13th World Conference on Earthquake Engineering
Vancouver, B.C., Canada
August 1-6, 2004
Paper No. 451

A SIMPLIFIED APPROACH TO THE ANALYSIS OF TORSIONAL EFFECTS IN ECCENTRIC SYSTEMS: THE ALPHA METHOD

Giada GASPARINI[1], Stefano SILVESTRI[2], and Tomaso TROMBETTI [3]

SUMMARY

Eccentric structures, characterized by non coincident center of mass and center of stiffness, when subjected to dynamic excitation, develop a coupled lateral-torsional response that may increase the local peak dynamic response of such a structure: this behaviour becomes particularly important for seismic isolated structures for which large displacements are developed in the isolators. The coupled lateral-torsional response can be estimated only through a three-dimensional analysis which is specifically carried out for a single structure subjected to a determined dynamic input. In this paper the authors present the analytical formulation of a simplified method which allows to understand, predict and govern the global trend of one-storey eccentric structures to develop a torsional response to dynamic inputs through the identification of a system key parameter named "alpha". This parameter can be easily used to effectively estimate the maximum rotational response of a given eccentric system under a dynamic excitation through a simple linear elastic analysis of the "equivalent" non-eccentric system. Moreover, the results of the analysis in the non-linear field show that the linear elastic value of "alpha" acts as an upper bound for the corresponding value of elastic-perfectly plastic systems. In summary, this paper proposes a physically-based general theory which frames the problem of torsional phenomena of one-storey eccentric systems subjected to dynamic inputs and immediately allows the quantification of the system torsional response and the identification of the structural parameters governing it.

INTRODUCTION

The dynamic behaviour of eccentric systems has been the object of extensive research works both in linear and non-linear domains. However, a number of issues remains unresolved as regarding the inelastic response [1] of the system and the development of a simplified, yet accurate and physically-based design

[1] PhD Student, Department of Civil Engineering DISTART, Università degli Studi di Bologna, Viale Risorgimento 2, 40136 Bologna, BO, Italy. Email: giada.gasparini@mail.ing.unibo.it
[2] PhD Student, Department of Civil Engineering DISTART, Università degli Studi di Bologna, Viale Risorgimento 2, 40136 Bologna, BO, Italy. Email: stefano.silvestri@mail.ing.unibo.it
[3] Researcher, Department of Civil Engineering DISTART, Università degli Studi di Bologna, Viale Risorgimento 2, 40136 Bologna, BO, Italy. Email: tommaso.trombetti@mail.ing.unibo.it

procedures. Actually, the research in lateral-torsional coupling performed to date can be subdivided into three categories:

1. Investigation of the linear elastic response of three-dimensional laterally-torsionally coupled structural systems [2,3].
2. Investigation of the non-linear response of three-dimensional one- and multi-storey eccentric systems [1,4,5,6].
3. Evaluation of design code provisions for torsional effects in building structures [1].

As far as base-isolated buildings are concerned, the analysis of their coupled lateral-torsional dynamic response can be fairly simplified for the following reasons:

1. most common seismic isolators are cylindrical elements with a well known lateral stiffness that is generally independent of the direction of deformation [3];
2. the dynamic behaviour of seismic isolated structures can be fairly well captured through a simplified linear analysis in spite of the inherent non-linear force-deformation relationship of most common seismic isolators [3];
3. under strong motion excitation, the deformations of a seismic isolated structure are localized mainly in the seismic isolators and are only marginally influenced by the dynamic interaction with the superstructure [3,5,6,7].

For conceptual design purposes, the dynamic analysis of seismic isolated structures can be effectively reduced to that of a one-storey three-dimensional linear structural system with a roof diaphragm/slab assumed infinitely stiff in its own plane (i.e., rigid diaphragm assumption). Since the maximum isolator deformation is the basic design parameter considered in design codes, it is essential to develop a simple, rational and reliable method to determine the local increase (as compared to non eccentric case) in the maximum isolator deformation due to the eccentricity-induced rotational response.

Nagarajaiah *et al.* [6] confirmed that the superstructure has a small influence on the maximum deformation of base isolators and investigated the effects of selected parameters of eccentric structures on the dynamic coupled lateral-torsional response of base isolated systems. These results, however, are presented for a specific structural system and are not extended into a general purpose simplified theory.

The new insight presented in this paper is based on the study of the free vibration response of linear elastic eccentric systems and directly derived from accurate study of the equations of motions. The resulting simplified dynamic analysis procedure applies to a wide range of base-isolated systems and provides both a qualitative understanding and an effective quantitative estimate (for practical engineering purposes) of their coupled lateral-torsional response both in linear and non-linear field. As immediate result, a simple physically-based formula is here provided to estimate the maximum rotational response of a given base isolated structure in terms of few identified dimensionless controlling parameters of the system.

THE ECCENTRIC DYNAMIC SYSTEM AND ITS EQUATIONS OF MOTION

Consider the three-dimensional one-storey system idealization with rigid in-plane diaphragm of a general base isolated structure with non-coincident center of mass (C.M.) and center of rigidity (C.R.) displayed in Figure 1 where the seismic isolators are considered axially inextensible and the degrees of freedom are attached to the center of mass of the system.

Under the following two hypotheses:

1. the lateral stiffness (k_i, $i = 1, \ldots, N$) of each one of the N base isolators does not depend on the direction of deformation,
2. the rotational response u_θ developed under dynamic (e.g. seismic) excitation is small enough such that $u_\theta \cong \sin(u_\theta) \cong \tan(u_\theta)$,

the dynamic coupled lateral-torsional response of the system under consideration is governed by the following set of coupled differential equations of motion [8,9]:

$$m \cdot \begin{bmatrix} \ddot{u}_x(t) \\ \ddot{u}_y(t) \\ \rho \cdot \ddot{u}_\theta(t) \end{bmatrix} + [C] \cdot \begin{bmatrix} \dot{u}_x(t) \\ \dot{u}_y(t) \\ \rho \cdot \dot{u}_\theta(t) \end{bmatrix} + m\omega_L^2 \cdot \begin{bmatrix} 1 & 0 & \left(-e_y\sqrt{12}\right) \\ 0 & 1 & \left(e_x\sqrt{12}\right) \\ \left(-e_y\sqrt{12}\right) & \left(e_x\sqrt{12}\right) & \gamma^2 \end{bmatrix} \cdot \begin{bmatrix} u_x(t) \\ u_y(t) \\ \rho \cdot u_\theta(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\rho \end{bmatrix} \cdot \begin{bmatrix} p_x(t) \\ p_y(t) \\ p_\theta(t) \end{bmatrix} \tag{1}$$

where:

