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# NON-ITERATIVE EQUIVALENT LINEAR METHOD FOR DISPLACEMENT-BASED DESIGN

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## SUMMARY

A method to estimate the maximum inelastic displacement demands on structures subjected to earthquake ground motions is presented. The method is based on equivalent linearization in which inelastic displacement demands are estimated by analyzing an equivalent linear elastic system with period and damping ratio larger than those of the inelastic system. The proposed method includes two new features not included in previous equivalent linear methods. The first one is that the properties of the equivalent system are obtained without knowledge of the ductility demand in the system and therefore iteration is avoided. The second is that period-dependent period shifts and equivalent damping ratio are used and therefore more accurate estimations of inelastic displacements demands are achieved.

## INTRODUCTION

In recent years several displacement-based seismic design procedures have been developed. The most important characteristic of these procedures is that seismic demands are lateral displacements as opposed to lateral forces that are commonly used as demand parameters in most building codes. One of the most important steps in these new design procedures is the estimation of the target lateral displacement demand in the structure. This target displacement is typically estimated by using a reference single-degree-of-freedom (SDOF) system whose properties are determined from a nonlinear static analysis of the structure. Furthermore, since the seismic hazard at a site is typically defined by elastic spectral ordinates, the peak inelastic deformations of the SDOF system are often estimated from the maximum deformation of elastic SDOF systems by using approximate procedures.

There are two basic types of methods that are used to estimate peak deformations of inelastic SDOF systems: displacement modification methods and equivalent linear methods. In displacement modification methods the maximum deformation of the inelastic system is approximated as the maximum deformation of an elastic system with the same stiffness and same damping ratio as the inelastic system times a displacement modification factor that depends on the lateral strength and period of vibration of the structure. No iteration is required to estimate the peak deformation of the inelastic SDOF system. The

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method implemented in FEMA-273 [1] and FEMA-356 [2] documents belongs to this type of approximate methods.

The second type of methods to estimate the peak deformation demands of inelastic SDOF systems are methods based on equivalent linearization in which the maximum deformation of an inelastic system is approximated as the maximum deformation of an elastic system with a lateral stiffness smaller than that of the initial stiffness of the inelastic system and with a damping ratio larger than that of the inelastic system. The method implemented in the ATC-40 document belongs to this type of methods.

In existing equivalent linear methods the period of vibration and the damping ratio of the equivalent period are computed as a function of the displacement ductility ratio, defined as the ratio of the maximum deformation to the yield deformation. However, the maximum deformation is unknown and is precisely the quantity that is being estimated, so in existing equivalent methods iteration is required. Besides being cumbersome and time consuming, iteration, if it is not adequately implemented, can lead to several problems such as lack of convergence or to multiple solutions [12]. Furthermore, recent evaluations of the accuracy of existing equivalent linear methods has shown that many of these methods can lead to significant overestimations of inelastic deformation demands of short period structures [4], [9]. The purpose of this work is to present a new equivalent linear method in which both of these disadvantages are eliminated.

## EQUIVALENT LINEAR METHODS

The maximum deformation of a nonlinear SDOF system subjected to an earthquake excitation can be obtained by solving the following equation:

$$\ddot{x} + 2\xi_0\omega_0\dot{x} + \omega_0^2 f(x)/k_0 = -\ddot{x}_g \tag{1}$$

in which $x$ = the lateral displacement of the mass relative to the ground; $\ddot{x}_g$ = the ground acceleration; $\xi_0$ = the viscous damping ratio; $f(x)$ = the restoring force of the system; $k_0$ = the initial stiffness of the system; and $\omega_0$ = the circular frequency of vibration of the system as shown below.

$$\omega_0 = \sqrt{\frac{k_0}{m}} = \frac{2\pi}{T_0} \tag{2}$$

where $m$ and $T_0$ = the mass and natural period of vibration of the system, respectively. In equivalent linear methods the maximum deformation of the inelastic system is approximated by computing the maximum deformation of an equivalent linear system using the following equation:

$$\ddot{x} + 2\xi_{eq}\omega_{eq}\dot{x} + \omega_{eq}^2 x = -\ddot{x}_g \tag{3}$$

where $\xi_{eq}$, $\omega_{eq} = 2\pi/T_{eq}$ and $T_{eq}$ are the equivalent viscous damping ratio, the equivalent circular frequency, and the equivalent natural period of vibration, respectively.

