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EFFICIENCY AND ACCURACY IMPROVEMENT OF INDIRECT BOUNDARY ELEMENT METHOD BY THE HIGHER ORDER BORN APPROXIMATION

Toshiaki Yokoi[1]

SUMMARY

The higher order Born approximation combined with the Indirect Boundary Element Method using the sparse matrix approximation and the conjugate gradient approach is proposed breaking through the difficulty that the boundary methods have. The sparse matrix approximation can be a countermeasure for the weakest point of the boundary methods, *i. e.*, requirement of huge memory allocation and unacceptable long CPU time consumption. The elimination of relatively small matrix elements, however, may result in a considerable distortion of calculated wave field. The higher order Born approximation is expected to be able to restore it. Therefore, their combination has a possibility to improve the efficiency and accuracy together for conventional IBEM.

It is shown that the third order Born approximation provides an enough accurate solution and at the same time reduces drastically the required CPU time and memory allocation in a case study and that the contribution of boundary elements nearer than the wavelength at the peak frequency have to be integrated. The calculated results show that the distortion of the main phases that correspond to the interaction between the boundary elements relatively near each other can be recovered. This implies that the refinement of synthetic seismograms may work better for the problems, in that surface waves and interface waves do not play important role. Possible example may be the body waves in relatively high frequency range that propagate through irregular interfaces, *e. g.*, acceleration observed on irregularly layered sediment that is important in Earthquake Engineering.

INTRODUCTION

For many years, researchers of boundary methods such as the Boundary Element Method and the Indirect Boundary Element Method (abbreviated IBEM, a detailed review is given by Yokoi [1]) have attempted to make them practically usable to compute complete wave fields for the problems of irregular interface and boundaries. Especially for the interest of Earthquake Engineering, the body waves coming up through irregular interfaces just beneath the observation point and the basin induced surface waves that come from

[1] Chief Research Scientist, International Institute of Seismology and Earthquake Engineering, Building Research Institute, Tsukuba, Japan, Email: tyokoi@kenken.go.jp

mountain-plain borders far away. Trying to handle both of them has made the requirement for the boundary methods unrealistic, because the body waves at the frequency range as high as the natural frequency of usual dwellings have to be calculated in huge sedimentary basins that corresponds a huge number of small boundary elements and huge Green's function matrices for simultaneous linear equations that require unacceptable long CPU time consumed. This is the reason why boundary methods are still in the way of research at present, while domain methods, such as the Finite Element (*e. g.*, Bielak [2]) and the Finite Difference Method (*e. g.* Graves [3] and Sato [4]) are being applied to realistic cases already. The boundary methods, however, are still attracting researchers, because of the possibility for substantial improvement in cost performance and accuracy, not only by advances in computer's hardware technology, but also by the wisdom and effort of the researchers.

POSSIBLE BREAK THROUGH

A possibility can be seen to break through this difficulty for the boundary methods in separation of wave types, *e. g.*, relatively large boundary elements required to cover a wide sedimentary basin for long period surface waves or relatively narrow spatial extent to be covered by small boundary elements for the body waves of short period.

Researchers of wave propagation in Earthquake Engineering have paid much attention on the former during 80's and 90's decade, because these are the hard evidences confronting to the myth in Earthquake Engineering, *i. e.*, the engineering bedrock that might provide laterally homogeneous input ground motion in regional extent to the shallowest sedimentary layers (*e. g.*, Bielak [2], Sato [4]).

The latter one has not been considered much, although it is widely recognized that the acceleration of ground motion at close epi-central distance is composed mainly of body waves and can be affected strongly by irregularities along the underground interfaces. 1D wave propagation in 1D velocity structure, *i. e.*, vertically upcoming S-wave in horizontally layered media has been and is being used popularly for the seismic response of the shallowest sediments over the engineering bedrock. For deeper underground velocity structure, 3D wave propagation in 1D structure is sometimes used with the velocity distribution beneath the observation point. They can provide the calculated results that are not bad except special extreme cases such as huge and deep sedimentary basin. Even these cases, the long period surface waves induced by basin do not affect to usual dwellings, because of the difference of frequency range. Therefore, it seems appropriate to concentrate attention to the directly coming body waves in consideration on seismic risk mitigation strategy. The next step for this approach may be to glade up the approximation for irregularly layered media.