$u_x(t), u_y(t), u_\theta(t)$	translations along the *x*- and *y*-directions and rotation along the *z*-axis, respectively, of the base isolated system;
m	total mass of the super-structure, i.e. total mass resting over the base isolators;
I_p	polar mass moment of inertia of the superstructure with respect to the *z*-axis which passes through the center of mass;
$\rho = \sqrt{\frac{I_p}{m}}$	mass radius of gyration of the superstructure with respect to the *z*-axis;
$[C]$	damping matrix (note that this matrix takes into account the particular degrees of freedom here selected);
x_i, y_i	*x*- and *y*-coordinates of the *i*-th isolator;
$k = \sum_{i=1}^{N} k_i$	lateral stiffness (in any direction) of the total base isolation system;
$E_x = \left(\sum_{i=1}^{N} k_i x_i\right) \Big/ k$, $E_y = \left(\sum_{i=1}^{N} k_i y_i\right) \Big/ k$	eccentricities of the center of stiffness of the total base isolation system with respect to the center of mass in the *x*- and *y*-directions, respectively;
$k_{\theta\theta} = \sum_{i=1}^{N} k_i \cdot \left(x_i^2 + y_i^2\right)$	rotational stiffness (about the *z*-axis) of the total base isolation system;
$p_x(t), p_y(t), p_\theta(t)$	external dynamic forces/moment applied along the *x*-, *y*- and *z*-directions;
$\omega_L = \sqrt{\frac{k}{m}}$	uncoupled lateral (longitudinal or transversal) natural circular frequency of vibration;
$\omega_\theta = \sqrt{\frac{k_{\theta\theta}}{I_p}}$	natural circular frequency of rotational vibration of a fictitious non-eccentric structure having the same rotational stiffness and mass moment of inertia (with respect to the *z*-axis) as the eccentric system considered here;
$\gamma = \frac{\omega_\theta}{\omega_L} = \sqrt{\frac{k_{\theta\theta}}{\rho^2 \cdot k}}$	this ratio identifies the properties of the isolators mesh. $\gamma > 1$ characterises torsional-rigid structures, whilst $\gamma < 1$ characterises torsional-flexible systems.

Hereafter we will refer only to the first case above mentioned ($\gamma > 1$). Parameter γ tends to unity as the number of seismic isolators increases within the given planar dimensions of the base-isolated system.

$D_e = \rho \cdot \sqrt{12}$ "equivalent diagonal" of the system which, for system with rectangular shape and uniform mass distribution, coincides with the actual length of the diagonal of the system;

$e_x = \dfrac{E_x}{D_e}$, $e_y = \dfrac{E_y}{D_e}$ relative eccentricities in the *x*- and *y*-directions, respectively.

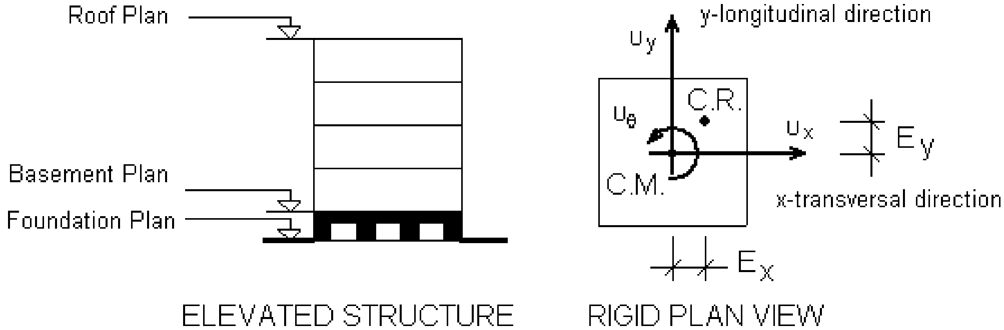


Figure 1: Three-dimensional one-storey system idealization

UNDAMPED FREE VIBRATIONS RESPONSE FROM A GIVEN TRANSLATIONAL DISPLACEMENT

Solving now Eq.1 specialized for undamped free vibration systems (i.e. $[C]=0$ and $p_x(t)=p_y(t)=p_\theta(t)=0$) with initial conditions:

$$\begin{bmatrix} u_x(0) \\ u_y(0) \\ \rho \cdot u_\theta(0) \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} \tag{2}$$

where a represents the given initial displacement along the *y*-direction (herein referred to as the longitudinal direction) and expressing the dynamic response of the system in modal coordinates, $Y_i(t)$, $i=1,2,3$, lead to uncoupled modal equations of motion, which can be integrated separately, and then recombined to form the free vibration response histories along the original degrees of freedom [8,9]:

$$u_x(t) = a\frac{e_x \cdot e_y}{e^2}\left\{-\frac{\Theta_3}{\Theta_3-\Theta_1}\cos(\omega_1 t)+\cos(\omega_2 t)-\frac{\Theta_1}{\Theta_1-\Theta_3}\cos(\omega_3 t)\right\} \tag{3a}$$

$$u_y(t) = a\frac{e_x^2}{e^2}\left\{\frac{\Theta_3}{\Theta_3-\Theta_1}\cos(\omega_1 t)+\frac{e_y^2}{e_x^2}\cos(\omega_2 t)+\frac{\Theta_1}{\Theta_1-\Theta_3}\cos(\omega_3 t)\right\} \tag{3b}$$

$$u_\theta(t) = \frac{a}{\sqrt{48}\cdot\rho}\cdot\frac{e_x}{e}\cdot\frac{\Theta_1\cdot\Theta_3}{\Theta_3-\Theta_1}\left\{\cos(\omega_1 t)-\cos(\omega_3 t)\right\} \tag{3c}$$

where ω_1, ω_2 and ω_3 are the undamped modal circular frequencies of the system:

$$\omega_1 = \omega_L\sqrt{1+\frac{e}{2}\Theta_1} \qquad \omega_2 = \omega_L \qquad \omega_3 = \omega_L\sqrt{1+\frac{e}{2}\Theta_3} \tag{4}$$

with:

$$e = \sqrt{e_x^2+e_y^2} \quad \text{and} \quad F = \frac{e}{\gamma^2-1} \tag{5}$$

$$\Theta_1 = \frac{1}{F}\cdot\left(1-\sqrt{1+48F^2}\right) \quad \text{and} \quad \Theta_3 = \frac{1}{F}\cdot\left(1+\sqrt{1+48F^2}\right) \tag{6}$$

