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THE COMPLEX-COMPLETE-QUADRATIC-COMBINATION (CCQC) METHOD FOR SEISMIC RESPONSES OF NON-CLASSICALLY DAMPED LINEAR MDOF SYSTEM

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SUMMARY

In this paper, for non-classically damped linear multiple degrees of freedom system with complex modes, the CCQC method for calculation of maximum seismic responses of structures based on response spectra is deduced following similar procedures as the well-known CQC method, in which new modal displacement-velocity and velocity correlation coefficients are involved besides the modal displacement correlation coefficient in normal CQC formula. The new real value form formula of CCQC method not only is as concise as that of normal CQC method but also has explicit physical meaning. The results obtained from CCQC approach are discussed and verified in example through step by step integration computation under a prescribed earthquake motion input. From examplary analyses, it may be pointed that the CCQC algorithm normally gives conservative outcome and that the forced mode uncoupling approach has good approximation even the discussed examplary structures are strongly non-proportional.

INTRODUCTION

The response-spectrum mode superposition method is usually used to calculate dynamic responses of structures subjected to earthquake ground motion in seismic design code of many earthquake prone countries. For the classically damped linear system in which all the modes are real, the square root of the sum of squares (SRSS) method and complete quadratic combination (CQC) method of combining maximum modal responses (Kiureghian[1]) are widely used to determine the maximum seismic responses for structures. Generally speaking, SRSS method has good accuracy when the modal frequencies are well separated. However when the frequencies of major contributing modes are very close together as that normally emerged in three dimensional systems, this method will give poor results and a more accurate CQC method is proposed. For non-classically damped general linear system, motion equation under earthquake excitation can be solved by using decoupled method suggested by Foss [2], which was cited and improved by numerous writers (Harris[3], Igusa[4], Hurty[5]...). It is noted that in the case of non-classically damped linear system, which has complex eigenvalues and eigenvectors, the seismic responses depend on not only modal displacement response but also modal velocity response. Following similar

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procedures as deduction of CQC algorithm for classically damped linear system, a closed-up formula of response-spectrum complex mode superposition for calculation of maximum seismic effects at any position of a linear elastic structure with complex modal characteristics is deduced, in which a new modal velocity correlation coefficient, together with a new modal displacement-velocity correlation coefficient are involved besides the well-known modal displacement correlation coefficient in normal CQC algorithm. We call the new algorithm for calculation of maximum seismic effects of structures with non-classical damping complex complete quadratic combination (CCQC) method. If the correlations among modal responses are ignored, the CCQC method is reduced to corresponding complex square root of the sum of squares (CSRSS) method for non-classically damped linear system. This new CCQC algorithm is as concise as real CQC and is very adequate to code application. The CCQC and CSRSS methods are examined and compared in some numerical examples including time history analysis. In examplary analyses, the forced uncoupling approach, which approximately handles non-classically damped system through neglecting all the off-diagonal elements in transformed damping matrix and is popularly adopted in seismic design codes in many countries, is also used to analyze and is modified by introducing exact modal periods and damping ratios. Limited examplary analysis results show that this approximate approach has good accuracy.

COMPLEX MODE SUPERPOSITION METHOD OF STRUCTURAL SEISMIC RESPONSEES

For a discrete system, having N degrees of freedom, the equations of motion under the earthquake motion input are expressed as:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = -\mathbf{M}\{\mathbf{I}\}\ddot{y}_g(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $N \times N$ mass, damping and stiffness matrices, $\mathbf{y}(\mathbf{t})$ is $N \times 1$ nodal displacement vector which describes the dynamic response of the structure, N is an arbitrarily large integer, $\{\mathbf{I}\}$ is unit vector, and $\ddot{y}_g(t)$ is arbitrary time history of ground acceleration.

Eq. (1) can be rewritten into a group of linear differential equations of one order as follows:

$$\mathbf{R}\dot{\mathbf{x}} + \mathbf{S}\mathbf{x} = -\ddot{y}_g(t)\mathbf{R}\{\mathbf{E}\} \tag{2}$$

where

$$\mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \dot{\mathbf{y}} \\ \mathbf{y} \end{bmatrix}, \quad \{\mathbf{E}\} = \begin{bmatrix} \{\mathbf{I}\} \\ \{\mathbf{0}\} \end{bmatrix} \tag{3}$$

Substituting Eq. (3) into Eq. (2), it is easy to find that Eq. (2) is coincided with Eq. (1).
Suppose:

$$\mathbf{x} = \begin{bmatrix} \dot{\mathbf{y}} \\ \mathbf{y} \end{bmatrix} = \sum_{n=1}^{2N} \mathbf{\Phi}^n q_n(t) \tag{4}$$

in which, $\mathbf{\Phi}^n = \left[\mu_n \phi^n, \phi^n\right]^T$ is the generic eigenvector, ϕ^n and μ_n are the complex mode and complex eigenvalue respectively. $q_n(t)$ is the generalized coordinate. Because matrix $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric in general and so the eigenvalues and the eigenvectors obtained from the free vibration equation described by Eq. (2) normally occur in complex conjugate pairs, but for highly damped systems, an even number of them can be real (Inmam[6]).

