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# STABILITY RATIO AND DYNAMIC P-Δ EFFECTS IN INELASTIC EARTHQUAKE RESPONSES

**Shinji YAMAZAKI[1] and Kazuaki ENDO[2]**

## SUMMARY

Stability coefficient, which is obtained by dividing the equivalent stiffness drop due to P-Δ effects by elastic stiffness, is often used as an index for the P-Δ effects. Although this is an effective index for the behavior of elastic structures, it is not always valid for inelastic structures. In this study, the stability ratio is proposed as an index regarding the P-Δ effects in inelastic earthquake responses. The stability ratio is an index which is determined by both the characteristics of structures and the degree of seismic input energy. The validity of this index was verified based on response analyses.

## INTRODUCTION

Many studies on the P-Δ effects have been carried out. Bernal [1] obtained amplification factors to account for P-Δ effects in inelastic dynamic response as the ratio of inelastic acceleration response spectra generated with and without P-Δ effects included. Akiyama [2] derived an equation indicating the relationship between the yield shear coefficient in the case of considering the P-Δ effects and that in the case of not considering them for shear type multi-story frames. Uetani [3] theoretically clarified a concentrated phenomenon of deformation of multi-story frames with weak beams in their dynamic collapse.

In this paper, as an index for estimating the concentration of strain energy to one side and the increasing phenomenon of the ductility factor caused by the P-Δ effects, the stability ratio is introduced. The validity of the stability ratio, which is an index including the characteristics of a vibration system and the degree of seismic input energy, is verified by carrying out inelastic response analyses.

## STABILITY RATIO

A one-mass model shown in Fig.1 is considered. The load-displacement relation of the spring is a bi-linear type as shown in Fig.2. There is no decrease in the strength of the model and the plastic stiffness coefficient of $\alpha$ is set at 0 or more.

[1]Professor, Tokyo Metropolitan University, Tokyo, Japan. Email: yamazaki@arch.metro-u.ac.jp
[2]NIHON SEKKEI, INC., Tokyo, Japan.

The vibration equation with consideration to the P-Δ effects can be expressed using the following equation.

$$m\ddot{x} + c\dot{x} + Q - k_{P\Delta}x = -m\ddot{x}_0 \quad \cdots \quad (1)$$

Where $m$: mass, $c$: damping coefficient, $Q$: reaction of spring, $k_{P\Delta}$: equivalent stiffness drop caused by P-Δ effects, $\ddot{x}_0$: ground motion acceleration

$Q - k_{P\Delta}x$ is a restoring force of this system. The $k_{P\Delta}$ can be obtained from the following equation.

$$k_{P\Delta} = \frac{mg}{l} \quad \cdots \quad (2)$$

Where $g$: gravity acceleration, $l$: mass point height (See Fig.1)

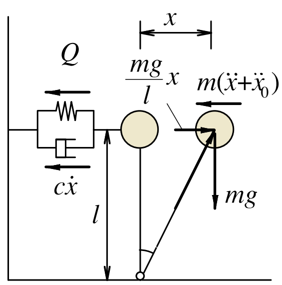



**Fig.1 One mass model**

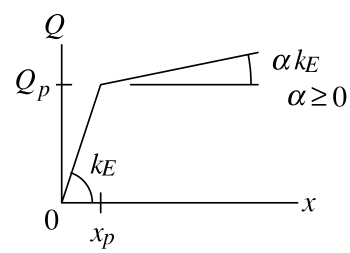



**Fig.2 Load-displacement relation of spring**

The stability coefficient $\theta$ which can be defined by the following equation is used as the index for the P-Δ effects.

$$\theta = \frac{k_{P\Delta}}{k_E} \quad \cdots \quad (3)$$

Where $k_E$: elastic stiffness of spring

The amount of energy input which contributes to damage is set as $E_D$. The $E_D$ is indicated using the cumulative plastic deformation ductility ratio of $\eta$ obtained from the following equation.

$$\eta = \frac{E_D}{Q_p x_p} \quad \cdots \quad (4)$$

Where $Q_p$: yield strength, $x_p$: yield displacement of spring (See Fig.2)

