

MODELING EARTHQUAKE RESPONSE OF STRUCTURES WITH HYSTERETIC DAMPING

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SUMMARY

The present paper examines differences between responses of single-degree-of-freedom (SDOF) systems with viscous and hysteretic damping to earthquake ground motions. Only responses within yield limits are considered. Structural behavior of SDOF systems with hysteretic damping is modeled by a generalized elasto-slip model of the Masing type. The model leads to a force-displacement relationship similar to that proposed by Ramberg and Osgood. Responses of the two systems are compared for a number of earthquake records. To achieve consistent comparison parameters of the linear system with viscous damping are determined by the secant stiffness method. In many cases significant differences in the peak displacement responses and energy values have been observed between the systems. This indicates that the prediction of the response of structures to earthquake ground motions even within elastic range by using linear models with viscous damping may be far from accurate and further research on this issue is needed.

INTRODUCTION

Performance-based seismic design implies the consideration of several performance levels, which are associated with damage suffered by structures as a result of earthquake ground motions (e.g., SEAOC [1]). At higher performance levels (such as fully operational and operational) very limited damage is allowed that means that values of structural response parameters (e.g., deflections, internal forces) should mainly stay within their elastic range. Usually, it is presumed that in such cases structural response can be predicted accurately by using well-known techniques of elastic analysis based on a linear structural model with viscous damping. However, this presumption is not necessarily correct since for traditional structural materials (e.g., steel, concrete, masonry) dissipated energy depends mainly on the amplitude of vibrations and not their frequency, i.e., damping is not viscous but hysteretic (e.g., Newmark [2]).

The present paper examines differences between responses of single-degree-of-freedom (SDOF) systems with viscous and hysteretic damping to earthquake ground motions. Only responses within elastic range are considered (i.e., the ductility ratio does not exceed unity). Behavior of SDOF systems with hysteretic

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damping is modeled by a generalized elasto-slip model of the Masing type (e.g., Lazan [3]). The model leads to a force-displacement relationship which can be described in terms of the Ramberg-Osgood expression (Ramberg [4]). Based on the model formulas, which allow to compute recoverable strain energy and energy dissipated by hysteretic damping at any point in time, have been derived.

Responses of SDOF systems with viscous and hysteretic damping and natural periods between 0.05 and 3 s are compared for a number of recorded earthquake ground motions. To achieve consistent comparison parameters of a linear system with viscous damping are determined by the secant stiffness method (e.g., Chopra [5]), i.e., its stiffness is set equal to the secant stiffness of the corresponding system with hysteretic damping and the damping ratio is obtained by equating the energy dissipated per cycle by the two systems. Methods employed in earthquake resistant design can be divided into (i) strength-based methods, (ii) displacement-based methods, and (iii) energy-based methods. Performance-based design puts emphasis on damage in structures, which is mainly associated with displacements and not strength. Thus, in this study responses of SDOF systems with viscous and hysteretic damping are compared in terms of peak displacement and energy values.

HYSTERETIC MODEL

Generalized elasto-slip model

Hysteretic behavior of a SDOF system is modeled using a generalized elasto-slip model (e.g., Lazan [3]). The model can be represented as two thin elastic bars having the same modulus of elasticity, E, and crosssectional areas of A_1 and A_2 (see Figure 1). At their ends the bars can be loaded by axial forces, f_S . The interaction between the bars along the contact zone of length L_c is described by Coulomb friction; there are also non-contact zones of length L_n . The maximum friction force between the bars determines the ultimate axial force which can be applied to the system, i.e., it can be treated as the yield strength of the system and it is denoted herein as f_y .



Figure 1. Generalized elasto-slip model

Load-deformation curves

Skeleton curve

When the system is first loaded by forces $f_S < f_y$ it causes slip between the bars at the ends of the contact zone. Diagrams of the axial forces forming in the bars as a result of this loading are shown in Figure 2, where $a_1=A_1/(A_1+A_2)$, $a_2=A_2/(A_1+A_2)$, $L_{sl1}=a_2L_cf_S/f_y$ and $L_{sl2}=a_1L_cf_S/f_y$ are the lengths of the slip zones, and $L_{sl}=L_c(1-f_S/f_y)$ is the length of the stick zone.