Substituting Eq. (4) into Eq. (2), and pre-multiplying by $(\mathbf{\Phi}^n)^T$, following equation is obtained:

$$R_n \dot{q}_n(t) + S_n q_n(t) = -\ddot{y}_g(t)(\mathbf{\Phi}^n)^T \mathbf{R}\{\mathbf{E}\} \tag{5}$$

where $R_n = (\mathbf{\Phi}^n)^T \mathbf{R}\mathbf{\Phi}^n$, $S_n = (\mathbf{\Phi}^n)^T \mathbf{S}\mathbf{\Phi}^n$

Suppose the n-th mode participation coefficient is:

$$\eta_n = \frac{(\Phi^n)^T \mathbf{R}\{\mathbf{E}\}}{R_n} \tag{6}$$

then Eq. (5) can be rewritten as:

$$\dot{q}_n(t) + \mu_n q_n(t) = -\eta_n \ddot{y}_g(t) \tag{7}$$

Suppose:

$$\phi^n = \varphi^n + i\psi^n, \ \mu_n = -\alpha_n + i\beta_n \tag{8}$$

where $\alpha_n = \zeta_n \omega_n, \beta_n = \omega_{Dn} = \omega_n\sqrt{1-\zeta_n^2}$ are damping coefficient and damped frequency of the n-th mode respectively, the free vibration frequency ω_n and the corresponding critical damping ratio ζ_n can be deduced from the following formulas:

$$\omega_n^{\ 2} = \frac{(\bar{\phi}^n)^T \mathbf{K}\phi^n}{(\bar{\phi}^n)^T \mathbf{M}\phi^n}, \ 2\zeta_n\omega_n = \frac{(\bar{\phi}^n)^T \mathbf{C}\phi^n}{(\bar{\phi}^n)^T \mathbf{M}\phi^n} \tag{9}$$

where $\bar{\phi}^n$ and ϕ^n are a pair of conjugate complex modes.

Substituting Eq. (8) into Eq. (6) and separating the the right part of Eq. (6) into real and imaginary parts, the following formula is available after some simplification

$$\eta_n = \frac{1}{a_n^{\ 2} + b_n^{\ 2}}\left[a_n(\varphi^n)^T\mathbf{M}\{\mathbf{I}\} + b_n(\psi^n)^T\mathbf{M}\{\mathbf{I}\} + i\left(a_n(\psi^n)^T\mathbf{M}\{\mathbf{I}\} - b_n(\varphi^n)^T\mathbf{M}\{\mathbf{I}\}\right)\right] \tag{10}$$

where

$$a_n = -2\alpha_n((\varphi^n)^T\mathbf{M}\varphi^n - (\psi^n)^T\mathbf{M}\psi^n) - 4\beta_n(\varphi^n)^T\mathbf{M}\psi^n + (\varphi^n)^T\mathbf{C}\varphi^n - (\psi^n)^T\mathbf{C}\psi^n \tag{11}$$

$$b_n = 2\beta_n\left((\varphi^n)^T\mathbf{M}\varphi^n - (\psi^n)^T\mathbf{M}\psi^n\right) - 4\alpha_n(\varphi^n)^T\mathbf{M}\psi^n + 2(\varphi^n)^T\mathbf{C}\psi^n \tag{12}$$

Substituting proceeding η_n given by Eq. (10) into Eq. (7) and combining the terms consisted of a pair of conjugated complex modes, the displacement response superposition formulas in time domain induced by time history of ground acceleration, $\ddot{y}_g(t)$, is deduced as follows(Zhou[7]):

$$\mathbf{y}(\mathbf{t}) = \sum_{n=1}^{N}\left[A_n q_n(t) + B_n \dot{q}_n(t)\right] \tag{13}$$

where

$$\mathbf{y}(\mathbf{t}) = \{y(t)\} = [y_1, y_2, \cdots, y_n]^T, A_n = [A_1, A_2, \cdots, A_n]^T, B_n = [B_1, B_2, \cdots, B_n]^T$$

$$A_n = -\frac{2}{a_n^2 + b_n^2}[(\zeta_n p_n + \sqrt{1-\zeta_n^2}w_n)\varphi^n + (\zeta_n w_n - \sqrt{1-\zeta_n^2}p_n)\psi^n]\omega_n$$

$$B_n = -\frac{2}{a_n^2 + b_n^2}\left(p_n\varphi^n + w_n\psi^n\right)$$

where, $\mathbf{y}(t)$ is nodal displacement vector of the MDOF linear vibration system with non-classical damping, and $p_n = a_n c_n + b_n d_n$, $w_n = b_n c_n - a_n d_n$, $c_n = (\varphi^n)^T\mathbf{M}\{\mathbf{I}\}$, $d_n = (\psi^n)^T\mathbf{M}\{\mathbf{I}\}$**,** in which a_n and b_n are determined by Eqs. (11) and (12) respectively, and $q_n(t)$ is the solution of following equation.

$$\ddot{q}_n(t) + 2\zeta_n\omega_n\dot{q}_n(t) + \omega_n^2 q_n(t) = -\ddot{y}_g(t) \tag{14}$$

i.e.

$$q_n(t) = \frac{-1}{\omega_{Dn}}\int_0^t e^{-\zeta_n\omega_n(t-\tau)}\sin\left(\sqrt{1-\zeta_n^2}\,\omega_n(t-\tau)\right)\ddot{y}_g(\tau)d\tau \tag{15}$$

By using impulse response function, Eq. (15) also can be written as:

$$q_n(t) = \frac{-1}{\omega_{Dn}}\int_0^t h(t-\tau)\ddot{y}_g(\tau)d\tau \tag{16}$$

where

$$h_n(t) = e^{-\zeta_n \omega_n t} \sin(\sqrt{1-\zeta_n^2}\,\omega_n t)$$

What can be seen from above deduction is that even for earthquake response of general damped system with non-classical damping matrix, it can be completely decoupled and regarded as sum of earthquake responses of the N independent single degree of freedom oscillators subjected to the same ground motion according to the complex mode superposition analysis method. But in this case each seismic response is composed of two parts of displacement and velocity response for each SDOF oscillator, which is the different place that distinguishes the non-classically damped systems with complex modes from proportionally damped system. The closed-form Eq. (13) provides practical method for accurately calculating time history responses of non-classically damped system based on complex mode superposition principle.