The vibration system is collapsed when its restoring force reaches 0. As one of the cases for collapse, a case, in which the plastic deformation increment accumulates completely in one direction and the restoring force becomes 0 by the end of an earthquake, is taken into consideration. Assuming that the broken line indicates the reaction force of the spring and the solid line indicates the restoring force of the system in Fig.3, in the above case the restoring force would reach point F in the figure. $\theta$ in this case is set as $\theta_{cr}$. Since the absorbed energy is in the area enclosed by the solid line and the axis of $x$, the value of $E_D$ in this case can be given by the following equation.

$$E_D = \frac{1}{2} Q_p x_p \frac{(1-\theta_{cr})(1-\alpha)}{\theta_{cr}-\alpha} \quad \cdots\cdots (5)$$

From equations (4) and (5), the following equation showing the relationship between $\theta_{cr}$ and $\eta$ can be derived.

$$\eta = \frac{(1-\theta_{cr})(1-\alpha)}{2(\theta_{cr}-\alpha)} \quad \cdots\cdots (6)$$

Solution of equation (6) for $\theta_{cr}$ results in the following equation.

$$\theta_{cr} = \frac{2\eta\alpha + 1 - \alpha}{2\eta + 1 - \alpha} \quad \cdots\cdots (7)$$

The stability ratio of $\tau$ is defined by

$$\tau = \frac{\theta}{\theta_{cr}} \quad \cdots\cdots (8)$$

When using equation (7), $\tau$ can be expressed by

$$\tau = \theta \frac{2\eta + 1 - \alpha}{2\eta\alpha + 1 - \alpha} \quad \cdots\cdots (9)$$

$\tau$ is an index which consists of the property of the vibration system ($\alpha$ and $\theta$) and the amount of input energy ($\eta$). Under the condition of the vibration system being fixed, $\tau$ is an index indicating the amount of input energy. When fixing the amount of input energy, it is an index indicating the equivalent stiffness for the P-Δ effects ($k_{P\Delta}$). In the case of $\tau$=1, the response does not necessarily reach point F in Fig.3. The response reaches point F only when the plastic deformation occurs completely on one side. In general, the value for the maximum displacement becomes less than $x_F$ in Fig.3.

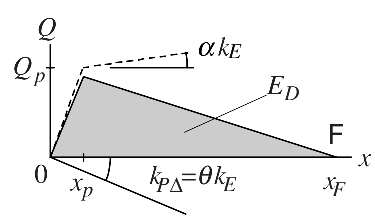



**Fig.3 Collapse condition**

The input energy of $E_D$ which contributes to damage can be indicated using the equivalent velocity $V_D$ defined by the following equation.

$$V_D = \sqrt{\frac{2E_D}{m}} \quad \cdots\cdots (10)$$

It is known that for one seismic motion $V_D$ is the value which is determined by a damping factor and a natural period only.

When using the relation of $Q_p = mgC_B$, the following equation can be derived from equations (4) and (10).

$$\eta = \frac{2\pi^2 V_D^2}{C_B^2 g^2 T^2} \quad \cdots\cdots \quad (11)$$

Where, $x_p = \frac{Q_p}{k_E}$, $k_E = \left(\frac{2\pi}{T}\right)^2 m$

$C_B$: base shear coefficient, $T$: natural period

Therefore, the stability ratio of $\tau$ can correspond to $V_D$.

## INELASTIC RESPONSE ANALYSES

### P-Δ Effects in Inelastic Responses

The input energy of a seismic motion is determined by mass and natural period and it does not vary when the restoring force characteristics are different. Therefore, the horizontal vibration induced input energy does not change due to the existence of the P-Δ effects. The P-Δ effects in the response causes the concentration of plastic energy in one direction, an increase in the ductility factor induced by this concentration and the possibility to be collapsed due to the loss of a restoring force.

When the value for $\tau$ is 1 or more, there is a possibility that the system is collapsed.

The relationship between the concentration of the plastic strain energy in one direction and $\tau$ and the relationship between the ductility factor and $\tau$ for the case of $0 \le \tau \le 1.0$ are investigated in this paper based on the numerical analyses.

### Analysis Conditions

It is set that the load-displacement relation of springs depends on the hysteresis law with consideration to the strain hardening effects of steel members (see Fig.4). In two regions (positive and negative) divided by axis $x$ which is the base line, the relation of $Q$-$x$ is independent. The curve obtained by translating the plastic increments of $Q$-$x$ in the hysteresis cycle on the positive or negative side parallel to the base line of axis $x$ and connecting the curves on the same side agrees to the relation of $Q$-$x$ under monotonic loading. The broken line in Fig.4 shows the relation of $Q$-$x$ under monotonic loading. C-D-E in the first cycle stage agrees to O-D'-E' in the relation of $Q$-$x$ under monotonic loading if it translates parallel to axis $x$. In the same manner as seen in the first cycle stage, F-G-H in the second cycle stage which translates parallel to axis $x$ agrees to C-B-H'.