Figure 2. Diagrams of axial forces in the bars - loading

Generally, the deformation between the opposite ends of the bars, u, can be calculated as

$$u = \int_{L} \frac{N(x)}{EA} dx \tag{1}$$

where N(x) is the axial force along the length of the bars. Substituting values of the axial force from the diagrams in Figure 2 with the corresponding lengths and cross-sectional areas into Eq. (1) and integrating gives (for both tension and compression)

$$u = f_{S} \frac{\left(1 + \frac{l}{a_{1}} + \frac{l}{a_{2}}\right)L_{c}}{E(A_{1} + A_{2})} \left[1 + \frac{\frac{l}{a_{1}} + \frac{l}{a_{2}} - 3}{2\left(1 + \frac{l}{a_{1}} + \frac{l}{a_{2}}\right)\left|\frac{f_{S}}{f_{y}}\right|}\right]$$
(2)

where $l=L_n/L_c$. Introducing

$$k_{0} = \frac{E(A_{1} + A_{2})}{\left(1 + \frac{l}{a_{1}} + \frac{l}{a_{2}}\right)L_{c}}$$
(3)

which can be considered as the initial stiffness of the system, and

$$\alpha = \frac{\frac{l_{a_1} + l_{a_2} - 3}{2\left(1 + \frac{l_{a_1} + l_{a_2}}{a_1}\right)} \tag{4}$$

reduces Eq. (2) to

$$u = \frac{f_s}{k_0} \left(1 + \alpha \left| \frac{f_s}{f_y} \right| \right)$$
(5)

This equation of the skeleton curve represents a particular case of the relationship proposed by Ramberg and Osgood (Ramberg [4]) for modeling nonlinear structural behavior.

Unloading/reloading curves

It is known that systems which represent different combinations of Coulomb friction and linear elastic elements exhibit the Masing type of behavior (Newmark [2]). Thus, for the system shown in Figure 1 unloading/reloading curves should be described by the following equation

$$\frac{u-u^{*}}{2} = \frac{f_{s} - f_{s}^{*}}{2k_{0}} \left(1 + \alpha \left| \frac{f_{s} - f_{s}^{*}}{2f_{y}} \right| \right)$$
(6)

where (u^*, f_S^*) is the last point at which the loading process was reversed. This can be shown using the approach that has been employed to derive Eq. (5) for the skeleton curve. As the applied load drops from its peak value f_S^* to f_S the length of the stick zone increases from $L_{st}^*=L_c(1-f_S^*/f_y)$ at f_S^* to $L_{st}=L_{st}^*+L_{st1}+L_{st2}$, where $L_{st1}=a_2L_c(f_S^*+f_S)/(2f_y)$ and $L_{st2}=a_1L_c(f_S^*+f_S)/(2f_y)$. Correspondingly, the lengths of the slip zones at the ends of the bars decrease from $L_{sl1}^*=a_2L_cf_S^*/f_y$ and $L_{sl2}^*=a_1L_cf_S^*/f_y$ to $L_{sl1}=a_2L_c(f_S^*-f_S)/(2f_y)$ so that $L_{st1}+L_{sl1}=L_{sl1}^*$ and $L_{st2}+L_{sl2}=L_{sl2}^*$. Diagrams of the axial forces forming in the bars after unloading are shown in Figure 3. Note that all inclined parts of the diagrams have the bars per unit length. Eq. (6) can be derived by substituting values of the axial force from the diagrams in Figure 3 with the corresponding lengths and cross-sectional areas into Eq. (1).