Yokoi [5] shows the way, with IBEM formulation, how to handle the waves passing through irregular interfaces. It follows almost similar formalism as that for stacked layers with flat interfaces (*e. g.*, Kennett [6]). These body waves accompanied with scattered ones by irregularities along interfaces can be separated from the long period surface waves propagating from far a way. The performance of calculation, however, remains un-improved. A set of simultaneous linear equations have to be solved for wave passing through each interfaces.

NUMERICAL APPROXIMATION

The main difficulty still remains in huge coefficient matrices resulting from the simultaneous linear equations composed in order that the boundary conditions in discrete form are fulfilled. An extensive memory is required to handle and store them, and a considerable CPU time to solve the corresponding simultaneous linear equations.

Few mathematical ways are proposed to make that solution relatively easier, such as the bi-conjugate gradient approach widely used today because of their efficiency against the conventional Gauss – Jordan approach. The sparse matrix technique (*e. g.* Press [7]) handles matrices with many zero elements faster than other techniques. Unfortunately, coefficient matrices used in the boundary element method, however, are full ones. Therefore, more approximation is necessary to apply it for IBEM.

SPARSE MATRIX APPROXIMATION

It is a matter of course that the elimination of relatively small matrix elements can speed up the solution of simultaneous linear equations. An example is shown for a boundary integral scheme using the threshold criteria based on the significant spatial decay (~1/r) of Green's functions, especially in three dimensional problems (Bouchon [8]). Órtiz-Alemán [9] takes this approximation with the frequency into account, and shows a threshold criteria to truncate the contributions from relatively far elements, for IBEM. The number of matrix elements can be reduced significantly by this combination and consequently the required memory and CPU-time for the overall solution.

A careless application of this approximation, however, may result in considerable distortion of the calculated seismic wave field, because the small matrix elements correspond to the interaction between the boundary elements far of each other that sometimes play important roles. Although the sparse matrix approximation is attractive by its calculation performance, a way to compensate the distortion of wave field is required.

HIGHER ORDER BORN APPROXIMATION

Igel [10] shows that the higher order Born approximation explained in the following can improve efficiently the accuracy of the solution for problems of wave propagation in global scales, obtained by the Finite Difference Method.

The simultaneous linear equations

$$Ac = -g, \tag{1}$$

may be solved by the iterative algorithm,

$$\begin{aligned} &A^{(0)}c^{(0)} = -g, \\ &A^{(0)}\delta c^{(1)} = -\delta A c^{(0)}, \\ &A^{(0)}\delta c^{(2)} = -\delta A \delta c^{(1)}, \\ &\cdots \\ &A^{(0)}\delta c^{(n)} = -\delta A \delta c^{(n-1)}, \\ &\cdots, \end{aligned} \tag{2}$$

where $A^{(0)}$ is the principal part of A, δA the difference of $A^{(0)}$ from A, $c^{(0)}$ the zero-order approximation of the solution c, $\delta c^{(1)}$ the first order one, $\delta c^{(2)}$ the second order one, and so on (Takeuchi [11]) .

The sum of both members of above equations gives

$$\left(A^{(0)} + \delta A\right)\left(c^{(0)} + \delta c^{(1)} + \delta c^{(2)} + \cdots\right) = -g. \tag{3}$$

The stack of the solutions $c^{(0)}$ and $\delta c^{(k)}$ gives an iterative improvement of c, if it converges.

$$c = c^{(0)} + \delta c^{(1)} + \delta c^{(2)} + \cdots. \tag{4}$$

The condition for convergence given by Igel [10] is that the absolute values of every eigen values of the matrix $\left\{\left(A^{(0)}\right)^{-1}\delta A\right\}$ must be less than unity.

Considering its theoretical base, it seems possible to apply it to the IBEM in cases that the elimination of small elements may affect the calculated wave field. The coefficient matrix used by Igel [10], however, corresponds to the differential operator, while that of the IBEM corresponds to the integral operator.

VALIDATION

A numerical validation check using a three dimensional semi-spherical canyon is performed in order to quantify the accuracy and efficiency of the IBEM incorporating the higher order Born approximation in threshold criteria to eliminate small matrix elements.