The plastic stiffness coefficient of $\alpha$ (Fig.2) is set as $0 \le \alpha \le 0.1$.

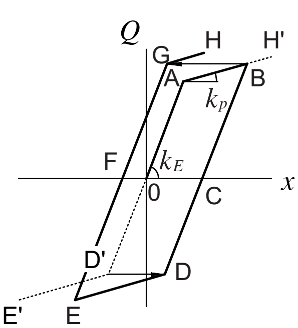



**Fig.4 Hysteresis model**

The natural period (in the case of $k_{P\Delta}$=0) is set at 1, 2, 3, 4 and 5sec, and the damping factor is 2%. As input seismic waves, 4 waves such as EL CENTRO 1940 (NS), TAFT 1952 (EW), HACHINOHE 1968 (NS) and JMA KOBE 1995 (NS) are used. The seismic wave is input so that the value for $\eta$ in the case of $\theta$=0 (namely: $k_{P\Delta}$=0) can be 10 and 20.

**Analysis Results**

Fig.5 shows the analysis results. Figs.5(1)~(4) show the results with regard to EL CENTRO, TAFT, HACHINOHE and JMA KOBE respectively. The abscissas in all figures show the stability ratio of $\tau$.

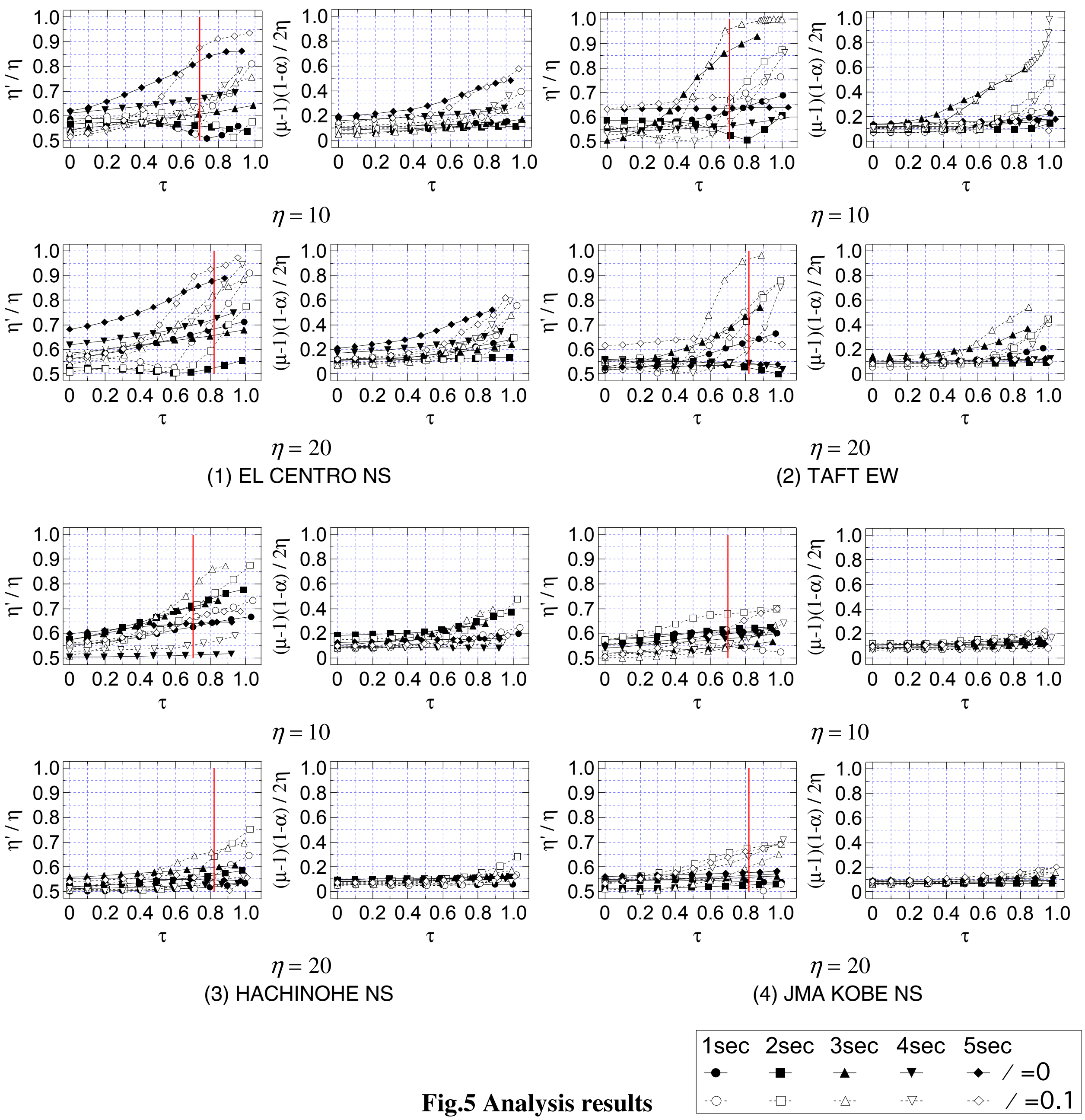



**Fig.5 Analysis results**

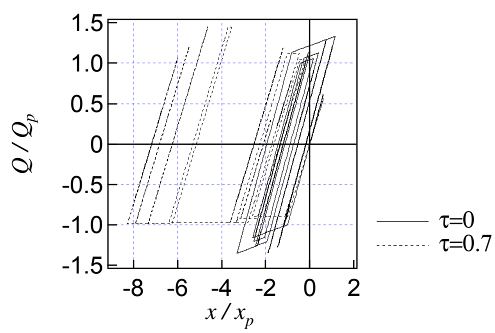
The upper figures of Fig.5(1) show the results obtained in the case of the value for $\eta$ being 10 and the lower figures show the results in the case of $\eta$ being 20. The left side of each figure indicates the deviation of energy and the right side indicates the ductility factor. The results shown for Figs.5(2)~(4) are the same as those for Figs.5(1). The analysis values are plotted in the cases of $\alpha$=0 and $\alpha$=0.1 for the natural period of 1 ~ 5 sec.

*Deviation of Strain Energy*

The values for the ordinate in the figure showing the deviation of energy are obtained by dividing the larger amount( $\eta'$ ) of strain energy absorbed in a positive or a negative sides by the total strain energy( $\eta$).

In the case of $\alpha$=0, the plastic stiffness of the restoring force characteristics for $\tau$>0 becomes negative. The vertical solid line in the figure shows the limit value for $\tau$ with which the plastic stiffness does not become negative in the case of $\alpha$=0.1. From the analysis results, it is made clear that the degree of the deviation of strain energy is not dominated by the sign of the plastic stiffness of the restoring force characteristics.