Figure 3. Diagrams of axial forces in the bars - unloading/reloading

Energy evaluation

Relationships of the type defined by Eqs. (5) and (6) are employed quite often to describe hysteretic behavior of nonlinear structural systems. Usually, they are used to formulate empirical models since with the proper selection of the parameters they provide a good approximation to experimentally obtained force-displacement (or stress-strain) curves. However, such an approach creates difficulties in evaluation of energy of a structural system. Although energy absorbed by the system (i.e., work performed by force f_S on displacements u) can be easily calculated by integration of Eqs. (5) and (6) it cannot be decomposed into two essential parts - recoverable elastic strain energy, E_S , and irrecoverable hysteretic energy, E_H , without additional assumptions (of course, except of extreme points at the end of loading cycles when E_H equals the area of completed hysteretic loops). A different approach is chosen in the present paper where formulas for the evaluation of E_S are E_H are directly derived using the generalized elasto-slip model (Figure 1) that allows calculating these energies at any point of cyclic loading.

Skeleton curve

The sum of the recoverable strain energy and the dissipated hysteretic energy equals work, W, performed by the force f_s as the displacement increases from 0 to u along the skeleton curve, i.e.,

$$W = E_s + E_H \tag{7}$$

The work can be calculated as

$$W = f_S u - \int_0^{f_S} u(f_S) df_S$$
(8)

Substituting Eq. (5) into Eq. (8) and integrating results in

$$W = \frac{f_s^2}{k_0} \left(\frac{1}{2} + \frac{2\alpha}{3} \left| \frac{f_s}{f_y} \right| \right)$$
(9)

The recoverable strain energy of the system shown in Figure 1 can be evaluated as

$$E_{S} = \int_{L} \frac{N^{2}(x)}{2EA} dx \tag{10}$$

Substituting values of the axial force from the diagrams in Figure 2 with the corresponding lengths and cross-sectional areas into Eq. (10) and integrating leads to

$$E_{s} = \frac{f_{s}^{2}}{k_{0}} \left(\frac{1}{2} + \frac{\alpha}{3} \left| \frac{f_{s}}{f_{y}} \right| \right)$$
(11)

Then using Eqs. (7), (9) and (11) the following formula for the dissipated hysteretic energy is obtained

$$E_{H} = \frac{\alpha f_{s}^{2}}{3k_{0}} \left| \frac{f_{s}}{f_{y}} \right|$$
(12)

Unloading/reloading curves

The recoverable strain energy of the system after unloading (the applied force decreases from its peak value f_s^* to f_s , see Figure 4) is calculated using Eq. (10) and the axial force diagrams presented in Figure 3 that gives

$$E_{s} = E_{s}^{*} - \frac{1}{2} (u^{*} - u) (f_{s}^{*} + f_{s})$$
(13)

where E_S^* is the recoverable strain energy at (u^*, f_S^*) .



Figure 4. Force-displacement curves

In order to evaluate hysteretic energy dissipated during unloading introduce a new coordinate system $u'-f_s'$ with the origin at (u^*, f_s^*) . The transformation of the coordinates between the two coordinate systems is

$$u' = u^* - u f_s' = f_s^* - f_s$$
(14)

According to Eq. (13) in the new coordinate system the change in the recoverable strain energy during unloading will be equal to

$$E_{s} - E_{s}^{*} = -\frac{1}{2}u'(2f_{s}^{*} - f_{s}')$$
(15)

At the same time in the new coordinate system $(E_S-E_S^*)$ should also be equal to the sum of the work done by the force f_S^* on the displacement increasing from 0 to u' and the recoverable strain energy, E_S' , supplied to the system due to the work, W', done by f_S' on the same displacement. The work done by f_S^* is negative (the direction of f_S^* is opposite to the direction of u') and its absolute value equals f_S^* times u' (f_S^* remains constant while the displacement changes) so that

$$E_{S} - E_{S}^{*} = -f_{S}^{*}u' + E_{S}'$$
(16)

The recoverable strain energy E_S' can be calculated as

$$E_{S}' = W' - E_{H}' \tag{17}$$

where E_{H} is the hysteretic energy dissipated during unloading. From Eqs. (15), (16) and (17) follows that

$$E_{H}' = W' - \frac{1}{2} f_{S}' u'$$
(18)