The geometry of the surface model and the element distribution are shown in Fig.1. The non-dimensional units that scale everything to the semi-spherical canyon's radius that is unit are employed. The shear wave velocity and the density of the material are 1.0 km/sec and 1.0 gr/cm^3, respectively. Poisson ratio is 0.25, and the quality factor of P and S waves $Q_\alpha = Q_\beta = 100$, respectively. Then, the threshold criterion given by Órtiz-Alemán [9] is applied,

$$r \approx \pi\eta_{\max} / \varepsilon N_I \,, \tag{5}$$

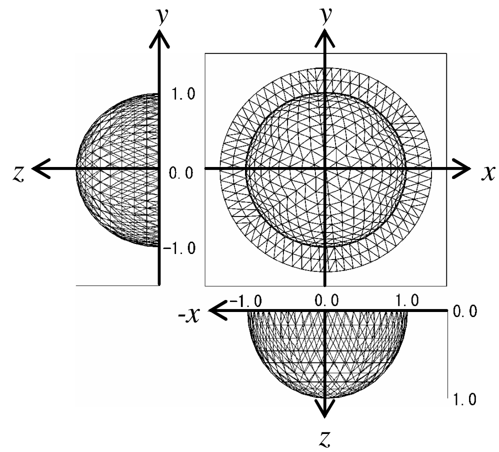

Fig.1 Boundary element model for the irregular surface S_I .

where $\eta_{\max} = \omega a / \pi\beta = 2a / \lambda_\beta$ is the normalized frequency, a radius of the canyon, λ_β the wavelength of the shear wave corresponding to the maximum frequency considered in the computation, N_I the total number of the boundary elements, β the shear wave velocity. The contributions of the boundary elements within this threshold distance are assembled in $A^{(0)}$, and those farther in δA. ε denotes the given control parameter for threshold.

The Ricker wavelet is used for the time dependency of the incident wave with the peak non-dimensional frequency $\eta_p = \omega_p a / \pi\beta = 2$. The incidence of plane S wave polarized in y-direction is of a null azimuth from the x-axis and incident angle 30 degrees from the vertical (positive down). Vertical components observed at stations along the line x=y are used for comparison, because this does not have the direct contribution of the incident wave and hence facilitates to check the accuracy of the solution.

The IBEM formulation is given as follows. The total wave field is composed of the reference wave field solution $(\bar{u}(x),\ \bar{t}(x))$ in a homogeneous half space V_0 upon the incident wave, and of the diffracted wave field given by the boundary integrals of the product of displacement and traction Green's function $g(x;\xi), h(x;\xi)$, and the imaginary force $\phi(\xi)$ distributed along the lower face of the free surface.

$$\begin{cases} u_i(x) = \bar{u}_i(x) + \int_{S_I} g(x;\xi)\phi(\xi)d\xi, \\ t_i(x) = \bar{t}_i(x) + \int_{S_I} h(x;\xi)\phi(\xi)d\xi, \end{cases} \quad x \in V. \tag{6}$$

The traction free boundary condition along the irregular surface S_I is given as

$$\bar{t}_i(x) + \int_{S_I} h(x;\xi)\phi(\xi)d\xi = 0,\ x \in S_I. \tag{7}$$

In discrete form, integral operator is substituted with matrix and the simultaneous linear equations to be solved are given as follows.

$$H_{II}\Phi = -\bar{t}_I. \tag{8}$$

The coefficient matrix H_{II} corresponds the original matrix A in Eq. (1). The higher order Born approximation (Eqs (2) and (3)) is applied to solve this simultaneous linear equations. The displacement at the surface is given by the following formula.

$$u_I = \bar{u}_I + G_{II}\Phi. \tag{9}$$

The formulation to calculate the elements of Green's function matrices is given by Sánchez-Sesma [12] and Yokoi [1] for three dimensional problems.