Inspection of Eq.3a through Eq.3c leads to the following considerations:

1. when $e_y = e_x$, the maximum absolute value, $|u_y|_{\max}$, of the longitudinal response $u_y(t)$ is equal to the maximum absolute value, $|u_x|_{\max}$, of the transversal response $u_x(t)$, i.e. $|u_y|_{\max} = |u_x|_{\max}$;
2. $u_x(t) \neq 0$ only when $e_y \neq 0$ and $e_x \neq 0$;
3. the maximum absolute value, $|u_\theta|_{\max}$, of the rotational response $u_\theta(t)$ is obtained when $e_y = 0$ and $e_x \neq 0$;
4. the longitudinal, transversal and rotational responses consist of the sum of trigonometric functions of various amplitudes and circular frequencies.

Note that, as plotted in Figure 2, Θ_1 is always smaller than Θ_3. Furthermore, for the range of *F* values typical of most structures ($10^{-2} \leq F \leq 1$), Θ_1 assumes relatively small values and Θ_3 assumes relatively large ones. As a consequence, in these cases, the third terms in Eq.3a and Eq.3b can be neglected and due to the fact that generally $e < 0.3$, $\omega_1 \cong \omega_2$ and $\omega_3 >> \omega_2$. At this point it is worth recalling that, the sum of two trigonometric functions of equal amplitudes and different frequencies produces a harmonic (fast mode) with harmonically modulated (slow mode) amplitude. Figure 3 shows, for the special case of $e_y = e_x$, that (a) the longitudinal and transversal displacements responses have a slow modulation of their amplitude due to the closeness of ω_1 and ω_2, while the rotational response exhibits a fast modulation of its amplitude, due to the fact that ω_1 and ω_3 are well separated; (b) when the envelope of the longitudinal displacements reaches its maximum, that of transversal displacements is at its minimum and vice versa, and (c) the rotational response has a high amplitude modulation frequency, which results in a strong interaction between the fast and slow modes (due to the closeness of the fast and slow mode frequencies).

For the special case of $e_y = 0$, which gives, for fixed e_x, the maximum rotational response in free vibration from an initial deformation along the *y*-direction, Eq.3a through Eq.3c reduce to:

$$u_x(t) = 0 \tag{7a}$$

$$u_y(t) = a \cdot \left\{ \frac{\Theta_3}{\Theta_3 - \Theta_1} \cos(\omega_1 t) + \frac{\Theta_1}{\Theta_1 - \Theta_3} \cos(\omega_3 t) \right\} \tag{7b}$$

$$u_\theta(t) = \frac{a}{\sqrt{48}\rho} \cdot \frac{\Theta_1 \cdot \Theta_3}{\Theta_3 - \Theta_1} \cdot \left\{ \cos(\omega_1 t) - \cos(\omega_3 t) \right\} \tag{7c}$$

Figure 4 shows the rotational response $u_\theta(t)$ versus the longitudinal displacement response $u_y(t)$: this plot reveals that, consistently with the fast modulation of the rotational response above observed, for every longitudinal cycle of vibration, the rotational response reaches a value close to its maximum, $|u_\theta|_{\max}$. Furthermore, it can be observed that the rotational maxima are developed at instants of time when the longitudinal displacement response is close but not equal to its maximum, $|u_y|_{\max}$. This fact can also be explained through inspection of the Argand diagram representation of Eq.7b and 7c [9].

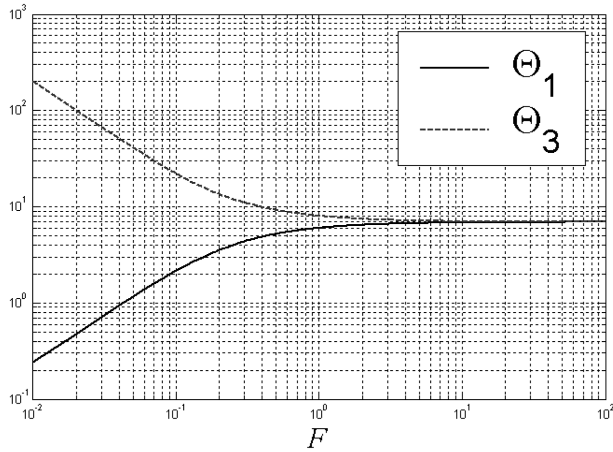


Figure 2: Θ_1 and Θ_3 as a function of *F*.

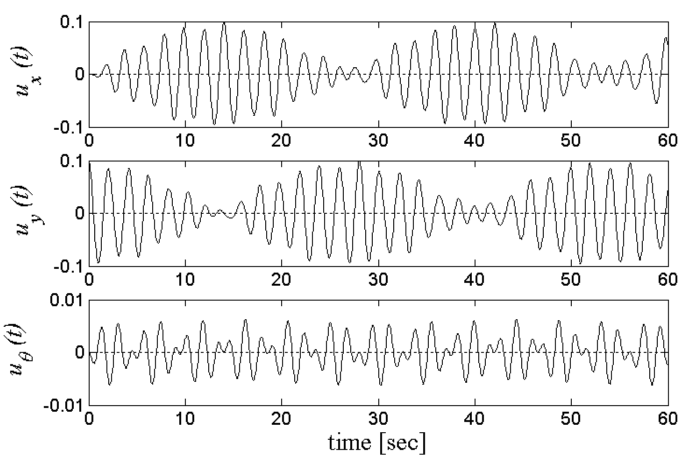
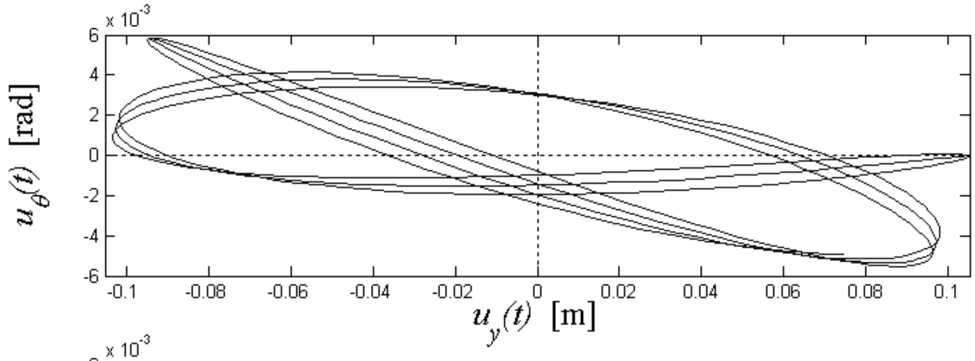