According to Eq. (6)

$$u' = \frac{f_s'}{k_0} \left(1 + \alpha \left| \frac{f_s'}{2f_y} \right| \right)$$
(19)

while W can be calculated using Eq. (8) with f_s and u replaced by f_s and u', respectively (see Figure 4)

$$W' = \frac{f_{s}'^{2}}{k_{0}} \left(\frac{1}{2} + \frac{\alpha}{3} \left| \frac{f_{s}'}{f_{y}} \right| \right)$$
(20)

Finally, substituting Eqs. (19) and (20) into Eq. (18) gives

$$E_{H}' = \frac{\alpha f_{s}'^{2}}{12k_{0}} \left| \frac{f_{s}'}{f_{y}} \right| = \frac{\alpha (f_{s} - f_{s}^{*})^{2}}{12k_{0}} \left| \frac{f_{s} - f_{s}^{*}}{f_{y}} \right|$$
(21)

EQUIVALENT LINEAR SYSTEM

The equation of motion of a nonlinear SDOF system with hysteretic damping subjected to earthquake ground motion is

$$m\ddot{u} + f_s(u) = -m\ddot{u}_g(t) \tag{22}$$

where *m* is the mass of the system, *u* the relative displacement, $f_S(u)$ the restoring force, and $\ddot{u}_g(t)$ the ground motion acceleration. The equation does not include a term representing viscous damping since for a structural system built from traditional construction materials (e.g., steel, concrete) and without specially installed viscous dampers the contribution of viscous damping to energy dissipation is negligible (e.g., Newark [2]). For the model considered above the relationship between the restoring force and the displacement can be derived for the skeleton curve from Eq. (5)

$$f_{s}(u) = -\frac{f_{y}}{2\alpha} \left(1 - \sqrt{1 + \frac{4\alpha k_{0}}{f_{y}}} u \operatorname{sgn} u \right) \operatorname{sgn} u$$
(23)

and for an unloading/reloading curve from Eq. (6)

$$f_{S}(u) = f_{S}^{*} - \frac{f_{y}}{\alpha} \left[1 - \sqrt{1 + \frac{2\alpha k_{0}}{f_{y}} (u - u^{*}) \operatorname{sgn}(u - u^{*})} \right] \operatorname{sgn}(u - u^{*})$$
(24)

The equation of motion of a linear SDOF system with viscous damping can be written as

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_{o}(t) \tag{25}$$

where *c* is the viscous damping coefficient and *k* the elastic stiffness of the system. To be able to compare responses of the two systems to an earthquake ground motion "equivalence" between the systems needs to be established, i.e., it should be defined how to select parameters of one of the systems when parameters of the other system are given. A number of methods have been proposed for the replacement of a nonlinear system with an "equivalent" linear system. The methods differ by the determination of two fundamental parameters of the equivalent linear system – the equivalent damping ratio, $\zeta_e = c_e/(2m_e\omega_e)$, and the natural frequency of vibrations, $\omega_e = \sqrt{k_e/m_e}$ (or the natural period of vibrations, $T_e = 2\pi/\omega_e$), where c_e , m_e and k_e denote the viscous damping coefficient, mass and elastic stiffness of the equivalent linear system, respectively. The methods can be divided into two groups – analytical methods based on harmonic loading (e.g., Iwan [6]) and empirical methods (e.g., Gulkan [7], Iwan [8]).

In this study responses of the two systems are compared within elastic range, i.e., when the ductility ratio $\mu = u_m/u_y$ does not exceed unity (u_m denotes the maximum displacement of the system and u_y a yield displacement, i.e., the displacement corresponding to f_y). The empirical methods have been developed for yielding systems and equations of these methods to estimate T_e and ζ_e are only applicable when $\mu > 1$. The analytical methods can be used for equivalent linearization within the whole range of deformations. There have been a number of studies on the performance of the different analytical methods for the linearization of nonlinear systems subjected to earthquake loading. However, these comparative studies were limited to the case of nonlinearity associated with yielding so that a nonlinear system was usually presented by a simple bilinear model (e.g., Jennings [9], Iwan [6], Hadjian [10]). Thus, no data are currently available which could give a clear indication what of the analytical linearization methods is the most suitable for the purpose of the present study.