Fig.6 compares the response in the case of $\tau$=0 to that in the case of $\tau$=0.7 under the conditions of $\alpha$=0.1, 5 sec for the natural period, EL CENTRO for the input seismic wave and $\eta$=10. In both cases, the plastic stiffness is positive, but in the case of $\tau$=0.7, it is clear that the deviation of strain energy occurs.




$\alpha$=0.1, 5 sec, EL CENTRO, $\eta$=10

**Fig.6 Effects of $\alpha$ on inelastic response**

*Ductility Factor*

The following equation can be formed for the deformation of $x_F$ at point F in Fig.3.

$$\frac{x_F}{x_p} - 1 = \frac{1-\theta_{cr}}{\theta_{cr} - \alpha}$$

When substituting equation (7) for $\theta_{cr}$ in the above equation, the following equation can be derived.

$$\frac{x_F}{x_p} - 1 = \frac{2\eta}{1-\alpha}$$

When setting at $x_F / x_p = \mu_F$ (the ductility factor at point F in Fig.3), the above equation can be expressed as the following equation.

$$\frac{(\mu_F - 1)(1-\alpha)}{2\eta} = 1 \quad \cdots\cdots (12)$$

In the figures indicating the ductility factor, the ductility factor of $\mu$ is obtained from the response results and the values for $(\mu-1)(1-\alpha)/2\eta$ are plotted. In the case of the conditions for point F in Fig.3, the plotted ordinate values are 1.0.

*Summary of Analysis Results*
It is clear from Fig.5 that both the variation in the degree of the deviation of plastic strain energy and the variation in the ductility factor for the same seismic wave are generally equal even in cases where either the input amount or the value for $\alpha$ varies. Therefore, $\tau$ is an appropriate index which can indicate the P-Δ effects.