Many studies have been proposed in the past to compute the parameters $T_{eq}$ and $\xi_{eq}$ of the equivalent system. Some of these previous studies have been reviewed by Iwan and Gates [3]. In a recent study Miranda and Ruiz-Garcia [4] evaluated some of the most commonly used equivalent linear methods. Of the equivalent linear methods evaluated in that study it was found that the method developed by Iwan [5] provided the best results. In his study, Iwan determined the properties of the equivalent linear system (period and damping ratio) by minimizing the root mean square of relative error between the maximum deformations computed through nonlinear response history analyses and those computed with the equivalent linearization. The errors were averaged over all records and over all periods of vibration. Equivalent periods and equivalent damping ratios were determined as a function of displacement ductility ratio. Finally regression analyses were conducted to obtain simplified equations to obtain the equivalent period and equivalent damping ratio as a function of the displacement ductility ratio as follows:

$$\frac{T_{eq}}{T_0} = 1 + 0.121(\mu - 1)^{0.939}, \tag{4}$$

$$\xi_{eq} = \xi_0 + 0.0587(\mu - 1)^{0.371} \tag{5}$$

where $\mu$ = displacement ductility ratio, $\xi_0$ = viscous damping ratio of the nonlinear system.

## OPTIMAL EQUIVALENT PERIOD AND OPTIMAL EQUIVALENT DAMPING RATIO

In the proposed equivalent linear method the parameters of the equivalent system were determined using an optimization procedure similar to the one used by Iwan [5]. However, instead of obtaining optimum $T_{eq}/T_0$ ratios and $\xi_{eq}$-$\xi_0$ with the root mean square of relative errors averaged over all periods and over all ground motions, in this study this ratios were obtained for each period of vibration.

Let $\varepsilon_k$ be the relative error between the maximum inelastic displacement computed with response history analysis of the nonlinear system using Eq.(1) and the approximate maximum elastic displacement of computed with Eq.(3) for the *k*-th earthquake,

$$\varepsilon_k = \frac{\Delta_{ap}(T_0(T_{eq}/T_0), \xi_{eq})_k}{\Delta_{ex}(T_0, \xi_0)_k} \text{-1} \tag{6}$$

where $\Delta_{ap}(T_0(T_{eq}/T_0), \xi_{eq})_k$. is the approximate displacement for the *k*th record and $\Delta_{ex}(T_0, \xi_{eq})_k$. is the displacement computed with nonlinear response history analysis for the *k*th record. Then, a measure of the error produced with the equivalent linear method when using the parameters $T_{eq}$ and $\xi_{eq}$ can be computed with the root-mean-square or all relative errors as follows

$$\varepsilon(T_{eq}/T_0), \xi_{eq}) = \sqrt{\frac{1}{n}\sum_{k=1}^{n}\left[\frac{\Delta_{ap}(T_0(T_{eq}/T_0), \xi_{eq})_k}{\Delta_{ex}(T_0, \xi_0)_k} - 1\right]^2} \tag{7}$$

where *n* is the number of earthquake records.

If $T_{eq}/T_0$ and $\xi_{eq}$ are varied over a wide range of values, a matrix consisting of errors ($\varepsilon$) corresponding to different levels of $T_{eq}/T_0$ and $\xi_{eq}$ can be obtained. The optimal period shift and optimal equivalent damping ratio are those that yield the minimum error.

Although the overall optimization procedure used here is similar to that previously used by Iwan [5], there are two important differences: (1) errors are computed for each period as opposed to averaging them over all periods hence leading to "specialized" equivalent properties for each period of vibration; and (2) "exact" inelastic displacements in Eq. (6) are not computed for SDOF systems undergoing predetermined displacement ductility ratios, but rather for systems having predetermined strength ratios, *R*, defined as

$$R = \frac{mS_a}{F_y} \tag{8}$$

where *m*= mass of systems, $S_a$ = ordinate of acceleration spectral, and $F_y$ = lateral yield strength of systems. It should be noted that the strength ratio is also equal to the ratio of the elastic spectral ordinate $S_d$ over the yield displacement of the system $\Delta_y$. Hence this parameter is simply a measure of the intensity of the ground motion relative to the yielding parameters of the SDOF system. By using this normalized intensity parameter instead of the displacement ductility ratio, an estimation of the maximum inelastic deformation can be obtained directly without iteration.