Figure 3: Free vibration response ($u_x(t)$, $u_y(t)$ and $u_\theta(t)$) of a system characterised by $e_y = e_x$, $e = 0.1$, $\gamma = 1.3$, $\omega_L = \pi$ rad/sec and $D_e = 28m$.



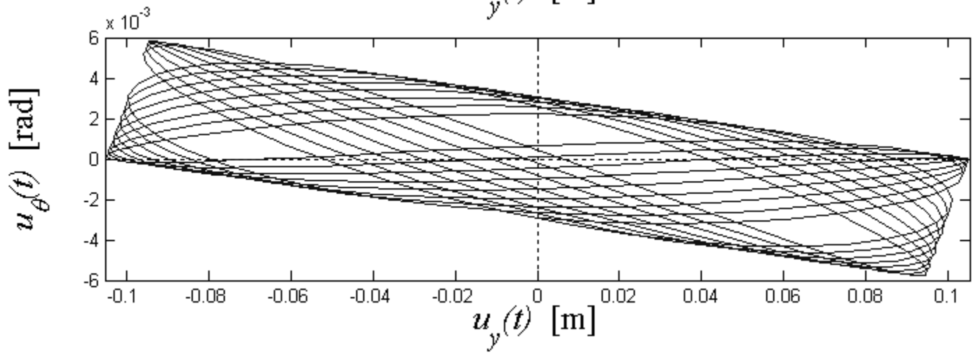


Figure 4: Time evolution of $u_\theta(t)$ versus $u_y(t)$ for an undamped eccentric structure characterised by $e_y = 0$, $D_e = 28m$, $\omega_L = \pi$ rad/sec, $a = 0.1m$, $\gamma = 1.3$, $e_x = 0.05$ (for $t = 10\sec$ and $t = 25\sec$).

MAXIMUM ROTATIONAL TO MAXIMUM LONGITUDINAL DISPLACEMENT RESPONSE RATIO OF UNDAMPED ECCENTRIC SYSTEMS IN FREE VIBRATION: THE "ALPHA" PARAMETER IN THE UNDAMPED CASE

The behavioural trend of the rotational response observed in the previous section suggests that the maximum rotational and maximum longitudinal displacement responses of eccentric systems might be strongly correlated, their ratio representing a basic property of eccentric systems, which could control their dynamic response also under general forced vibration conditions.

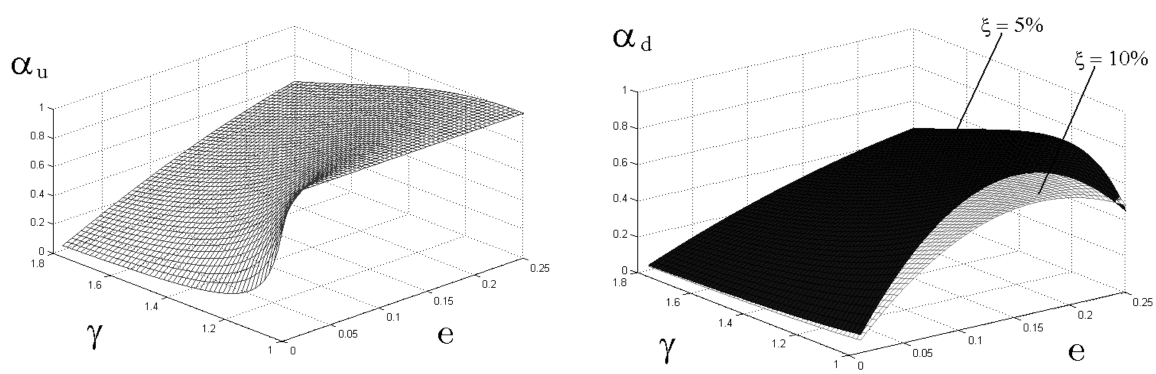
For undamped eccentric structures, in the special case of $e_y = 0$, the ratio $\left(\frac{|u_\theta|_{\max}}{|u_y|_{\max}}\right)_{free,u}$ can be expressed in closed-form from Eq.7a, Eq.7c, Eq.6 and Eq.5 as follows:

$$\left(\frac{|u_\theta|_{\max}}{|u_y|_{\max}}\right)_{free,u} = \frac{1}{\rho}\frac{\sqrt{48F^2}}{\sqrt{48F^2+1}} = \frac{4e\sqrt{3}}{\rho\sqrt{\left(\gamma^2-1\right)^2+48e^2}} \tag{8}$$

Eq.8 shows that $\left(\frac{|u_\theta|_{\max}}{|u_y|_{\max}}\right)_{free,u}$ is inversely proportional to the mass radius of gyration ρ of the structure; thus suggesting the definition of the following dimensionless rotational parameter:

$$\alpha_u \overset{def}{=} \rho \cdot \left(\frac{|u_\theta|_{\max}}{|u_y|_{\max}} \right)_{free,u} = \frac{\sqrt{48F^2}}{\sqrt{48F^2+1}} = \frac{4e\sqrt{3}}{\sqrt{\left(\gamma^2-1\right)^2+48e^2}} \tag{9}$$