CCQC METHOD FOR SEISMIC RESPONSES BASED ON RESPONSE SPECTRA

In order to employ the general time history superposition formula for the non-classically damped system with complex modes to determine the maximum response of structure, we have to deal with the random combination problem just like the classical mode superposition method for classically damped system. For the non-classically damped system with complex modes, the seismic modal response not only depends upon the displacement response of the separated oscillator but also relates to the corresponding velocity response that makes the mode superposition method more difficult than that of classical one. However the superposition method of earthquake response for non-classically damped system with complex modes is able to be deduced according to following steps.

The deviation or mean square response of $\mathbf{y}(\mathbf{t})$ descried by Eq. (13) becomes：

$$\begin{aligned} E\left[\mathbf{y}^2(t)\right] &= E\left[\sum_{n=1}^{N} A_n q_n(t) + B_n \dot{q}_n(t)\right]^2 = E\left[\sum_{n=1}^{N} A_n q_n(t) + \sum_{n=1}^{N} B_n \dot{q}_n(t)\right]^2 \\ &= \sum_{n=1}^{N}\sum_{m=1}^{N}\left[A_n A_m < q_n(t) q_m(t) > + B_n B_m < \dot{q}_n(t)\dot{q}_m(t) > + 2 B_n A_m < \dot{q}_n(t) q_m(t) >\right] \end{aligned} \tag{17}$$

Now let us calculate covariance $< q_n(t) q_m(t) >$ of the two separated modal response $q_n(t)$ and $q_m(t)$. Considering the expression given by Eq. (16) and noticing that the impulse response function is deterministic, hence we have:

$$< q_n(t) q_m(t) >= \left[\int_0^t \int_0^t h_n(t-\tau) h_m(t-s) < \ddot{y}_{gn}(\tau) \ddot{y}_{gm}(s) > d\tau ds\right] \tag{18}$$

where the symbol < > represents operation of calculation of average. In consideration of these relations and excitation $\ddot{y}_g(t)$ commencing at t=0, the low limit of the above integral can be extended to native infinity, i.e. we have

$$< q_n(t) q_m(t) >= \left[\int_{-\infty}^t \int_{-\infty}^t h_n(t-\tau) h_m(t-s) < \ddot{y}_{gn}(\tau) \ddot{y}_{gm}(s) > d\tau ds\right] \tag{19}$$

Assuming ground motion excitation, $\ddot{y}_g(t)$, involved in Eq. (14) is stationary white noise with zero mean value, hence we have $< \ddot{y}_{gn}(\tau)\ddot{y}_{gm}(s) >= 2\pi S_0 \delta(\tau - s)$, where S_0 is severity of the ground motion $\ddot{y}_g(t)$, $\delta(\tau - s)$ is *Dirac delta* function which gives 0 when $\tau \neq s$ and $\int_{-\infty}^{\infty} \delta(\tau - s) d\tau = 1$.

Using Fourier inverse transform formula of the *Dirac delta* function $\delta(\tau)$ and completing integration to s and τ and only retaining steady state terms when s and τ infinitively increase, the correlation coefficient of modes m and n becomes

$$R_{nm}^{DD}(\tau - s) = < q_n(\infty) q_m(\infty) > = S_0 \int_{-\infty}^{\infty} H_n(i\omega) H_m(-i\omega) e^{-i\omega(\tau - s)} d\omega \tag{20}$$

Let $\tau = s$ in Eq. (20), we get the stationary state covariance of modes m and n :

$$I_{nm}^{DD} = R_{nm}^{DD}(0) = S_0 \int_{-\infty}^{\infty} H_n(i\omega) H_m(-i\omega) d\omega \tag{21}$$

where

$$H_n(i\omega) = -\frac{1}{\omega_n^2 - \omega^2 + 2i\zeta_n \omega_n \omega} \tag{22}$$

And the covariance of displacement response produced by modes m and n can be obtained by using contour integration method or following the deduction procedures proposed by Elishakaff [8] based on partial fraction expression:

$$I_{nm}^{DD} = S_0 \int_{-\infty}^{\infty} H_n(i\omega) H_m(-i\omega) d\omega = \frac{\pi S_0}{2} \frac{1}{\omega_n \omega_m \sqrt{\omega_n \omega_m \zeta_n \zeta_m}} \rho_{nm}^{DD} \tag{23}$$

where

$$\rho_{nm}^{DD} = \frac{8\sqrt{\zeta_n \zeta_m}(r\zeta_n + \zeta_m) r^{3/2}}{(1-r^2)^2 + 4\zeta_n \zeta_m r(1+r^2) + 4(\zeta_n^2 + \zeta_m^2) r^2} \quad (r = \omega_n / \omega_m) \tag{24}$$