## EARTHQUAKE GROUND MOTIONS USED IN THE STUDY

A total of 72 earthquake acceleration time histories recorded in the state of California from 9 different earthquakes were used to determine optimum parameters of the equivalent linear system. All the ground motions selected have the following characteristics: (1) they were recorded on firm sites; (2) they were recorded on free field stations or in the first floor of low-rise buildings with negligible soil-structure interaction effects; (3) they were recorded in earthquakes with surface wave magnitudes ($M_s$) between 5.8 and 7.5; and (4) the records had the peak ground accelerations greater than 45 cm/s$^2$. For detailed information about these ground motions, the reader is referred to Ruiz-Garcia and Miranda [10].

## RESULTS OF STATISTICAL STUDY

The inelastic displacements in the denominator of equation (6) were computed for elasto-plastic SDOF systems with a viscous damping ratio of 5% and seven levels of the strength ratios (*R*=1, 1.5, 2, 3, 4, 5 and 6). For each earthquake record and each strength ratio, the inelastic displacements were calculated for a set of 48 periods of vibration between 0.15 and 3.0 sec with an increment of 0.05 sec. Approximate displacements in numerator of Eq.(6) were computed for a wide range of normalized equivalent periods ($T_{eq}/T_0$) between 0.7 and 10.0 having an interval of 0.05 and with a equivalent viscous damping ratios ($\xi_{eq}$) ranging from 5% to 99% having a step of 2%.

Matrices of errors for different normalized equivalent periods ($T_{eq}/T_0$) different equivalent viscous damping ratios ($\xi_{eq}$) were obtained for each period of vibration and each strength ratio *R* (for a total of 288 error matrices). A contour plot can be constructed for each of these matrices. An example corresponding to a system with *R*=6 and $T_o$=2.8 s is shown in Figure 1. From this figure it can be seen that the optimum parameters corresponding to *R*=6 and $T_o$=2.8 s (those that minimize the error) are $T_{eq}/T_0$=1.75 and $\xi_{eq}$=21%.

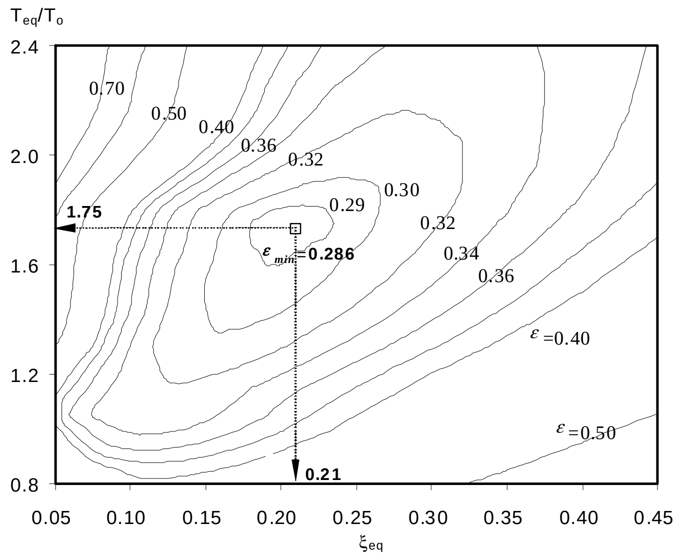


**Figure 1. Root-mean-squared errors of displacement ($\varepsilon$) for determining the optimal period shift ($T_{eq}/T_o$) and equivalent damping ratio ($\xi_{eq}$). $T_o$=2.8 s and R=6**

Figure 2 shows the relationship between the optimal normalized equivalent periods ($T_{eq}/T_0$) and the natural period ($T_o$) for various strength ratios (*R*). It can be seen that these optimal values depend on both natural period and strength reduction factor. It can be observed that for all periods and all strength ratios, $T_{eq}/T_0 \geq 1.0$, which means that the equivalent linear system will always have a period greater than that of the corresponding nonlinear system. Furthermore, the optimal period shifts increase significantly for systems with natural periods less than 0.8 s, whereas they are small for periods greater than 0.8s. It is also interesting to note that for strength ratios smaller than four and periods of vibration greater than 1.0s, the optimal normalized equivalent periods do not exhibit large variations with changes in period of vibration.