As per Eq.9, the rotational parameter α_u (where the subscript *u* stands for "undamped") depends on the pair of system parameter *e* and γ or, even better, on the single structural parameter $F = e/\left(\gamma^2-1\right)$. Structures characterized by large values of α_u are prone to develop large rotational dynamic response, while structures with small values of α_u are less prone to rotate when responding dynamically. Figure 5a shows the graphical representation of α_u as a function of *e* and γ: it can be observed that for the same initial longitudinal displacement, eccentric systems can develop very different values of maximum rotational response depending on the system characteristics. In detail, Figure 5a indicates that the rotational parameter α_u increases for increasing values of *e* and decreasing values of γ. It can be seen that for low values of *e* and γ, α_u strongly increases for decreasing γ and for increasing *e*. Note that high values of γ (torsionally-rigid systems) characterize structures with sparse isolators mesh, e.g. with few isolators located on the perimeter. The small values of α typical of structures characterised by large γ indicate that, in analogy to a flexional resistant section (where it is worth to centrifugate the masses in order to obtain a major inertia), in base isolated buildings is better to centrifugate isolators in order to limit the rotations of the storey (compatibly with vertical loads). Furthermore, Eq.9 and Figure 5a indicate that the rotational parameter α_u is bounded between zero and one ($0 \le \alpha_u \le 1$), thus limiting the maximum rotational response $|u_\theta|_{\max}$ that can be developed in free vibration by any eccentric system to $\frac{|u_y|_{\max}}{\rho} = \frac{|u_y|_{\max}\sqrt{12}}{D_e} = \frac{a\sqrt{12}}{D_e}$. This is a fundamental result. Note also that for $\gamma = 1$, the maximum rotational response reaches this upper bound independently of the eccentricity *e*.



(a) (b)

Figure 5: (a) The α_u and (b) the α_d ($\xi = 5,\ 10\%$) parameters as a function of *e* and γ.

DAMPED FREE VIBRATIONS RESPONSE FROM A GIVEN TRASLATIONAL DISPLACEMENT

The free vibration response histories of classically damped eccentric systems from a given initial displacement *a* along the *y*-direction and assuming equal viscous damping ratios for each one of the three modes of vibration, i.e. $\xi_1 = \xi_2 = \xi_3 = \xi$, are given by:

$$u_x(t) = a\frac{e_x \cdot e_y}{e^2}\Lambda \cdot \exp(-\xi\omega_1 t)\left\{-\frac{\Theta_3}{\Theta_3 - \Theta_1}\cos(\omega_{D1}t+\theta) + \frac{e_y^2}{e_x^2}\cos(\omega_{D2}t+\theta)\exp(-\xi(\omega_2-\omega_1)t) - \frac{\Theta_1}{\Theta_1-\Theta_3}\cos(\omega_{D3}t+\theta)\exp(-\xi(\omega_3-\omega_1)t)\right\} \quad (10a)$$

$$u_y(t) = a\frac{e_x^2}{e^2}\Lambda \cdot \exp(-\xi\omega_1 t)\left\{\frac{\Theta_3}{\Theta_3 - \Theta_1}\cos(\omega_{D1}t+\theta) + \frac{e_y^2}{e_x^2}\cos(\omega_{D2}t+\theta)\exp(-\xi(\omega_2-\omega_1)t) + A_3\cos(\omega_{D3}t+\theta)\exp(-\xi(\omega_3-\omega_1)t)\right\} \quad (10b)$$

$$u_\theta(t) = \frac{a}{\sqrt{48}\rho}\frac{e_x}{e}\frac{\Theta_1 \cdot \Theta_3}{\Theta_3 - \Theta_1}\Lambda \cdot \exp(-\xi\omega_1 t)\left\{\cos(\omega_{D1}t+\theta) - \cos(\omega_{D3}t+\theta)\exp(-\xi(\omega_3-\omega_1)t)\right\} \quad (10c)$$

where $\omega_{Di} = \omega_i\sqrt{1-\xi_i^2}$ ($i = 1,2,3$) are the damped modal circular frequencies, $\Lambda = \left(1 + \frac{\xi}{\sqrt{1-\xi^2}}\right)^{1/2}$ and $\theta = -\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)$. The similar structure of the equations of motion for undamped and damped eccentric systems and response history simulation studies indicate that undamped and damped eccentric systems in free vibration follow similar behavioral patterns [8,9].

MAXIMUM ROTATIONAL TO MAXIMUM LONGITUDINAL DISPLACEMENT RESPONSE RATIO OF DAMPED ECCENTRIC SYSTEMS IN FREE VIBRATION: THE "ALPHA" PARAMETER IN THE DAMPED CASE

For damped eccentric structures, due to the exponential decay in time of the amplitude of the various harmonic components of the damped free vibration response (as given in Eq.10b and Eq.10c), it is not possible to obtain a simple exact closed-form expression for the maximum rotational to maximum longitudinal displacement response ratio (here referred to as α_d, where the subscript *d* stands for "damped") as done for the undamped case. However, an upper bound analysis yields the following result [9]:

$$\alpha_d \stackrel{def}{=} \rho \cdot \left(\frac{|u_\theta|_{\max}}{|u_y|_{\max}}\right)_{free,d} \le \frac{\Lambda}{2}\alpha_u \max\left\{1\ ,\ \exp\left(-\xi\frac{\pi}{2}\sqrt{\frac{\Omega_1}{\Omega_3}}\right) + \exp\left(-\xi\frac{\pi}{2}\right)\right\} \quad (11)$$