Meanwhile the covariance of velocity response, I_{nm}^{VV} , and the covariance of velocity-displacement response, I_{nm}^{VD}, produced by modes m and n can be deduced respectively by the same method as that of displacement response:

$$I_{nm}^{VV} = < \dot{q}_n(\infty) \dot{q}_m(\infty) > = S_0 \int_{-\infty}^{\infty} \omega^2 H_n(i\omega) H_m(-i\omega) d\omega = \frac{\pi S_0}{2} \frac{1}{\sqrt{\omega_n \omega_m \zeta_n \zeta_m}} \rho_{nm}^{VV} \tag{25}$$

where

$$\rho_{nm}^{VV} = \frac{8\sqrt{\zeta_n \zeta_m}(\zeta_n + r\zeta_m) r^{3/2}}{(1-r^2)^2 + 4\zeta_n \zeta_m r(1+r^2) + 4(\zeta_n^2 + \zeta_m^2) r^2} \tag{26}$$

and

$$I_{nm}^{VD} = < \dot{q}_n(\infty) q_m(\infty) > = S_0 \int_{-\infty}^{\infty} i\omega H_n(i\omega) H_m(-i\omega) d\omega = \frac{\pi S_0}{2} \frac{1}{\omega_m \sqrt{\omega_n \omega_m \zeta_n \zeta_m}} \rho_{nm}^{VD} \tag{27}$$

where

$$\rho_{nm}^{VD} = \frac{4\sqrt{\zeta_n \zeta_m}(1-r^2) r^{1/2}}{(1-r^2)^2 + 4\zeta_n \zeta_m r(1+r^2) + 4(\zeta_n^2 + \zeta_m^2) r^2} \tag{28}$$

It can be seen from Eqs. (24), (26) and (28) that three modal correlation coefficients, ρ_{nm}^{DD} , ρ_{nm}^{VV} and ρ_{nm}^{VD} , are fractional formals with common denominator. If the ratio $r = 1.0$ in Eqs. (24), (26) and (28), that means, $\omega_n = \omega_m$, the self-correlation function of velocity and displacement response, I_{nn}^{VD} , is equal to 0, meanwhile the self-correlation functions of displacement response, I_{nn}^{DD} , and velocity response, I_{nn}^{VV} , can be deduced as follows:

$$I_{nn}^{DD} = R_{nn}^{DD}(0) = < q_n^2(\infty) > = S_0 \int_{-\infty}^{\infty} \left| H_n(i\omega) \right|^2 d\omega = \frac{\pi S_0}{2} \frac{1}{\zeta_n \omega_n^3} \tag{29}$$

and

$$I_{nn}^{VV} = R_{nn}^{VV}(0) = < \dot{q}_n^2(\infty) > = S_0 \int_{-\infty}^{\infty} \omega^2 \left| H_n(i\omega) \right|^2 d\omega = \frac{\pi S_0}{2} \frac{1}{\zeta_n \omega_n} \tag{30}$$

From Eqs. (29) and (30), the following relation is obtained:

$$I_{nn}^{VV} = \omega_n^2 I_{nn}^{DD} \tag{31}$$

It should be noted that the displacement correlation coefficient for different two modes m and n , ρ_{nm}^{DD} , is reported in references and even is used in seismic design code and regulation, but for that of ρ_{nm}^{VV} and ρ_{nm}^{VD} , they are newly deduced in this study.

Substituting Eqs. (23), (25) and (27) into Eq. (17) and taking into account relations shown by Eq. (31), we further get:

$$E[\mathbf{y}^2(t)] = \sum_{n=1}^{N}\sum_{m=1}^{N}[A_n A_m \rho_{nm}^{DD} + B_n B_m \omega_n \omega_m \rho_{nm}^{VV} + 2B_n A_m \omega_n \rho_{nm}^{VD}] < q_n(t)^2 >^{1/2} < q_m(t)^2 >^{1/2} \tag{32}$$

If we assume as usual that the maximum response $|y(t)|_{\max}$ is proportional to the root of the mean square response, the following closed-form formula of complex mode response spectrum superposition for calculation of maximum response of the non-classically damped system, i.e. the complex complete quadratic combination (CCQC) formula is deduced:

$$|y(t)|_{\max} = \left[\sum_{n=1}^{N}\sum_{m=1}^{N}\rho_{nm}^{DD}(A_n A_m + \lambda_{nm} B_n B_m \omega_n \omega_m + 2\gamma_{nm} B_n A_m \omega_n)|q_n(t)|_{\max}|q_m(t)|_{\max}\right]^{1/2} \tag{33}$$

where λ_{nm} and γ_{nm} are ratios of $\rho_{nm}^{VV}/\rho_{nm}^{DD}$ and $\rho_{nm}^{VD}/\rho_{nm}^{DD}$ respectively, that is:

$$\lambda_{nm} = \frac{\rho_{nm}^{VV}}{\rho_{nm}^{DD}} = \left(\frac{\zeta_n}{\zeta_m} + \frac{\omega_n}{\omega_m}\right)\bigg/\left(1 + \frac{\zeta_n \omega_n}{\zeta_m \omega_m}\right), \quad \gamma_{nm} = \frac{\rho_{nm}^{VD}}{\rho_{nm}^{DD}} = \left[1 - \left(\frac{\omega_n}{\omega_m}\right)^2\right]\bigg/\left[2\zeta_m \frac{\omega_n}{\omega_m}\left(\frac{\omega_n \zeta_n}{\omega_m \zeta_m} + 1\right)\right]$$

Fig. 1 describes the relations of λ_{nm} and ω_n/ω_m when damping ratio ζ_n/ζ_m is equal to 0.2, 0.25, 0.75, 1.0, 4/3, 2.0 and 5.0, meanwhile the changes of γ_{nm} when $\zeta_m = 0.05$ are shown in Fig. 2.