The secant stiffness method is the most commonly used among the analytical linearization methods (e.g., Miranda [11], Kwan [12]) and will be used herein. In this method the mass of the equivalent linear system is taken as the mass of the original nonlinear system. The stiffness of the linearized system is determined as the secant stiffness of the nonlinear system at the maximum displacement. Since in this study the systems are compared only within elastic range, the maximum displacement is equal to the yield displacement (i.e., $u_m=u_y$). The secant stiffness at u_y can be found from Eq. (5)

$$k_e = \frac{k_0}{1 + \alpha} \tag{26}$$

The natural frequency of vibration of the equivalent linear system is then

$$\omega_e = \sqrt{\frac{k_e}{m}} = \frac{\omega_0}{\sqrt{1+\alpha}} \tag{27}$$

where $\omega_0 = \sqrt{k_0/m}$ can be considered as the initial instant natural frequency of the nonlinear system. The equivalent damping ratio is found by equating the energy dissipated per cycle of harmonic vibrations with amplitude of u_y by the equivalent linear system with viscous damping and by the nonlinear system with hysteretic damping. The energy dissipated per cycle by the equivalent linear system, ΔE_D , is (e.g., Chopra [5])

$$\Delta E_D = 2\pi \zeta_e \frac{\omega}{\omega_e} k_e u_y^2 \tag{28}$$

where ω is the exciting frequency. The energy dissipated per cycle by the nonlinear system (i.e., the area enclosed by a hysteretic loop), ΔE_H , can be obtained using Eq. (21)

$$\Delta E_H = \frac{4\alpha f_y^2}{3k_0} \tag{29}$$

Substituting Eqs. (5), (26) and (27) into Eq. (28) and equating it to Eq. (29) gives

$$\zeta_e = \frac{2\alpha}{3\pi (1+\alpha)^{3/2}} \frac{\omega_0}{\omega}$$
(30)

In Eq. (30) damping of the nonlinear system is represented by the parameter α . However, damping is usually measured by the loss factor, ξ , which is defined as fractional part of the strain energy, E_{Sm} (i.e., strain energy at the maximum displacement which herein equals u_y), dissipated during one cycle of motion and divided by 2π

$$\xi = \frac{1}{2\pi} \frac{\Delta E_H}{E_{Sm}} \tag{31}$$

Finding E_{Sm} from Eq. (11) (when $f_S=f_y$), ΔE_H from Eq. (29) and substituting that into Eq. (31) allows to obtain the following relationship between α and ξ

$$\alpha = \frac{3\pi\xi}{2(2-\pi\xi)} \tag{32}$$

Substituting Eq. (32) into Eq. (30) results in

$$\zeta_{e} = \frac{\xi \sqrt{2 - \pi \xi}}{(2 + \pi \xi/2)^{3/2}} \frac{\omega_{0}}{\omega}$$
(33)

As can be seen the equivalent damping ratio depends on the exciting frequency. Thus, in order to evaluate the equivalent damping ratio for a nonlinear system subjected to an earthquake ground motion the frequency content of this motion should be characterize with a single parameter. Several frequency content parameters have been proposed (see Rathje [13]). In this study the frequency content will be represented by the average period, T_{av} , which is defined using a fast Fourier transformation (FFT) of the ground motion accelerogram as

$$T_{av} = \frac{\sum C_i(1/f_i)}{\sum C_i}$$
(34)

where C_i 's are the Fourier amplitudes of the accelerogram, and f_i 's the discrete Fourier transform frequencies. Eq. (33) then becomes

$$\zeta_{e} = \frac{\xi \sqrt{2 - \pi \xi}}{\left(2 + \pi \xi/2\right)^{3/2}} \frac{T_{av}}{T_{0}}$$
(35)

where $T_0=2\pi/\omega_0$.