Fig.7 shows the results obtained by superposing the plots for the deviation of strain energy and the plots for the ductility factor indicated in Fig.5. The P-Δ effects induced deviation of strain energy and the variation in the ductility factor can be estimated on the conservative side using the following equations.

Deviation of strain energy

$$\frac{\eta'}{\eta}=0.3\tau+0.7 \cdots\cdots \quad (13)$$

Ductility factor

$$\frac{(\mu-1)(1-\alpha)}{2\eta}=0.8\tau^2+0.2 \cdots\cdots \quad (14)$$

Under the condition of $0\leq\alpha\leq0.1$.

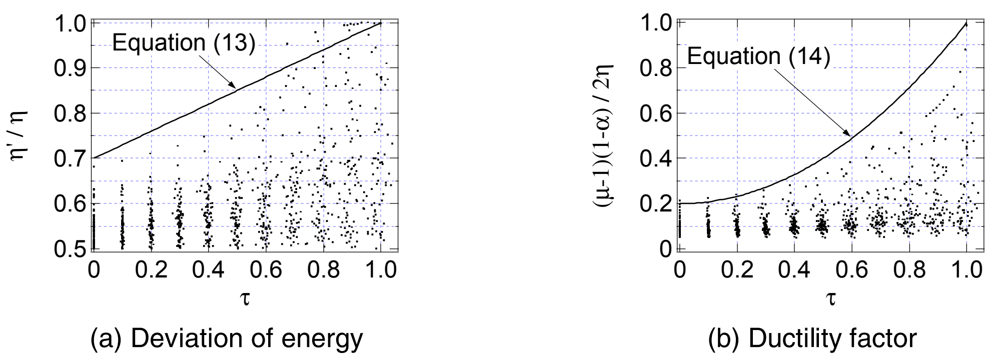



**Fig.7 Superposed plots of analysis results**

## COMPARISON TO BERNAL'S ESTIMTION EQUATION

Bernal obtains the yield shear force in which the ductility factor satisfies the target value for a one-mass model with elastic-perfectly-plastic restoring force characteristics, in both cases where the P-Δ effects are taken into account and they are ignored. The ratio of the yield shear force obtained in each case was defined as an amplification factor and an estimation equation for this factor was proposed.

In our study, the ductility factor with the consideration of the P-Δ effects when the yield strength is set to be constant is obtained. Therefore, the amplification factor that Bernal defines is derived using our results obtained above in order to compare the factor with Bernal's estimation equation.

Assuming that $\eta$ in the case of neglecting the P-Δ effects is set as $\eta_0$ and $\eta$ in the case of considering the

P-Δ effects is set as $\eta_{P\Delta}$, in both cases the target ductility factor is $\mu$, the following equations can be obtained from equations (14) and (9).

$$\frac{(\mu-1)(1-\alpha)}{2\eta_0}=0.2 \quad \cdots \quad (15)$$

$$\frac{(\mu-1)(1-\alpha)}{2\eta_{P\Delta}}=0.8\left\{\frac{\theta(2\eta_{P\Delta}+1-\alpha)}{2\eta_{P\Delta}\alpha+1-\alpha}\right\}^2+0.2 \quad \cdots \quad (16)$$

If the ductility factor of the response in the case of considering the P-Δ effects is equal to that in the case of neglecting the P-Δ effects, the maximum displacement are related to each other as shown with point A and point B in Fig.8. The input amount of $\eta_0$ and $\eta_{P\Delta}$ satisfies equations (15) and (16). Next, assuming that the ductility factor of the response in the case of considering the P-Δ effects for the same amount of input as $\eta_0$ is $\mu$ when the yield strength of this system is multiplied by $a_f$, the maximum displacement point is shown with C in Fig.8. The response values for both the displacement and the restoring force in this case are $a_f$ times the response values when the maximum response point is located at A in Fig.8. Therefore, the square root of the ratio of each strain energy is equal to $a_f$. Namely,

$$a_f=\sqrt{\frac{\eta_0}{\eta_{P\Delta}}} \quad \cdots \quad (17)$$

The relationship between $\theta$ and $a_f$ for a given value of $\mu$ can be obtained using equations (15), (16) and (17).