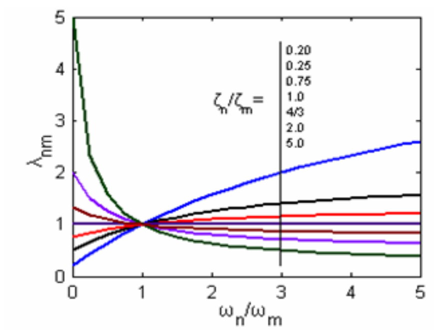

Fig. 1. The relations of λ_{nm} and ω_n/ω_m

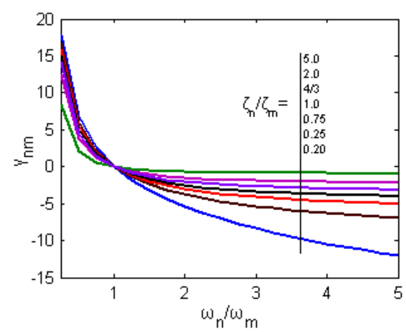

Fig. 2. The relations of γ_{nm} and ω_n/ω_m

If assuming $\rho_{nm}^{DD} = \rho_{nm}^{VV} = 0$ when $n \neq m$, complex squared root of the sum of squares (CSRSS) formula will be reduced

$$|y(t)|_{\max} = \left[\sum_{n=1}^{N}(A_n^2 + \omega_n^2 B_n^2)|q_n(t)|_{\max}|q_n(t)|_{\max}\right]^{1/2} \tag{34}$$

FORCED UNCOUPLING APPROACH OF NON-CLASSILLY DAMPED SYSTEM

Now using $\mathbf{Y}$, the mode shape matrix of corresponding non-damping or proportional damping system described by Eq. (1) to transform it to normal coordinates by pre-and post-multiplying by $\mathbf{Y}$ leads to following modal coordinate equations of motion:

$$\mathbf{M}^*\ddot{\mathbf{y}} + \mathbf{C}^*\dot{\mathbf{y}} + \mathbf{K}^*\mathbf{y} = \mathbf{P}(\mathbf{t}) \tag{35}$$

where $\mathbf{M}^*$ and $\mathbf{K}^*$ are the diagonal modal coordinate mass and stiffness matrices and $\mathbf{P}(\mathbf{t})$ is the standard modal coordinate load vector. However, the modal coordinate damping matrix

$$\mathbf{C}^* = \mathbf{Y}^T\mathbf{C}\mathbf{Y} \tag{36}$$

is not diagonal but includes non-zero modal coupling coefficients c_{ij}^* $(i \neq j)$ because the damping matrix $\mathbf{C}$ is non-proportional, that is

$$\mathbf{C}^* = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \cdots & \cdots & \cdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

The so called forced uncoupling approach may be attained by ignoring all the off-diagonal coupling coefficients of the modal damping matrix $\mathbf{C}^*$, let $\mathbf{C}_d^*$ replace $\mathbf{C}^*$, i.e.

$$C_d^* = \begin{bmatrix} c_{11} & & \\ & \ddots & \\ & & c_{nn} \end{bmatrix}$$

hence we have $\zeta_n = c_{nn}^* / 2\omega_n^* M_{nn}^*$, here $\omega_n^* = \sqrt{K_{nn}^* / M_{nn}^*}$ is the n -th frequency, and then to solve the resulting uncoupled equation as a normal mode superposition analysis (Clough[9]). Now back transforming $\mathbf{C}_d^*$ to original displacement coordinates, a new equivalent matrix $\mathbf{C}'$ which usually is a full matrix is formed. The forced uncoupling method actually is using $\mathbf{C}'$ to replace $\mathbf{C}$. This approach, being as an empirical characteristic, has been widely used in seismic design codes for its simplicity. Thus it is utilized to calculate the approximate seismic responses of non-classically damped linear system in following example.

THE COMPATIBILITY OF MODE SUPERPOSITION METHOD OF SEISMIC RESPONSE FOR NON-CLASSICALLY AND CLASSICALLY DAMPED SYSTEMS

The mode superposition method of seismic response described in preceding paragraph not only adapts to calculate seismic response for classically damped system but also is compatible to classically damped system. In fact the classically damped system can be regarded as a particular case where the real part of the eigenvector equals to zero, that means $\varphi = 0$ in Eq. (8). In this case all the forms corresponding to real parts in Eq. (9) are vanished, and then we have

$$\omega_n^2 = \frac{(\psi^n)^T \mathbf{K} \psi^n}{(\psi^n)^T \mathbf{M} \psi^n}, \quad 2\zeta_n \omega_n = \frac{(\psi^n)^T \mathbf{C} \psi^n}{(\psi^n)^T \mathbf{M} \psi^n} \tag{37}$$

These formulas are known expressions of vibration characteristics for classically damped system， and ψ^n is nothing but mode shape of the corresponding un-damped system.

It can be seen that in this particular case

$$a_n = w_n = 0 \tag{38}$$

and

$$b_n = 2\omega_n \sqrt{1-\zeta_n^2} (\psi^n)^T \mathbf{M} \psi^n, \quad p_n = 2\omega_n \sqrt{1-\zeta_n^2} \left((\psi^n)^T \mathbf{M} \psi^n\right)\left((\psi^n)^T \mathbf{M}\{\mathbf{I}\}\right) \tag{39}$$

In consideration of the relations shown by Eqs. (38) and (39), and $\varphi = 0$ in this case, Eq. (13) is simplified as:

$$y(t) = \sum_{n=1}^{N} \frac{-(\psi^n)^T \mathbf{M}\{\mathbf{I}\}}{\omega_n \sqrt{1-\zeta_n^2} (\psi^n)^T \mathbf{M} \psi^n} \int_0^t \ddot{y}_g(t) e^{-\zeta_n \omega_n (t-\tau)} \sin\left(\sqrt{1-\zeta_n^2}\, \omega_n (t-\tau)\right) d\tau \tag{40}$$

This right is the well-known mode superposition formula of calculating seismic response for classically damped system and it has been widely used in seismic design codes.

COMPARISON AND ANALYSIS OF EXAMPLE

Numerical example has been conducted in order to examine the results getting from the proposed complex mode superposition analysis for non-classically damped systems and compare with that from step by step integration technology. All the computations in the listed example are carried out in platform of MATLAB.