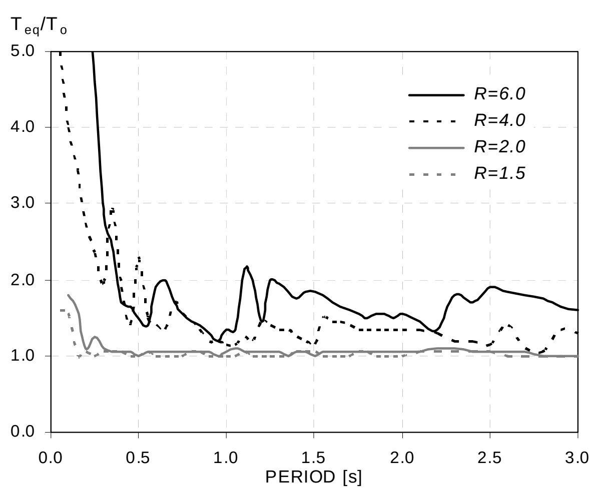


**Figure 2 Optimal period-dependent normalized equivalent periods for different levels of R.**

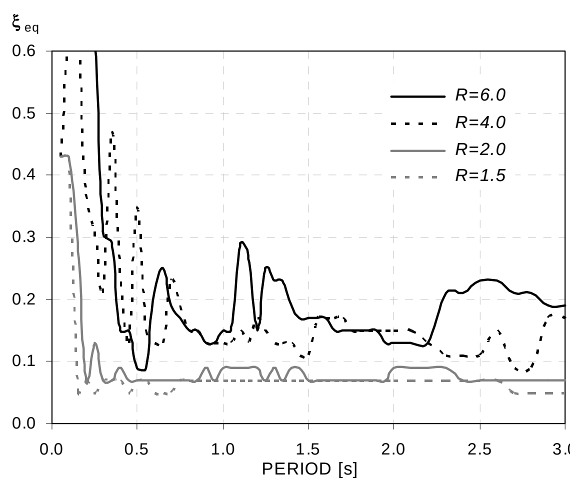


**Figure 3. Optimal period-dependent equivalent damping ratios for different levels of R.**

The relationship between the optimal equivalent viscous damping ratio ($\xi_{eq}$) and the natural period is presented in Figure 3 for different strength ratios. The observations are similar to those of optimum normalized equivalent periods. In particular, it can be observed that optimal equivalent damping ratios are relatively large for periods of vibration smaller than 0.8s, and that are more sensitive to the lateral strength ratio for short periods of vibration.

Iwan [5] previously noted that error countour plots tended to have an elliptical shape, indicating that equivalent linear methods using either large periods shift and large equivalent damping ratios or small period shift and small equivalent damping ratios tended to produce better results than methods that proposed large period shifts in combination with small equivalent damping ratios or methods that used small period shifts in combination with large damping ratios. However, those observations based were based on error countours averaged over all periods. Results from this investigation indicate than in the short period region better deformation estimates are obtained by using large period shifts and large equivalent damping ratios. Similarly, for medium and long periods better estimates are obtained by using small period shifts and small equivalent damping ratios.

## NONLINEAR REGRESSION ANALYSES

The above statistical results and observations were used to derived improved equations to obtain the equivalent period and equivalent damping ratio to estimate inelastic deformations. A two-stage nonlinear regression analysis [11] was conducted to obtain simplified expressions of the equivalent period and of equivalent damping ratio as a function of the strength ratio *R* and the period of vibration $T_0$. The resulting equations for $T_{eq}/T_0$ and $\xi_{eq}$ are given as follows:

$$\frac{T_{eq}}{T_0} = 1 + (R^{1.8} - 1)(0.027 + \frac{0.01}{T_0^{1.6}}) \tag{9}$$