COMPARISON OF THE SYSTEMS

Responses of the two systems are compared for three earthquake records: (1) Pacoima, USA, 196°, 02/09/1971, T_{av} =0.3420 s; (2) El Centro, USA, 270°, 05/18/1940, T_{av} =0.4623 s; and (3) Mexico City – Station 1, Mexico, 270°, 09/19/1985, T_{av} =1.6312 s (source of the records: NISEE, U.C. Berkeley, CA, USA; the average periods were computed in this study). Parameters of the system response being compared are: the maximum displacement, the maximum input energy, the dissipated energy, and the maximum sum of the elastic strain energy and the kinetic energy. The maximum displacement obtained for the nonlinear system is always equal to u_y . This is achieved by adjusting the yield strength of the system. For both of the systems the input energy, E_I , is evaluated as

$$E_I = -\int m\ddot{u}_{\rho} \dot{u} dt \tag{36}$$

and the kinetic energy, E_K , as

$$E_K = \frac{m\dot{u}^2}{2} \tag{37}$$

For the nonlinear system the energy dissipated in hysteretic damping, E_H , is estimated using Eqs. (12) and (21), and the elastic strain energy, E_S , by Eqs. (11) and (13). For the linear system the energy dissipated in viscous damping, E_D , is defined as

$$E_D = \int c_e \dot{u}^2 dt \tag{38}$$

where $c_e=2\zeta_e m\omega_e$, and the elastic strain energy as

$$E_S = \frac{ku^2}{2} \tag{39}$$

When the same symbol is used to denote a parameter, for example, like E_I for the input energy, in order to distinguish between the two systems in the following subscripts h and v will be used for the symbols associated with the nonlinear system with hysteretic damping and the linear system with viscous damping, respectively. For each of the earthquake records responses of the systems are compared for the initial period of vibration of the nonlinear system, T_0 , ranging from 0.05 s to 3.0 s with period increments of 0.05 s.

In the first series of analyses the loss factor, ξ , representing damping of the nonlinear system, is set equal to 0.05 which is a typical value for steel elements subjected to cyclic loading with the amplitude of f_y (e.g., Lazan [3]). The equivalent damping ratio of the linear system is defined by Eq. (35), i.e., ζ_e changes depending on T_0 . Results of the analyses are shown in Figure 5.



Figure 5. Ratios between response parameters of linear and hysteretic systems (ζ_e variable)

In the second series of analyses the damping ratio of the linear system, ζ_e , is assumed to be constant and equal to 0.02. The corresponding value of the loss factor for the nonlinear system, ξ , found from Eq. (35) for $T_{av}/T_0=1$ is 0.0436 ($\alpha=0.1102$). The aim of these analyses is to check if it is possible using a single

value of the damping factor (i.e., independent of the system period of vibration) to obtain a good approximation of the response of a nonlinear hysteretic system to an earthquake ground motion. Results of the analyses are presented in Figure 6.



Figure 6. Ratios between response parameters of linear and hysteretic systems (ζ_e constant)

According to the results presented in Figures 5 and 6 better agreement between responses of the systems is observed when the damping ratio of the linear system was taken as a constant value, independent of the period of vibration. Especially, this concerns the prediction of the maximum displacement. However, even in this case at certain values of T_0 the difference between the maximum displacements obtained for the two systems exceeds 20%. Moreover, as can be seen in Figure 6 the ratios between the responses of the systems fluctuate without any pattern. This shows that no matter what value of the damping ratio is selected if it is constant (i.e., ζ_e , does not depend on the period of a nonlinear system) there are always periods at which differences between the responses of the systems are significant. It can be also noted that the largest differences are observed between the responses of the systems to the Mexico City earthquake record, which has a very long average period.

CONCLUSIONS

Responses within yield limits (the ductility ratio did not exceed unity) of SDOF systems with viscous and hysteretic damping to earthquake ground motions were compared. In many cases significant differences between the peak displacements and energy values obtained for the hysteretic system and the equivalent linear system were observed. This indicates that the prediction of the response of structures to earthquake ground motions even within elastic range by using linear models with viscous damping may be far from accurate and further research on this issue is needed.

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