}^{p-1} - \frac{\Delta t^2}{M_{i,j}}\left(\mathbf{F}_{i,j}^{p} + \sum_{m}\sum_{n} K_{ij,mn} u_{m,n}^{p}\right) \tag{4}$$

Equation (4) is the displacement calculation formula of the interior nodes.

Displacement calculation of the boundary nodes

To obtain the finite computational area from the semi-infinite space for finite-element analysis, the artificial boundaries must be introduced. Multi-Transmitting Formulae (MTF) is a kind of artificial boundary conditions, it directly simulate the procedure of the wave transmits from the inner of the finite element model to the outside through the boundary, and the accurate simulation is realized based on multi-transmission of the outgoing error wave (Liao，1984).

Multi-Transmitting Formulae (MTF)

Considering a node O on the artificial boundary, the boundary normal line is took as the x-axis, and the point O is took as the original point of x-axis. Assuming the displacement of the x-axis is u(t, x). The displacement of the node $x = -j\Delta x$ (Δx is the space-step in the artificial boundary area) at t=p Δt can be denoted by $\mathbf{u}_j^p = \mathbf{u}(p\Delta t, -j\Delta x)$. The Multi-Transmitting Formulae (MTF) with 2-order precision is:

$$\begin{aligned} &\mathbf{u}_0^{p+1} = 2(t_1\mathbf{u}_0^p + t_2\mathbf{u}_1^p + t_3\mathbf{u}_2^p) - (t_4\mathbf{u}_0^{p-1} + t_5\mathbf{u}_1^{p-1} + t_6\mathbf{u}_2^{p-1}) \\ &t_1 = (S-2)(S-1)/2 \quad t_2 = S(2-S) \quad t_3 = S(S-1)/2 \\ &t_4 = (S-1)(2S-1) \quad t_5 = 4S(1-S) \quad t_6 = S(2S-1) \end{aligned} \tag{5}$$

Where $S = c_a \Delta t / \Delta x$, and c_a is the artificial wave velocity.

Displacement calculation of the boundary nodes

For obliquely incident seismic wave (for example towards top right), both the incident displacement of the nodes in bottom artificial boundary area and the total free field displacement of the nodes in incident side boundary area (for example left-side) are ingoing waves, and must be eliminated from the total wave displacement to obtain the scattering displacement. Applying MTF, the scattering displacement of boundary nodes at (p+1) Δt can be calculated. Adding the scattering displacement at (p+1) Δt with the ingoing displacement at (p+1) Δt, the total displacement of boundary nodes at (p+1) Δt can be obtained.

The free field motion of layered elastic half-space can be computed by the Haskell-Thomson method and Fourier Transform, and then transmitted to finite element analysis in the form of dimension or file. Here, in order to simplify the simulation of the free field motion, a time-domain calculation method is presented.

THE TIME-STEP NUMERICAL SIMULATION OF FREE FIELD MOTION

The computational area used for free field motion calculation is the incident side boundary area including three rows of nodes. The free field displacement of three bottom boundary nodes at (p+1) Δt can be calculated by MTF, the free field displacement of interior nodes at (p+1) Δt can be calculated by Eq.(4). Now, the problem is how to calculate the free field displacement of the incident side boundary nodes at (p+1) Δt by use of the displacements of the side boundary and the adjacent interior nodes at p Δt and (p-1) Δt, and the displacements of the bottom boundary nodes and interior nodes near side boundary at (p+1) Δt.

Setting a Cartesian coordinate oxy, the original point is at the left down corner of the computational area (Assuming the seismic wave is obliquely incident to top right). The size of the finite elements is assumed to be $\Delta x \times \Delta y$, the left three rows of the finite-element nodes are shown in Fig. 3. The free-field displacement $\mathbf{u}_f(t, x, y)$ at the discrete nodes is denoted by $\mathbf{u}_{i,j}^p = \mathbf{u}_f(p\Delta t, x_i, y_j)$ $(i, j = 0, 1, 2, \cdots, N, \text{are positive intergers})$.

Interpolation equations of the free field motion

The incident side boundary (left artificial boundary) is took as y-axis, and a side boundary node (0, j) is took as the original position of x-axis. So the coordinates of node (i, j) are $x_i = i\Delta x,\ y_j = j\Delta y$, the free field displacement is denoted by $\mathbf{u}_{i,j}^{p}$. In order to interpolate $\mathbf{u}_{0,j}^{p+1}$ and $\mathbf{u}_{2,j}^{p+1}$, three auxiliary calculation points i_0, i_1 and i_2 whose coordinates are $x_{i0} = -V_x\Delta t + \Delta x$, $x_{i1} = V_x\Delta t$ and $x_{i2} = 2V_x\Delta t$ are introduced (V_x is the apparent horizontal velocity, which depends on the velocity of the incident media and the incident angle). Fig. 4 shows the position of discrete nodes (empty circle) and the auxiliary calculation points (solid circle).