$$\xi_{eq} = \xi_0 + (R - 1)(0.02 + \frac{0.002}{T_0^{2.4}}) \tag{10}$$

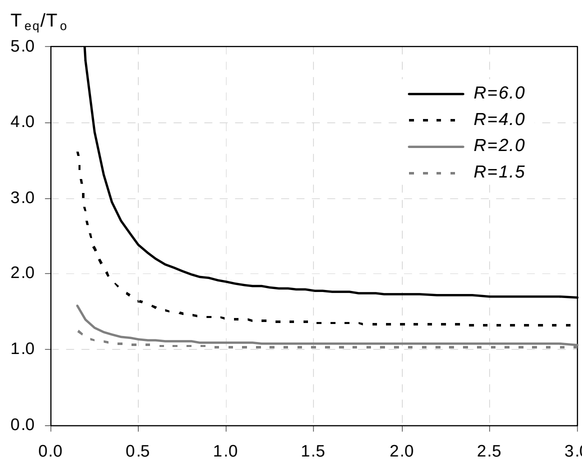


**Figure 4. Proposed period shifts for equivalent linear method.**

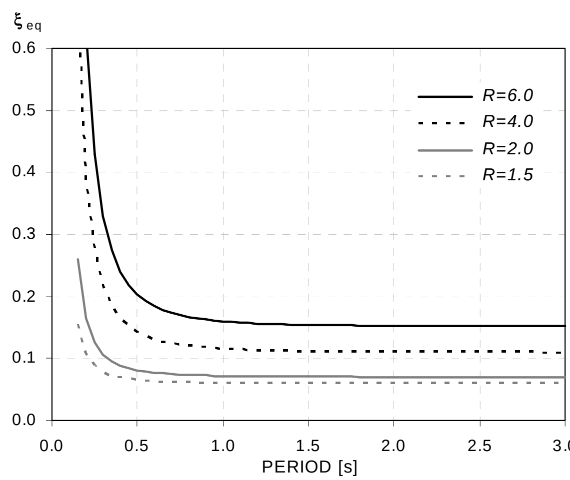


**Figure 5. Proposed equivalent damping ratios for equivalent linear method.**

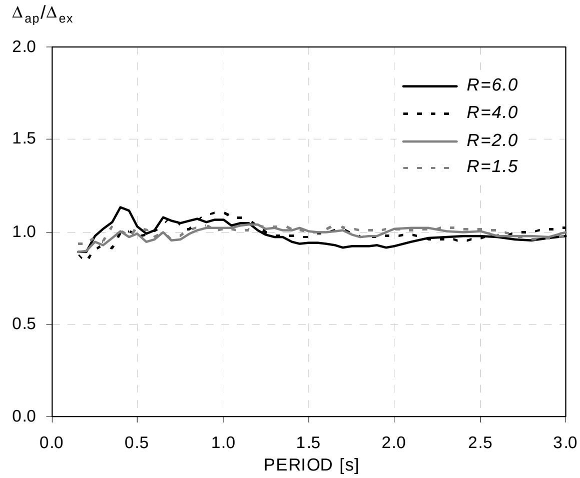


**Figure 6. Mean approximate to exact displacement ratio under 72 earthquake records.**

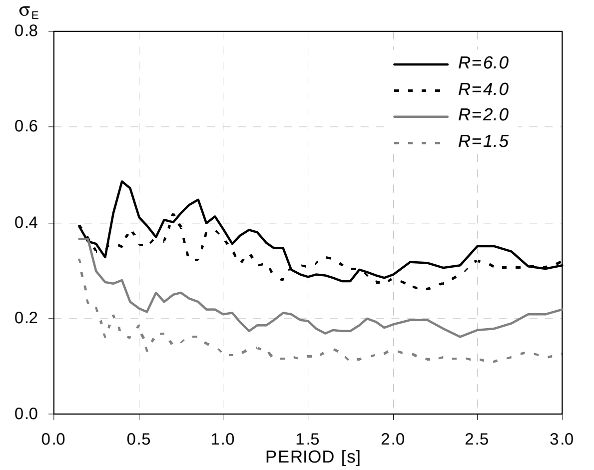


**Figure 7. Standard error for $\Delta_{ap}/\Delta_{ex}$ under 72 earthquake records.**

where $\xi_0$ = the viscous damping ratio of the inelastic systems, which in this study was assumed to be 5%. Eqs. (9) and (10) are plotted in Figures 4 and 5. It can be seen that in the short period region, both parameters are strongly period dependent, while for period longer than about 1.0, these parameters are approximately period independent as originally proposed by Iwan [5]. Equations (9) and (10) are relatively simple and have the advantage of permitting an estimate of the maximum deformation without having to iterate, and as will be shown later, lead to better results than those of other equivalent linear methods.