Due to the lack of an analytical expression for the α_d parameter, a number of α_d values were computed, for the special case of $e_y = 0$, through extensive numerical simulations carried out over a wide range of system parameter values, namely $0.02 \le e \le 0.22$, $1.05 \le \gamma \le 1.80$, $2\% \le \xi \le 12\%$. As expected, the maximum rotational response in free vibration due to a given initial displacement diminishes with increasing damping ratio. Least square fit method was then used in order to obtain the following analytical approximated expressions for α_d, also represented in Fig. 5b:

$$\alpha_d = -1.74 \cdot e + 15.71 \cdot \frac{e}{\gamma^2} - 51.17 \cdot \frac{e^2}{\gamma^4} \qquad \xi = 2\% \qquad (12a)$$

$$\alpha_d = -1.11 \cdot e + 12.55 \cdot \frac{e}{\gamma^2} - 39.18 \cdot \frac{e^2}{\gamma^4} \qquad \xi = 4\% \qquad (12b)$$

$$\alpha_d = -0.88 \cdot e + 11.30 \cdot \frac{e}{\gamma^2} - 34.58 \cdot \frac{e^2}{\gamma^4} \qquad \xi = 5\% \qquad (12c)$$

$$\alpha_d = -0.70 \cdot e + 10.25 \cdot \frac{e}{\gamma^2} - 30.70 \cdot \frac{e^2}{\gamma^4} \qquad \xi = 6\% \qquad (12d)$$

$$\alpha_d = -0.46 \cdot e + 8.64 \cdot \frac{e}{\gamma^2} - 24.95 \cdot \frac{e^2}{\gamma^4} \qquad \xi = 8\% \qquad (12e)$$

$$\alpha_d = -0.31 \cdot e + 7.45 \cdot \frac{e}{\gamma^2} - 20.95 \cdot \frac{e^2}{\gamma^4} \qquad \xi = 10\% \qquad (12f)$$

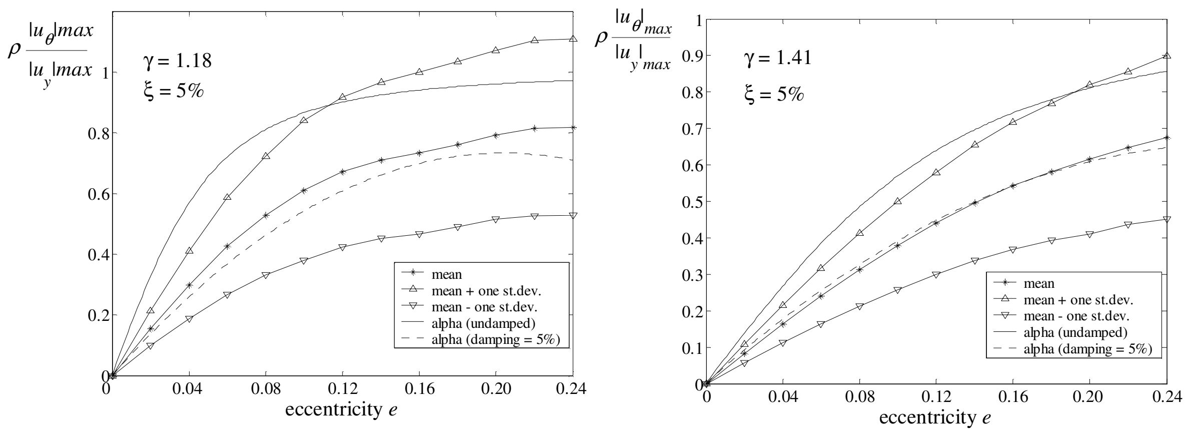
$$\alpha_d = -0.23 \cdot e + 6.59 \cdot \frac{e}{\gamma^2} - 18.19 \cdot \frac{e^2}{\gamma^4} \qquad \xi = 12\% \qquad (12g)$$

MAXIMUM ROTATIONAL TO MAXIMUM LONGITUDINAL DISPLACEMENT RESPONSE RATIO IN FORCED VIBRATION FOR LINEAR ELASTIC SYSTEMS

The interesting analytical (undamped case) and numerical (damped case) results presented in the previous section suggest to investigate the values taken by the $\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}}$ ratio under forced vibration conditions. With this aim, extensive numerical earthquake response simulations were performed for eccentric systems over a wide range of system parameters, namely $0.02 \le e \le 0.24$ (with $e_y = 0$, i.e. $e = e_x$), $1.05 \le \gamma \le 1.80$, $2\% \le \xi \le 12\%$, and using a set of 100 historical ground motion records as earthquake forcing functions. The seismic excitation is always supposed to be applied along the *y*-direction. The results are synthetically represented (in terms of mean, mean +/- one standard deviation) in Figures 6a and 6b for two structures respectively characterised by $\gamma = 1.18$ and $\gamma = 1.41$. These plots show that the response ratio $\left(\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}} \right)_{eqke}$ for earthquake excitation remains close to its counterpart for free vibration, i.e.:

$$\left(\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}} \right)_{eqke} \cong \alpha_d \qquad (13)$$

Moreover, it is observed that the rotational parameter α_u, available in closed-form from Eq.9 and function of only parameter *F*, appears to provide a good upper bound for the $\left(\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}}\right)_{eqke}$ response ratio induced by earthquake excitation.



(a) (b)

Figure 6: Values of $\left(\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}}\right)_{eqke}$ **as a function of the relative eccentricity** e **obtained from earthquake response simulations based on a set of 100 historical earthquake input records: (a)** $\gamma = 1.18$ **and (b)** $\gamma = 1.41$**.**

MAXIMUM ROTATIONAL TO MAXIMUM LONGITUDINAL DISPLACEMENT RESPONSE RATIO IN FORCED VIBRATION FOR NON-LINEAR SYSTEMS

In extending the research to the non-linear field [10], bilinear systems were here analysed. The eccentricities were computed with reference to the center of mass and the center of rigidity in linear elastic conditions. Systems characterized by transversal eccentricity e_x only are considered. The new parameters, which have to be introduced in the bilinear field, are [10]:

$\Psi = \frac{a}{b}$ — the shape factor of the building plan, where *a* and *b* are the side dimensions of the plan;

$Ip = \frac{\delta_{\max} - \delta_y}{\delta_y}$ — the plastic index, which indicates the level of the plastic excursion, where δ_y is the elastic limit displacement and $\delta_{\max}$ is the maximum displacement which is reached by the system;

T_L — the "effective" longitudinal period of the bilinear system, here assumed to be equal to 2 sec with $Ip = 5$;

$SHR = \frac{k_2}{k_1}$ the strain hardening ratio, between the plastic (k_2) and the elastic (k_1) stiffness, here assumed to be equal to 0.1.