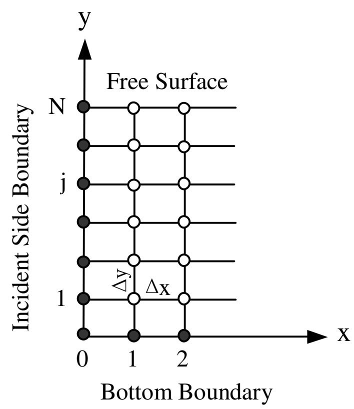

Fig.3 Nodes near incident side boundary

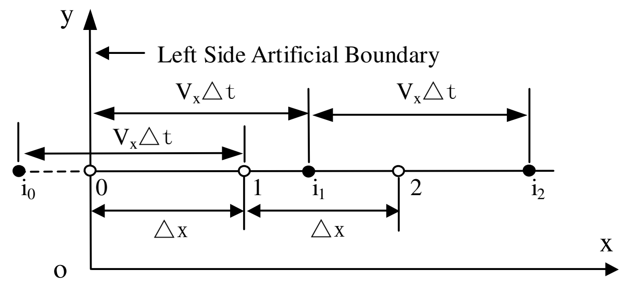

Fig.4 Sketch map of discrete nodes and auxiliary points for calculation

Assuming $V_x\Delta t \geq \Delta x$ (Which is tenable in many cases, because Δt is usually took a larger value that satisfies the stability criterion $\Delta t \leq \dfrac{\Delta x}{V}$). From Fig.4, we can see that the displacement $\mathbf{u}_{0,j}^{p+1}$ of the boundary node (0，j) at (p+1) Δt can be interpolated by $\mathbf{u}_{i0,j}^{p+1}$, $\mathbf{u}_{1,j}^{p+1}$ and $\mathbf{u}_{i1,j}^{p+1}$,

$$\mathbf{u}_{0,j}^{p+1} = d_1\mathbf{u}_{i0,j}^{p+1} + d_2\mathbf{u}_{1,j}^{p+1} + d_3\mathbf{u}_{i1,j}^{p+1} \tag{6}$$

Where d_1, d_2 and d_3 are interpolation coefficients. According to that the free field displacement propagates at the knowing velocity V_x along the x-axis from the incident boundary node (left) to the interior nodes (right), we can obtains $\mathbf{u}_{i0,j}^{p+1} = \mathbf{u}_{1,j}^{p+2}$ and $\mathbf{u}_{i1.j}^{p+1} = \mathbf{u}_{0,j}^{p}$, and then Eq.(6) is rewritten as

$$\mathbf{u}_{0.j}^{p+1} = d_1\mathbf{u}_{1,j}^{p+2} + d_2\mathbf{u}_{1,j}^{p+1} + d_3\mathbf{u}_{0,j}^{p} \tag{7}$$

Where $\mathbf{u}_{0,j}^{p+1}$ and $\mathbf{u}_{1,j}^{p+2}$ are unknown variables.

Similarly, the displacement $\mathbf{u}_{2,j}^{p+1}$ of the interior node (2, j) at (p+1) Δt can be interpolated by $\mathbf{u}_{1,j}^{p+1}$, $\mathbf{u}_{i1,j}^{p+1}$ and $\mathbf{u}_{i2,j}^{p+1}$. Considering $\mathbf{u}_{i1.j}^{p+1} = \mathbf{u}_{0,j}^{p}$ and $\mathbf{u}_{i2.j}^{p+1} = \mathbf{u}_{0,j}^{p-1}$, we can get

$$\mathbf{u}_{2,j}^{p+1} = e_1\mathbf{u}_{1,j}^{p+1} + e_2\mathbf{u}_{0,j}^{p} + e_3\mathbf{u}_{0,j}^{p-1} \tag{8}$$

Where e_1, e_2 and e_3 are interpolation coefficients.

Dynamic equations of the free field motion

Based on the finite-element dynamic equation, another relation equation between $\mathbf{u}_{0,j}^{p+1}$ and $\mathbf{u}_{1,j}^{p+2}$ can be developed.

According to the displacement calculation formulae (Eq.4), the free field displacement of any interior node (1, j) near the incident side boundary (left) at (p+2)Δt can be written as

$$
\begin{aligned}
\mathbf{u}_{1,j}^{p+2} &= 2\mathbf{u}_{1,j}^{p+1} - \mathbf{u}_{1,j}^{p} - \frac{\Delta t^2}{M_{1,j}} \sum_m \sum_n K_{1j,mn} \mathbf{u}_{m,n}^{p+1} \\
&= 2\mathbf{u}_{1,j}^{p+1} - \mathbf{u}_{1,j}^{p} - \frac{\Delta t^2}{M_{1,j}} \left[\sum_n K_{1j,0n} \mathbf{u}_{0,n}^{p+1} + \sum_n \left(K_{1j,1n} \mathbf{u}_{1,n}^{p+1} + K_{1j,2n} \mathbf{u}_{2,n}^{p+1} \right) \right]
\end{aligned} \tag{9}
$$