## EVALUATION OF THE PROPOSED EQUATIONS

In this section, the accuracy of the proposed period shifts (Eq.9) and equivalent viscous damping ratios (Eq.10) derived from the nonlinear regression analyses of the above section will be evaluated by defining the following error term:

$$E_{T_0,R,k} = \frac{\Delta_{ap}(T_{eq},\xi_{eq})_k}{\Delta_{ex}(T_0,\xi_0,R)_k} = \frac{\Delta_{ap,k}}{\Delta_{ex,k}} \tag{11}$$

where $E_{T_0,R,k}$ is the ratio of approximate $\Delta_{ap}(T_{eq},\xi_{eq})$ to exact $\Delta_{ex}(T_0,\xi_0,R)$ maximum inelastic displacement at the period of vibration ($T_0$) for a strength reduction factor (*R*) under the *k*-th earthquake. From Eq. (11), it is realized that the equivalent linear systems, $\Delta_{ap}(T_{eq},\xi_{eq})$, will give the best estimation if $E_{T_0,R,k}$ equals or approaches to 1.0. Errors were computed for each period, each strength ratio and each ground motion and then averaged for all ground motion to obtain bias factors. Mean errors corresponding to all ground motions as a function of the period of vibration and strength ratio R and shown in figure 6. It can be seen that despite some departures from the optimum values, the proposed equations for the parameters of the equivalent linear systems predict the mean maximum inelastic deformation of these systems very well. All mean approximate to exact displacement ratios fall around 1.0 no matter what strength ratio and natural period the systems have. This is because if the errors surfaces are relatively flat around the minimum error (optimum parameters) small deviations from the optimum parameters do not lead to large errors.

In order to evaluate the dispersion of the errors produced in the proposed method, standard error $\sigma_E(T_0, R)$ were computed for each given $T_0$ and $R$ as follows:

$$\sigma_E(T_0, R) = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}\left[\frac{\Delta_{ap,k}}{\Delta_{ex,k}} - 1\right]^2} \tag{12}$$

The standard error can quantify the variability of the approximate maximum inelastic displacements around the exact counterparts. As the quality of the approximate inelastic displacements increases, the standard errors approach zero. Note that the quantity is different from the standard deviation or the error, which quantifies the variability of the data around the mean (not around the exact value). Figure 7 shows the standard error for the 72 earthquake ground records. For a given level of *R*, this figure reveals that the dispersion is very uniform as systems in the mid and long period regions. For short periods the standard error exhibits slightly larger errors. It can be seen that the error increases as the strength ratio increases. But, standard errors tend to saturate for strength ratios are greater than about 3.0.

## COMPARISONS WITH OTHER METHODS

The accuracy of the proposed method was compared to that of methods proposed by Iwan [5], Kowalsky [6] and Iwan and Guyader [7]. As mentioned early, these methods the period shift ($T_{eq}/T_0$) and the equivalent damping ratio are period independent. Furthermore, these parameters are a function of the displacement ductility ratio and therefore require iteration..

### Equivalent period and equivalent damping ratio

Figure 8 compares period shifts for different methods assuming a post-elastic stiffness ratio $\alpha$=0 for a system with a period of 2s. In this figure, curves corresponding to the methods proposed by Iwan [5], Kowalsky [6] and Iwan and Guyader [7]. are plotted as a function of $\mu$, while Eq. 9 is plotted as a function of the lateral strength ratio R. In this period range, the strength ratio is on average approximately equal to the ductility ratio, according to the rule of equal displacement. Hence, they can be compared in