The example is taken from appendix A of reference(Hanson[10]) ,which is a planar five-storey shear-type structure with constant mass and stiffness coefficients for each storey: $m = 900/386.4$ (kips-sec. squared per in), $k = 1000$ (kip per in). Suppose the mass and stiffness matrices are denoted by M and K respectively and the damping matrix C complies following Rayleigh rule, i.e. $C = \alpha M + \beta K$, where $\alpha = 0.1757$ (1/sec.), $\beta = 0.00173$ (sec), then the first two damping ratio will be 0.02 and 0.02 .

Now transform it to a non-proportionally damped system by equipping a supplemental damper on the first inter-storey, which results in abrupt changes both in the stiffness and damping. The corresponding changes to K and C matrices are such that:

$$K'(1,1) = 1.05K(1,1), C'(1,1) = 31.0C(1,1) .$$

It is worth pointing out that serial number of masses in this example is from bottom to top of the example building, differs from the original example in reference. In this paper, the EW component of the El-Centro earthquake acceleration recorded on May 18, 1940 earthquake in California, which contains energy over a broad range of frequencies, is used as a ground motion input. The acceleration and displacement response spectrum for different damping ratios which are calculated by Eq. (9) are illustrated in Fig. 3 and Fig. 4 respectively, in which corresponding natural periods are labeled by symbol '*' at response spectral curves. It should be noted that the spectral acceleration shown in Fig. 3 is pseudo-acceleration, which can not be replaced by absolute acceleration in case of large damping ratio, i.e. calculated by equation $SA = \omega^2 \times SD$, SD represents spectral displacement.

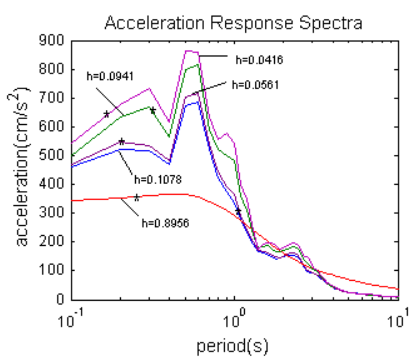

Fig. 3. Acceleration response spectra of complex mode responses

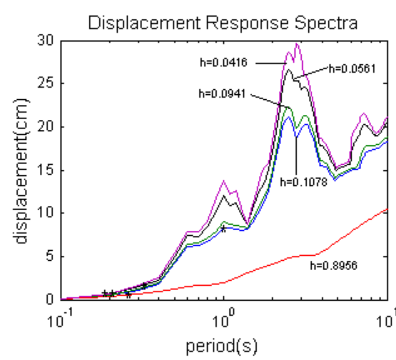

Fig. 4. Displacement response spectra of complex mode response

Via complex mode analysis procedure, the modal properties of the damper-added structure are obtained, as given in columns 2 and 4 in Table 1. It can be seen that there is a particularly high damping ratio for the third mode. Columns 3 and 5 of Table 1 show modal properties of the damper-added structure if forced uncoupling approach is used. These results illustrate that the errors of damping ratio coming from the simplifying calculation of non-proportional damping are significant in this case, particularly in the second mode and the third mode, in which the damping ratios are either greatly overestimated or underestimated respectively although their natural periods are approximate. Because forced uncoupling approach is used popularly in seismic design codes or regulations in many countries, we will give the corresponding results calculated from this simplified method in the example and modify it. Furthermore,

Fig. 5 shows the former four mode shapes including their real and imaginary parts for this example, in which the corresponding non-damping or proportionally damped modes are indicated by dotted lines which seem to be comparable to imaginary parts of modes of the non-proportionally damped system.

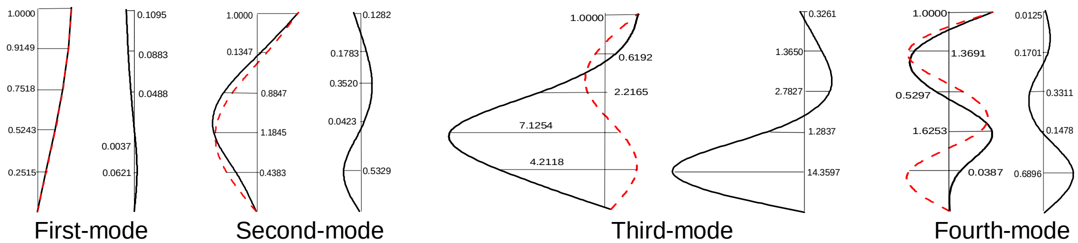

Fig. 5. The former four mode shapes. (the left is imaginary part of modes including proportionally damped modes indicated by dotted lines, the right is real part of modes)

Table 1. Modal properties of the non-proportional damping structure

Mode Number	Modal Periods (sec.)		Modal Damping Ratios (%)	
	Exact	Forced Uncoupling Method	Exact	Forced Uncoupling Method
1	1.0021	1.0481	10.78	12.45
2	0.3088	0.3599	9.41	28.90
3	0.2479	0.2292	89.56	34.21
4	0.1977	0.1793	5.61	25.89
5	0.1612	0.1578	4.16	11.19

To compare and analyze methods introduced in this paper, the following cases are considered.
Case A: Using complex mode superposition method to analyze seismic response of non-proportionally damping structure.
Case B: Using forced uncoupling method to analyze seismic response of non-proportionally damping structure.
Case C: Using modified forced uncoupling method to analyze seismic response of non-proportionally damping structure, that is, modal periods and damping ratios are determined by complex mode theory, whereas the mode shapes are determined from the un-damped structure as in Case B.