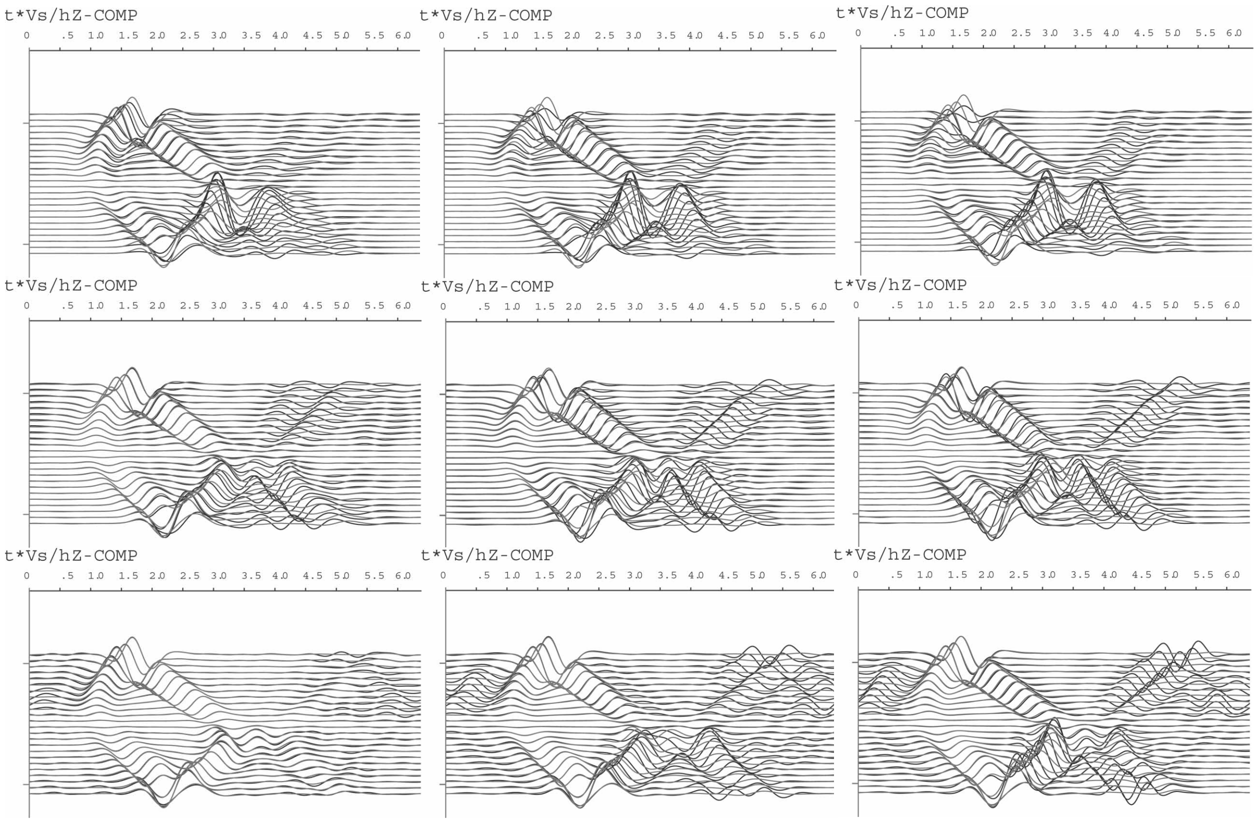

Fig. 2 Solution with elimination of small elements (*left*: *ε = 0.1*, *center*: *ε=0.2*, *right*: *ε=0.5*). *Top*: zero order, *middle*: first order, *bottom*: third order Born approximation applied. Vertical components observed along the line *x=y* are shown. Ticks on the vertical axis show the edge of the semi-spherical canyon. Grey traces denote "the exact solution", *i. e.*, the full matrix solution (*ε=0.0*).

Table 1 Performance of calculation

Value of ε	0.0	0.1			0.2			0.5		
Storage (%)	100	6.1			1.6			0.4		
Order	0	0	1	3	0	1	3	0	1	3
CPU time (%)	100	0.6	2.7	6.8	0.4	2.6	6.8	0.3	2.6	6.8
Error (%)	0.0	51.3	34.0	22.7	63.1	47.3	51.6	63.1	49.9	75.4

First, a calculation without elimination of small elements ($\varepsilon = 0.0$) is performed and the solution is used as "the exact solution", *i. e.*, the full matrix solution, for comparison with other cases. The full coefficient matrix A has 2592 X 2592 elements that correspond to 864 triangular boundary elements shown in Fig. 1.

Then, a series of calculation are performed using the zero, first and third order Born approximations, for the three values of $\varepsilon = 0.1, 0.2$ and 0.5. The ratio of the critical distance, within that the contribution of the boundary elements is taken into account, to average size of the boundary elements is *4.1*, *2.0* and *0.8*, respectively. In the extreme case of $\varepsilon = 0.5$, *e. g.*, only the contribution from the neighboring boundary elements is taken into account to compose $A^{(0)}$.

Fig. 2 show the vertical components of synthetic seismograms observed along the line $x=y$ that correspond to $\varepsilon = 0.1, 0.2$ and 0.5, respectively, in comparison with “the exact solution” (drawn in gray). The incident wave does not appear in the vertical component and the diffracted wave field obtained by IBEM is clearly seen in them. The all of three *top* panels clearly show the consequence of elimination of small elements. The discrepancy is considerable not only in later phases of small amplitude, but also around the main phases.
For all three values of ε, the waveforms due to the higher order Born approximation show the smaller deviations from "the exact solution", and these deviations appear enhanced later along the time axis. The small deviations before the first arrival are artifact due to FFT wrap around.