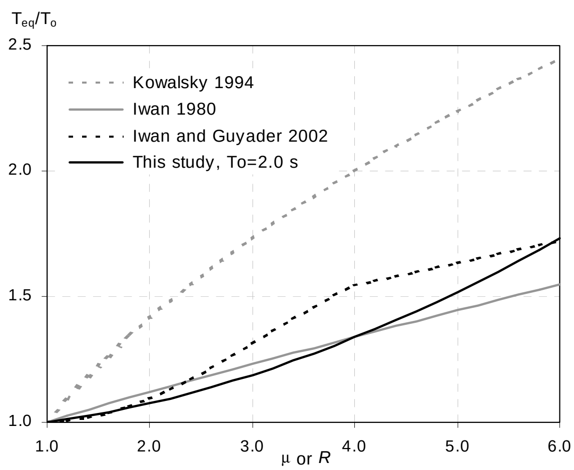


**Figure 8. Comparison of period shifts in various equivalent linear methods.**

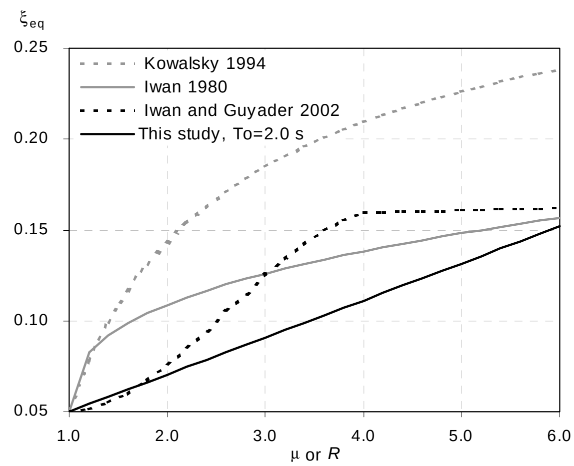


**Figure 9. Comparison of equivalent viscous damping ratios in various equivalent linear methods.**

the same figure. From Figure 8, it is seen that the period shifts from this study are close to those used by Iwan, and by Iwan and Guyader when $T_0$=2.0 sec, while those obtained by Kowalsky's method, which is based on secant stiffness, are larger. The difference between this method and the other methods becomes greater with increasing levels of $\mu$ or *R*. For displacement ductility ratios over three, $T_{eq}/T_0$ computed with Kowalsky's method becomes 40% percent larger than those calculated by the other three methods.

Figure 9 compares equivalent damping ratios for the four equivalent linearization methods. It can be seen that again Kowalsky's equivalent linear method yields values larger than those of the other methods. As shown in this figure for a given period of vibration, the equivalent damping ratio computed with equation 10 linearly increase as the strength ratio *R* increases. For systems with periods of vibration shorter than 0.8s, equations 9 and 10 lead to significantly larger parameters than those of the other three equivalent linear methods.

**Mean displacement error and its dispersion**

Figure 10 shows comparisons of the mean ratios of approximate $\Delta_{ap}(T_{eq},\xi_{eq})$ to exact $\Delta_{ex}(T_0,\xi_0)$ maximum inelastic displacement for *R*=3 and *R*=6. For each period of vibration and each value of *R*, these ratios were computed by averaging individuals errors of 72 ground motions. It is emphasized that the equivalent linear systems computed using Eqs. 9 and 10 of this study estimate the approximate maximum inelastic deformations $\Delta_{ap}(T_{eq},\xi_{eq})$ directly while iterative procedures are needed for methods proposed by Iwan, Kowalsky, and Iwan and Guyader. This figure shows that for periods longer than about 1.0s mean errors are similar more all four methods. In particular, the four methods produce mean error close to one, indicating that on average the maximum inelastic displacement computed with these methods will be close (with 5 or 10%) of those computed with inelastic response history analyses.

For periods shorter than 0.6s significant differences in mean errors are observed in figure 10. In particular it is shown that, as previously noted by Akkar and Miranda [9] existing equivalent linear methods tend to overestimate inelastic deformations In some cases these tendencies to overestimate exceed 50%.