Table 2 presents the results obtained from Case A. The columns 2 and 3 of Table 2 are the results by using the complex mode superposition formulas Eq. (13) in time domain, and the direct numerical integration has been employed to characterize the dynamic behavior of the non-proportionally damped structural system accurately. The peak values of the storey displacements calculated from Newmark direct numerical integration are shown in columns 4 of Table 2. Obviously, the both computation results from complex mode superposition method and direct integration by Newmark *beta* algorithm are well coincided.

Furthermore, based on the response spectra shown in Fig. 4, the maximum storey displacements obtained according to CCQC formula (33) are listed in column 5 of Table 2. The calculation results obtained from CSRSS method are shown in columns 6 and 7 in Table 2. As an intuitionistic comparison, Fig. 6 shows the process of complex mode superposition, in which the bottom curve represents superposition result of all the upper mode responses at the same time coordinate and uses symbol ‘o’ to denote the maximum displacement of the first floor calculated by CCQC method, ‘+’ to present the maximum displacement of the first-floor calculated by CSRSS method. It can be seen Fig. 6 that composite peak value do not locate at the time points at which the first-mode response absolute maximum value occurs in this example even though the first mode plays the most important part.

It can be seen from above analysis that the results of CCQC and CSRSS methods, which are based complex mode theory, are all close to exact value for this relatively structure because effect of the first-mode is significant and the mode frequencies are separated, but the results of CCOC method are more accurate.

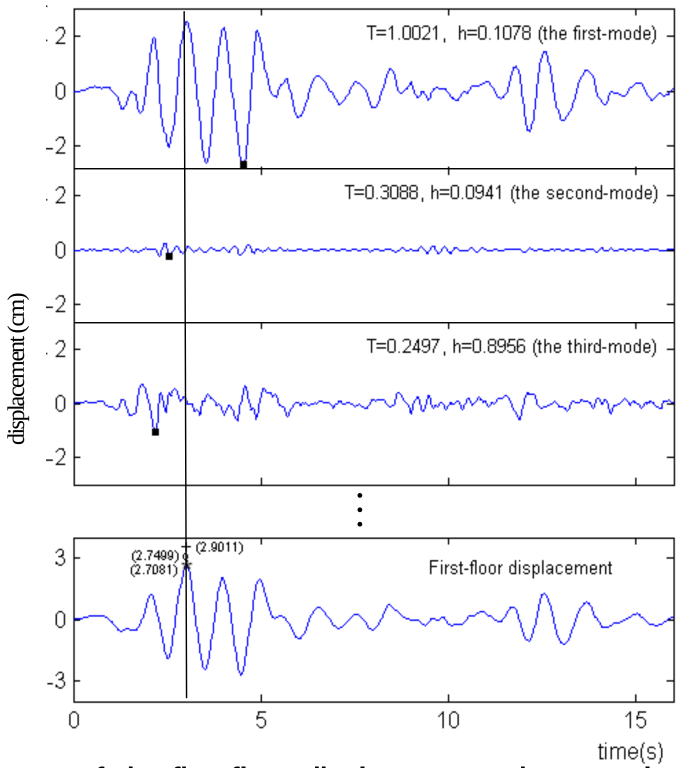

Fig. 6. Calculation process of the first-floor displacement using complex mode superposition method. ('▪'---peak value at individual mode response time history, '*'---the maximum displacement of the first-floor obtained from complex mode response time history superposition method, 'o'---the maximum displacement of the first-floor calculated by CCQC method, '+'--- the maximum displacement of the first-floor calculated by CSRSS method)

Table 2. Maximum storey displacement obtained from Case A (cm)

Storey	Using Formula (13)		Newmark-β	Using Formula (33) (CCQC Method)	Using Formula (34) (CSRSS Method)	
	Total-mode Incorporated	First-mode		Total-mode Incorporated	Total-mode Incorporated	First-mode
5	11.044	11.276	11.0439	11.2497	11.269	11.257
4	10.167	10.249	10.1670	10.2232	10.227	10.226
3	8.4863	8.2859	8.4863	8.3078	8.2871	8.267
2	5.9063	5.6139	5.9062	5.6978	5.6582	5.592
1	2.7081	2.8129	2.7081	2.7499	2.9011	2.6379

If forced uncoupling method, **Case B**, is adopted, the corresponding results calculated from this assumption will be changed and the results based on this simplified method are listed in Table 3, which includes the results of using mode response time history superposition method, CQC method and SRSS method. It can be figured out that the results coming from different approaches are close to the exact value even though many assumptions have been introduced. Fig. 7 gives the process of the corresponding mode superposition. It can be seen from Fig. 6 and Fig. 7 that the contributions of the second-mode and the third-mode are different.

The columns 5, 6 and 7 in Table 3 also give the results obtained from **Case C**, the modified forced uncoupling method , in which exact modal periods and damping ratios are adopted in forced uncoupling method. It can be seen the results are more conservative than that of **Case B**.