Where $M_{1,j}$ is the lumped mass of (1, j), $K_{1j,mn}$ is the stiffness coefficient of elastic restoring force of node (1, j) and adjacent nodes (m, n). The definitions of m and n are

$$
\begin{aligned}
& m = 0, 1, 2 \\
& n = \begin{cases} j-1, j, j+1 & j = 1, 2, \cdots, N-1 \\ N-1, N & j = N \end{cases}
\end{aligned} \tag{10}
$$

In the right part of Eq.(9), $\mathbf{u}_{1,j}^{p}$ and $\mathbf{u}_{1,j}^{p+1}$ are known, $\mathbf{u}_{2,j}^{p+1}$ can be calculated directly by Eq.(8), thus only $\mathbf{u}_{0,j}^{p+1}$ is unknown. Substitution of Eq.(7) into Eq.(9) leads to

$$
\mathbf{A}\mathbf{u}^{p+2} = \mathbf{b} \tag{11}
$$

Where

$$
\left.
\begin{aligned}
& \mathbf{u}^{p+2} = \left(u_{1,1}^{p+2}, u_{1,2}^{p+2}, \cdots, u_{1,N}^{p+2} \right)^T \\
& \mathbf{A} = \left[a_{j,n} \right]_{N \times N} \\
& a_{j,n} = \begin{cases} \dfrac{\Delta t^2}{m_{1,j}} d_1 K_{1j,0n} + \delta_{jn} & \text{n values prescribed in Eq. (8)} \\ 0 & \text{the other n values} \end{cases} \\
& \mathbf{b} = \left(b_1, b_2, \cdots, b_N \right)^T \\
& b_j = 2\mathbf{u}_{1,j}^{p+1} - \mathbf{u}_{1,j}^{p} - \frac{\Delta t^2}{m_{1,j}} \sum_n \left[K_{1j,0n} \left(d_2 \mathbf{u}_{1,n}^{p+1} + d_3 \mathbf{u}_{0,n}^{p} \right) + \sum_n \left(K_{1j,1n} \mathbf{u}_{1,n}^{p+1} + K_{1j,2n} \mathbf{u}_{2,n}^{p+1} \right) \right]
\end{aligned}
\right\} \tag{12}
$$

Because there are three unknown variables in each equation (two variables when j=1 and N), all the motion equations of the interior nodes near side boundary form a sparse linear equation set. Solving this linear equation set (Eq. 11), the free field displacement of all interior nodes near side boundary ((1, j), j=1, 2, …, N) at (p+2)Δt can be obtained, and then the free field displacement $\mathbf{u}_{0,j}^{p+1}$ of all left artificial boundary nodes ((0, j), j=1, 2, …, N) at (p+1)Δt can be interpolated by Eq.(7).

For out-plane wave obliquely incidence, the displacement vector only includes x-component u_x. For in-plane wave obliquely incidence, the displacement vector includes x-component u_x and y-component u_y.

PRECISION ANALYSIS AND APPLICABILITY

Precision analysis

Because the apparent horizontal velocity of the free field motion in the layered half-space can be calculated exactly, the interpolation formula based on this velocity may be regarded as an analytical formula. Thus, the precision of the free field motion calculation method in the time domain only depends on that of the explicit finite-element. In other words, this method is a numerical exact solution in the explicit finite element sense. It has been verified by numerical experiments that the explicit finite element has very high precision (Liao, 2002), so the free field motion time domain calculation method does not need further numerical verifications.

Applicability

The assumption of $c_x \Delta t \geq \Delta x$ is introduced when developing the interpolation formula. Substituting the apparent velocity $c_x = c/\sin\theta$ and the stability criterion of finite element analysis $\Delta x \leq c_{max} \Delta t$ (c_{max} is the maximum physical velocity in the media) into this assumption, we can obtain $\sin\theta \leq c/c_{max}$ (θ is the incident angle, c is the incident wave velocity at bottom half-space). Thus, the suggested wave incidence method can be used only when the incident angle is less than certain value (the critical angle). According to wave motion theory, the essence of this restriction condition is to ensure that there is no non-even wave that propagates along the interface and attenuates with the depth.

CONCLUSIONS

In the paper, an exact interpolation formula calculating the free field motion of the incident side boundary nodes is established. By the combination of this formula and the wave motion equations of the explicit finite element, a time-step numerical simulation method to calculate the free field motion of layered half-space subjected to obliquely incident body wave is developed. This method has several advantages: it has the same computational precision as the scattering field numerical simulation, the calculation is independent on the media layers, and it needs less computer memory or hard-disk space to transfer data. More importantly, this method provides a possible approach for synchronously simulation of the free field motion and the scattering field motion in time domain.

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