To study the dynamic response of these systems, non-linear numerical simulations must be performed as no closed-form exact solutions for the $\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}}$ ratio are available. Furthermore, in this case the free vibration response from given initial displacement (in the following figures referred to as "linear and then constant force" response) looses the reference meaning seen for the elastic case and therefore no non-linear α can be defined. Nonetheless, the value of the ratio $\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}}$ for a number of dynamic inputs has been computed to investigate if α_u and α_d could still provide any useful information on the rotational response of non-linear eccentric systems. For this reason the following base dynamic inputs have been considered: (a) free vibrations from given initial displacement (simulated with a first linear and then constant force), (b) free vibration from a given initial velocity, (c) harmonic excitation, (d) white noise excitation and (e) earthquake excitation (average of 11 records). Figure 7 shows the results. Note that: (a) the values of the ratio obtained for seismic excitation and for harmonic loading provide an upper bound and (b) when the plastic index *Ip* reaches high values, all loadings here considered produce tightly bounded values of the $\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}}$ ratio.

Figure 8 shows that, for the case of $\gamma = 1.1832$ (similar results were obtained for other values of γ), the elastic value ($Ip = 0$) of $\left(\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}} \right)_{eqke}$ is almost always larger than any plastic one. This is a very important result as it conservatively allows a linear modelling of bilinear isolators. Only systems characterized by large γ values (rare isolators mesh) and very low *Ip* values (little plastic excursion) make exception to this rule [10]. Moreover, the curves confirm that plasticity plays the role of hysteretic damping as they become lower and lower for increasing values of *Ip* (compare Figure 8 with Figure 5b).

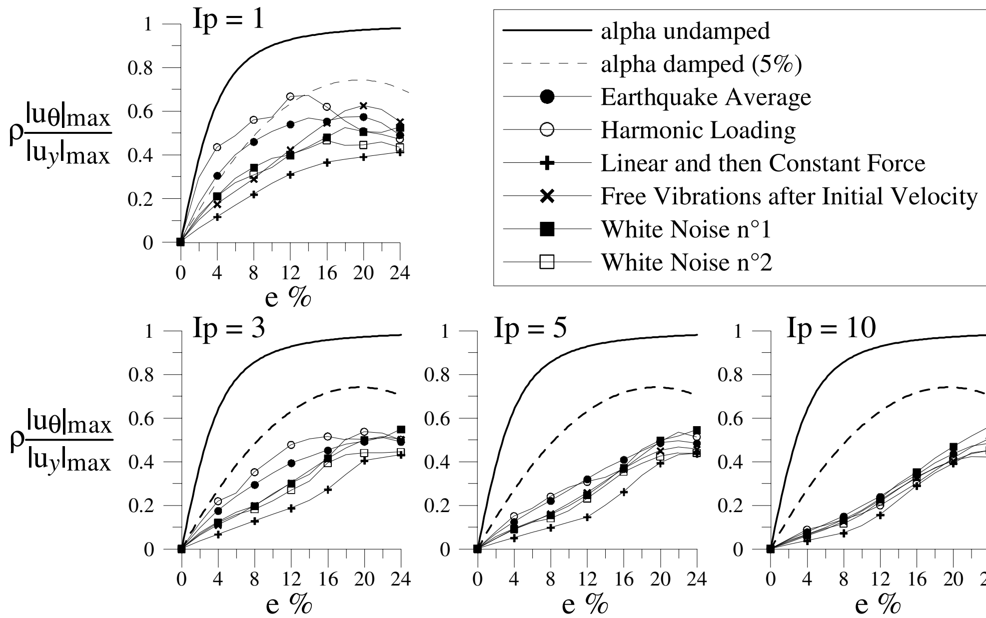


Figure 7: Responses under different base inputs for a square-shape structure with $\gamma = 1.1547$.

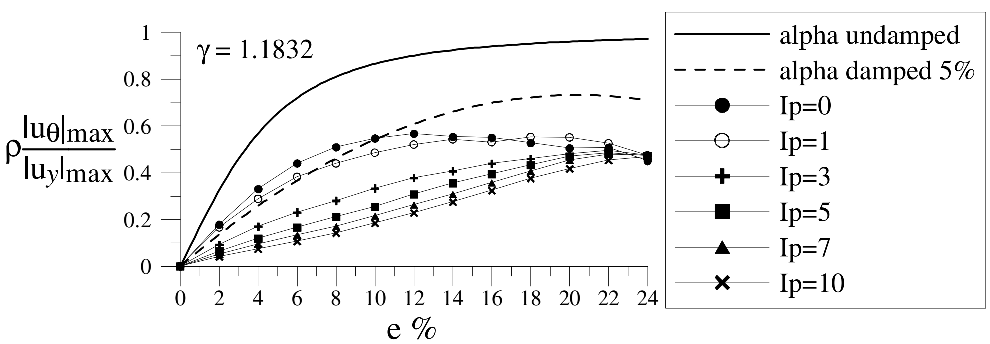


Figure 8: Plastic behavior of the $\left(\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}}\right)_{eqke}$ ratio for a square-shape structure with $\gamma = 1.1832$.

ALPHA METHOD FOR PREDICTION OF MAXIMUM ROTATIONAL RESPONSE OF ECCENTRIC SYSTEMS

The results presented in the previous sections strongly indicate that the response ratio $\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}}$ is a robust, low-variability response parameter, which is only weakly excitation dependent and therefore mainly system dependent. This fundamental property forms the basis of the here proposed simplified analysis procedure. The dimensionless response ratio $\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}}$, as obtained for free vibration conditions and named α_u and α_d for undamped and damped eccentric systems respectively, will be hereafter referred in general to as the α parameter. The limited difference between α and the corresponding value of the ratio $\rho \cdot \frac{|u_\theta|_{\max}}{|u_y|_{\max}}$ developed under forced vibration conditions suggests the following simple relationship between $|u_\theta|_{\max}$ and $|u_y|_{\max}$:

$$|u_\theta|_{\max} \cong \alpha \cdot \frac{|u_y|_{\max}}{\rho} \tag{14}$$

Other research works [5,6] show that the maximum longitudinal displacement response, $|u_y|_{\max}$, developed by an eccentric structure does not differ significantly from the maximum longitudinal displacement, $|u_y|_{\max-ne}$, developed by the "equivalent" SDOF oscillator, defined as the structure with equivalent dynamic characteristics (same mass and same longitudinal period), but with no eccentricity. In the case of earthquake excitation, $|u_y|_{\max-ne}$ can be readily obtained from a response spectrum. Thus, the approximation $|u_y|_{\max} \cong |u_y|_{\max-ne}$, together with Eq.14, gives the following useful formula for maximum rotational response prediction, which provides a powerful tool for simplified (code-like) analysis of the torsional response of eccentric structures:

$$|u_\theta|_{\max} \cong \alpha \cdot \frac{|u_y|_{\max-ne}}{\rho} \tag{15}$$

CONCLUSIONS

The analytical and numerical investigations presented in this paper provide (a) new insight into the understanding of dynamic lateral-torsional coupling in linear elastic one-storey *3*-DOF eccentric systems, (b) sensitivity of coupled lateral-torsional response to structural key system parameter and (c) a new, physically-based, simplified analysis procedure to predict the maximum rotational response of eccentric systems.