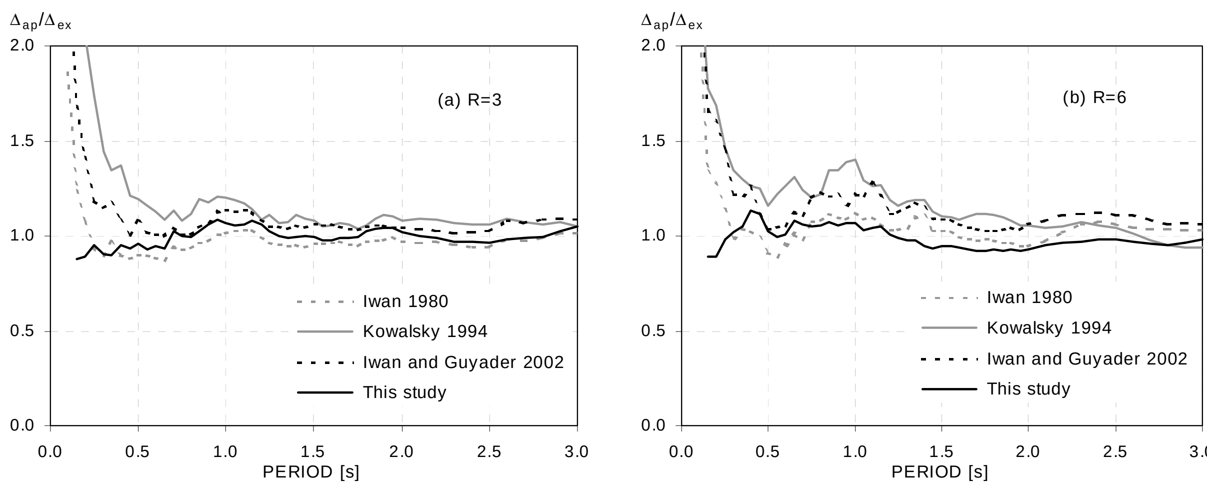


**Figure 10. Comparison of mean approximate to exact displacement ratio according to various equivalent linear methods. (a). *R*=3, (b). *R*=6.**

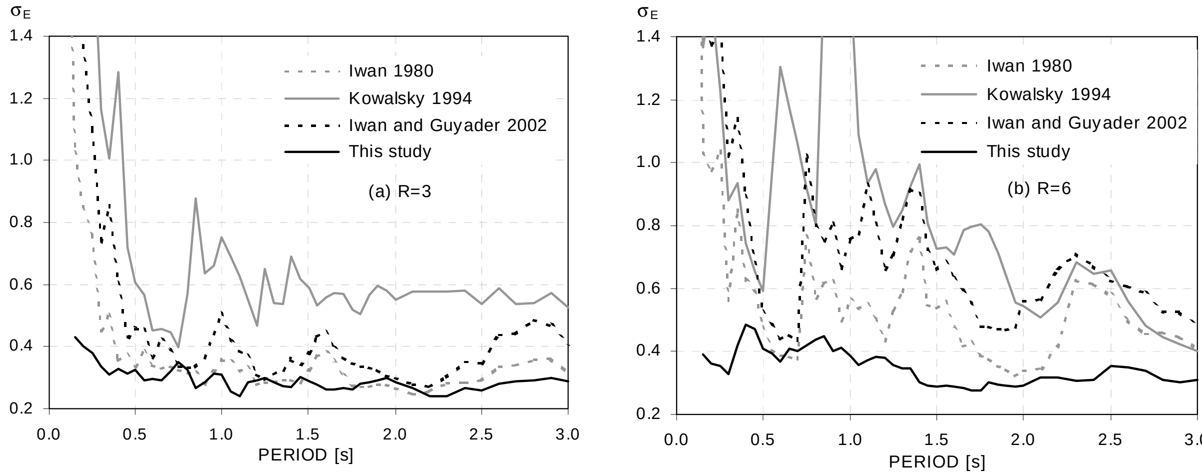


**Figure 11. Comparison of standard error of $\Delta_{ap}/\Delta_{ex}$ for various equivalent linear methods. (a). R=3, (b). R=6.**

While evaluating biases to over or underestimate inelastic deformation is critical, it is also very important to evaluate the dispersion of the error and in particular parameter that measure the size of error that may be produced by the approximate methods. Figure 11 presents a comparison of the standard errors (Eq. 12) for the four equivalent linear methods computed from the 72 earthquake records. Results are presented for lateral strength ratios $R$=3 and $R$=6. These figures show that the proposed equations not only lead to estimations that are adequate on average but that dispersion is also significantly reduced. Standard errors for existing equivalent linear methods are particularly large for short period systems.

## CONCLUSIONS

A method to estimate the maximum inelastic displacement demands on structures subjected to earthquake ground motions has been presented. The method is based on equivalent linearization in which inelastic displacement demands are estimated by analyzing an equivalent linear elastic system with a period and damping ratio larger than those of the nonlinear system whose maximum deformation is being estimated. The proposed method incorporates two improvements with respect to existing equivalent linear methods. The first improvement is that the properties of the equivalent system are obtained without knowledge of the ductility demand and therefore iteration is avoided. The second is that period-dependent period shifts and equivalent damping ratios are used therefore more accurate estimations of inelastic displacements demands are achieved.

The proposed method was evaluated by comparing its results with those computed with nonlinear response history analyses using an ensemble of 72 recorded earthquake ground motions. Mean ratios of approximate to exact deformations and standard errors were computed as a function of period and lateral strength ratio with standard errors. Similar results were computed for three previously proposed equivalent linear methods. It is shown that the proposed method can provide a similar accuracy, and in many cases a better accuracy, than previously proposed equivalent linear methods but without having to iterate.

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