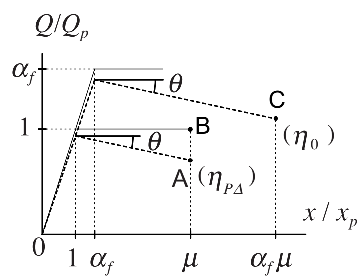



**Fig.8 *Q-x* relation**

Bernal proposes the following equation as an estimation of the amplification factor of $a_f$ based on the results of the earthquake response analyses.

$$a_f=\frac{1+\gamma\theta}{1-\theta} \;,\quad \gamma=1.87(\mu-1) \quad \cdots \quad (18)$$

Furthermore, Bernal sets the value of $\theta$ provided by the following equation as the limit value.

$$\theta=\frac{1}{\lambda\mu} \;,\quad \lambda=2.5 \quad \cdots \quad (19)$$

As for a region exceeding the limit value, the amplification factor is set under the condition that the ductility factor in the case of neglecting the P-Δ effects is $\mu$ and the ductility factor in the case of considering them is $1/(\lambda\theta)$. Therefore, the amplification factor in this study is obtained based on the same condition as above.

Fig.9 shows the amplification factor obtained from Bernal's estimation and from this study in the case of $\mu$=3~6. In the study by Bernal, only the case of $\alpha$=0 is subjected to the study, but in our study the results obtained in both cases of $\alpha$=0 and $\alpha$=0.1 are shown because the value for $\alpha$ is not necessarily 0 in our study.

Bernal's estimation for the amplification factor $a_f$ corresponds quite well to the results obtained in the case of $\alpha$=0 for our study. Moreover, it can be seen that the amplification factor in the case of $\alpha$>0 is smaller than that in the case of $\alpha$=0.

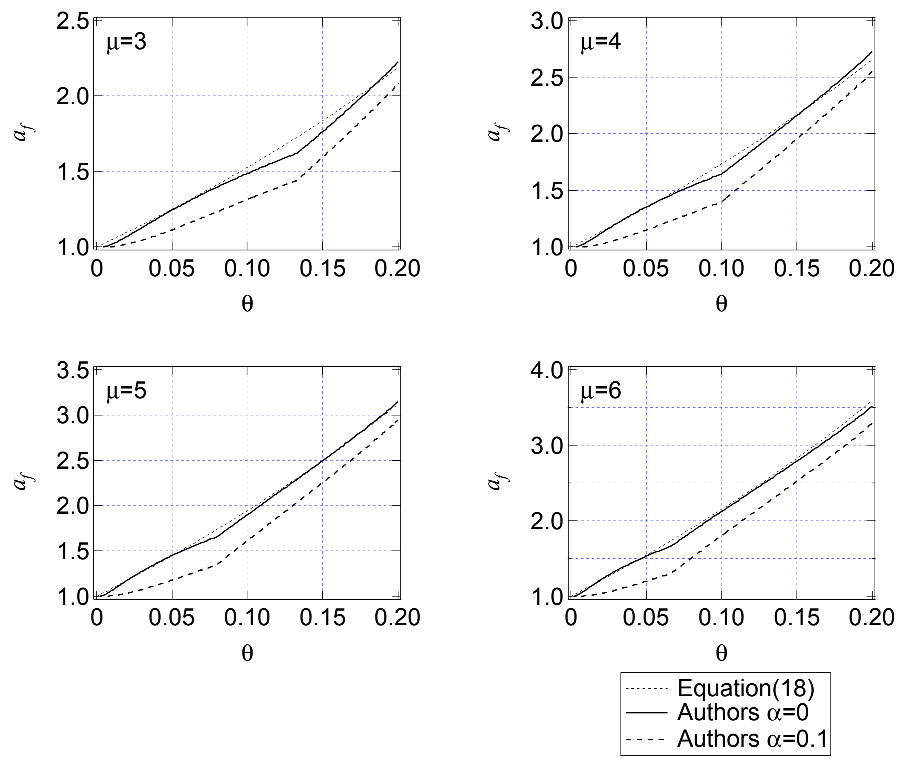



**Fig.9 $\theta$ versus $a_f$ relationship**

## CONCLUSIONS

As an index indicating the P-Δ effects in inelastic earthquake responses, the stability ratio of $\tau$ was introduced. The stability ratio is an index which indicates the characteristics of a vibration system and the amount of input energy. The P-Δ effects induced deviation of strain energy and variation of the ductility factor can be estimated using $\tau$.

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