An important dimensionless response parameter, called the "alpha" parameter, is here identified as the product of the mass radius of gyration of the structure and the ratio between the maximum rotational and the maximum longitudinal displacement response developed by one-storey eccentric systems in free vibration, i.e. $\alpha = \rho \cdot \left(\frac{|u_\theta|_{\max}}{|u_y|_{\max}} \right)_{free}$. It is found that the α parameter depends on only two condensed system parameters, namely the relative eccentricity e of the system and $\gamma = \frac{\omega_\theta}{\omega_L}$. Structures characterized by large values of α are prone to develop large rotational dynamic response, while structures characterized by small values of α are less prone to rotate when responding dynamically. Thus, sensitivity analysis of the α parameter with respect to physical structural parameters is crucial in understanding the dynamic behaviour of laterally-torsionally coupled systems. The sensitivity analysis performed in this paper leads to the following conclusions:

1. α decreases for increasing values of the longitudinal eccentricity (in the direction of the dynamic forcing function). Thus, zero longitudinal eccentricity for a given transversal eccentricity gives the maximum rotational response.
2. α increases as γ tends to unity (from above as for most eccentric structures γ is larger than one).
3. α decreases slower with increasing value of the modal damping ratio than the maximum longitudinal displacement.
4. α is bounded by one from above, thus limiting the maximum rotational response developed in free vibration by any eccentric system to the maximum longitudinal displacement at the center of mass divided by the mass radius of gyration of the structure (i.e. $\left|u_\theta\right|_{\max} \leq \frac{\left|u_y\right|_{\max}}{\rho}$).

A compact exact closed-form expression for the α parameter is given for the undamped case (α_u), and approximate empirical analytical expressions based on a least squares fitting of numerical dynamic simulations data are provided for the damped case (α_d).

Furthermore, numerical results have shown that the α values obtained in free vibration are almost the same of those obtained in forced vibration. In fact, it is shown that the corresponding dimensionless parameter under forced vibration, $\rho \cdot \left(\frac{\left|u_\theta\right|_{\max}}{\left|u_y\right|_{\max}} \right)_{forced}$, is only weakly excitation dependent and therefore mainly system dependent. Therefore, the maximum rotational response of a given system due to any dynamic excitation can be predicted through a simple code-like formula, as $\left|u_\theta\right|_{\max} \cong \alpha \cdot \frac{\left|u_y\right|_{\max-ne}}{\rho}$ where $\left|u_y\right|_{\max-ne}$ is the maximum longitudinal displacement developed by the "equivalent" SDOF oscillator. The procedure for the prediction of the maximum rotational response of eccentric dynamic systems is here called "alpha method".

Furthermore, it has been shown that this approach allows a general comprehension of the torsional response behaviour of laterally-torsionally coupled dynamic systems both in elastic and in plastic range. It is found that the "elastic" value of α represents an upper bound for the $\rho \cdot \frac{\left|u_\theta\right|_{\max}}{\left|u_y\right|_{\max}}$ ratio of the corresponding bilinear system. Estimations of maximum rotation of an elastic-plastic system obtained through α are therefore conservative.

REFERENCES

1. Rutenberg, A. Behavior of irregular and complex structures: State-of-the-art report - seismic nonlinear response of code-designed asymmetric structures. EAEE Task Group (TG) 8, CD-ROM Proceedings of the Eleventh European Conference on Earthquake Engineering, A.A. Balkema, Rotterdam, 1998.
2. Hejal, R. & Chopra, A.K. 1987 *Earthquake response of torsionally coupled buildings.* Report UCB/EERC-87/20, Earthquake Engineering Research Center, Univesity of California, Berkeley, CA, Dec 1987.
3. Naeim, F., Kelly, J. M., *Design of Seismic Isolated Structures - From Theory to Practice*, John Wiley & Sons, 1999.

4. Goel, R.K. & Chopra, A.K. 1990. *Inelastic seismic response of one-storey, asymmetric-plan systems: effects of stiffness and strength distribution.* Earthquake Engng. Struct. Dyn., 19(3), 949-970.
5. Nagarajaiah, S., Reinhorn, A. M., and Constantinou, M. C. Torsional coupling in sliding isolated structures. *Journal of the Structural Division*, ASCE, 119 (1), 130-149, 1993.
6. Nagarajaiah, S., Reinhorn, A. M., and Constantinou, M. C. Torsion in base isolated structures with elastomeric isolation systems. *Journal of Structural Engineering*, ASCE, 119(10), 2932-2951, 1993.
7. Kelly, J. M. *Earthquake-Resistant Design with Rubber.* Springer Verlag, London, 1993.
8. Trombetti, T. Un approccio semplificato all'analisi dei problemi torsionali negli edifici isolati sismicamente alla base. *Giornale del Genio Civile*, 132, 10-12, 243-267, Oct.-Dec. 1994.
9. Trombetti, T. L. and Conte, J. P. New insight into and simplified approach to analysis of laterally-torsionally coupled one-storey systems: Part I. Formulation. Submitted for possible publication in *Earthquake Engineering and Structural Dynamics*, 2000.
10. Trombetti T., Ceccoli C., Silvestri S. 2001. A Simplified Approach to the Analysis of Torsional Problems in Seismic Base Isolated Structures. *First International Structural Engineering and Construction Conference (ISEC-01), Honolulu, Hawaii, January 2001.*