The quantitative estimation of performance is summarized in Table 1. The full matrix A is too big to fit the main memory of the computer, whereas its principal part $A^{(0)}$ can be stored in it. The matrix δA, however, has similar size as A and must be calculated in each iteration as well as A. The storage percentage shown in Table 1 is the ratio of the number of non zero elements in $A^{(0)}$ to the number of all elements of the original full matrix A. The error is the value biggest among the maximum values of the ratios calculated for each synthetic seismogram for whole trace, *i. e.*, the maximum absolute value of the deviation divided by the maximum absolute amplitude of "the exact solution", where the deviation is "the exact solution" minus the solution given by the Born approximation.

For the zero order approximation shown in the *top* panels, the errors are more than a half of the maximum absolute amplitude of "the exact solution" (Table 1). Therefore, these results are not acceptable at all, although the CPU-time consumption and storage are substantially reduced.

The discrepancy is suppressed more in the case of the higher order of Born approximation. The waveforms of the third order approximation for $\varepsilon = 0.1$ show the best fitting with "the exact solution" among the panels of Fig. 2. The error is 22.7% for whole traces, however, 9.0% for the time window from $t=0.5$ to $t=3.5$. This means that main phases are approximated well, as shown in the *bottom* panel. This tendency can be observed in the panels for $\varepsilon = 0.2$. The big false pulse that appears clearly at around $t=3.0$ in the *top* panel is suppressed in the *bottom* panel and the fitting of the main phases is bettered by the third order Born approximation, whereas the discrepancy in the latter phases remains. The traces for $\varepsilon = 0.5$, however, are not acceptable, because the above mentioned false pulse is kept in the traces obtained by the third order approximation.

This numerical example shows that the higher order Born approximation introduced for the simultaneous linear equations corresponding to differential operator (Igel [10]) is applicable also to those corresponding to the integral operators for IBEM. The distortion of calculated waveform can be restored by the third order Born approximation except extreme cases like $\varepsilon=0.5$.

DISCUSSION AND CONCLUSION

It is shown by a numerical example that the higher order Born approximation can improve the accuracy of the IBEM affected by the elimination of small matrix elements, but it can not work well in the case of an extreme elimination, such as $\varepsilon = 0.5$. The case of the best fit ($\varepsilon=0.1$, third order) corresponds to the ratio of the distance, within that the contribution of the boundary elements is taken into account, to average size of the boundary elements *4.1*. Considering the configuration of boundary elements in Fig. 1, this means that the contribution of the boundary elements within the distance approximately equal to the wavelength at the peak frequency is taken into account for composing $A^{(0)}$. The CPU time consumed and storage required are approximately 1 sixteenth of those necessary for the full matrix solution.

The performance of CPU-time consumption has a possibility for further improvement. In the case of a huge number of the boundary elements, the time to calculate δA is large, because of the huge size of δA. The strategy to reduce the number of cases, *i. e.*, the number of combinations between sources and receivers for calculation of Green's functions would be so-called Fast Multipole Method (Fujiwara [13]). The combination, IBEM - the higher order Born approximation - Fast Multipole Method would make the calculation of IBEM faster, and should be considered for future work.

The elimination of relatively small boundary elements means the truncation of the interaction between boundary elements relatively far each other, in term of wave propagation theory. Typical examples of this interaction are the surface waves secondary generated at the irregularity in topographic problems and the waves propagating along the buried interfaces in problems of irregularly layered media. The slow reduction of discrepancy in the latter phases clearly shown in Fig. 3 (*left* and *center*) supports this consideration. The distortion of the main phases that correspond to the interaction between the boundary elements relatively near each other can be recovered by the higher order Born approximation as well as shown in Fig. 3 (*left* and *center*), except such extreme case as shown in Fig. 3 (*right*). This implies that the refinement of synthetic seismograms by the higher order Born approximation may work better for the problems, in that surface waves and interface waves do not play important role. Possible example may be the body waves in relatively high frequency range that propagate through irregular interfaces and distorted by the irregularities, *e. g.*, acceleration ground motion observed on irregularly layered sediments, that is important in Earthquake Engineering.

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