Table 3. Maximum storey displacement obtained from Case B and Case C (cm)

Storey	Forced uncoupling method (Case B)			Modified Forced uncoupling method (Case C)		
	MRTHS Method	CQC Method	SRSS Method	MRTHS Method	CQC Method	SRSS Method
5	9.6624	9.6076	9.6573	10.2591	10.3754	10.3696
4	8.8347	8.8226	8.8402	9.4560	9.4775	9.4931
3	7.3101	7.3212	7.2975	7.9322	7.8189	7.8359
2	5.2301	5.2159	5.1576	5.7844	5.5563	5.5368
1	2.6306	2.6334	2.5731	3.0075	2.8020	2.7614

*MRTHS--- Mode Response Time History Superposition

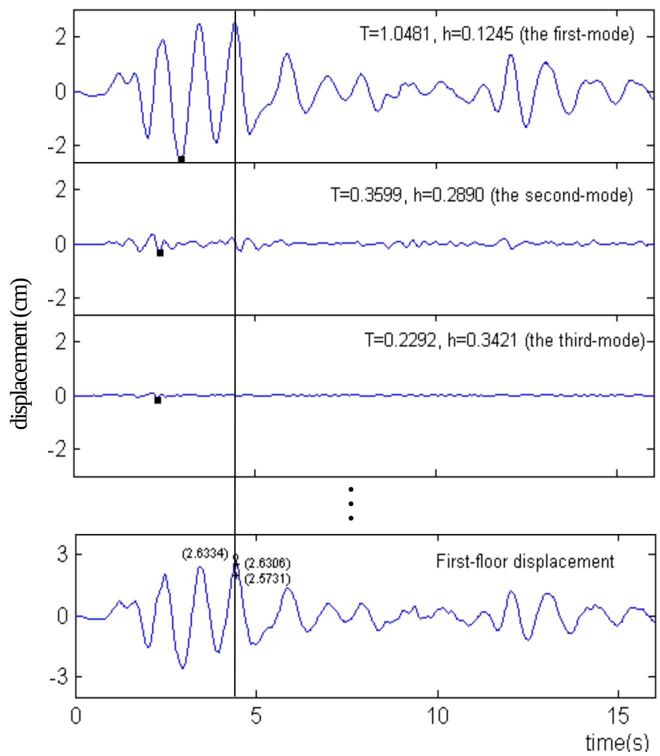

Fig. 7. Calculation process for the first-floor displacement of proportional damping assumed using mode superposition method. ('▪'---peak value at individual mode response time history, '*'--- the maximum displacement of the first-floor obtained from mode response time history superposition method, 'o'---the maximum displacement of the first-floor calculated by CQC method, '+'--- the maximum displacement of the first-floor calculated by SRSS method)

Comparisons between **Case A** and **Case B** for the first-floor displacement are pictured in Fig. 8; meanwhile Fig. 9 shows the displacement time history curves of **Case A** and **Case C**. It can be seen that the time history curve obtained from modified forced uncoupling method is close to exact values. Comparison of the errors between exact values and different approximate results for the first storey displacement are given in Table 4.

From this example, it can be seen that the complex mode superposition method for seismic response based on response spectrum, which has been suggested in this paper, possesses fairly good precision. Introducing the response spectrum, this method can be used to analyze complex structures added dampers. Furthermore, there is no obvious error involved in various simplified methods because the contribution of the first- mode is significant in modal combination process in this example.

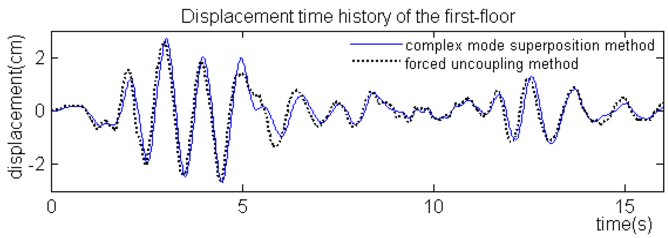

Fig. 8. First-floor displacement comparisons: complex mode superposition method, Case A and forced uncoupling method, Case B

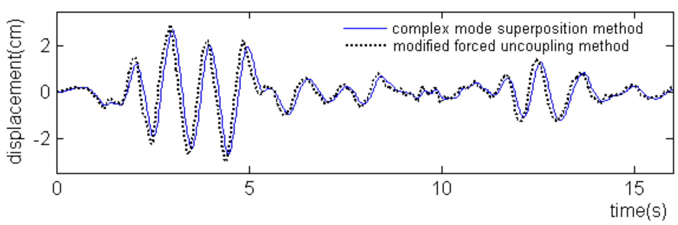

Fig. 9. First-floor displacement comparisons: complex mode superposition method, Case A and modified forced uncoupling method, Case C

Table 4. Errors of first inter-storey displacements in different analysis methods (%)

Case A		Case B			Case C		
CCQC Method	CSRSS Method	MRTHS Method	CQC Method	SRSS Method	MRTHS Method	CQC Method	SRSS Method
1.54	7.13	2.86	2.76	4.99	11.06	3.47	1.97

* MRTHS=Mode Response Time History Superposition

CONCLUSIONS AND CONCLUDING REMARKS

According to theoretical analysis and numerical examination in this paper, some important conclusions are obtained as follows:

1) A closed-form formula for calculation of seismic response of general damped linear vibration system with complex eigenvalues and eigenvectors in time domain has been deduced. It is an accurate solution in form of real values without imaginary terms appeared and is as compact and concise as that of proportionally damped system. Compared with step by step integration analysis, the proposed complex mode superposition algorithm is more effective and requires less computing time consumption.

2) Based on the proposed real form complex mode superposition formula for calculating seismic response of non-proportionally damped system in time domain, a new response spectrum complex complete quadratic combination (CCQC) algorithm is deduced. It is similar to normal CQC method for proportionally damped system, and the new modal velocity correlation coefficient, together with new displacement-velocity correlation coefficient have been involved. The proposed CCQC algorithm does not require any additional assumptions except what have been involved in normal CQC method. Hence it

is expected that the accuracy of CCQC algorithm is the same as normal CQC. In addition, the CCQC algorithm is nearly as concise as normal CQC method, thus can be grasped easily by engineers. If the correlations of displacement as well as velocity are neglected in CCQC algorithm, it can be automatically reduced to more compact form of CSRSS method which can be used to calculate maximum responses of non-proportional damped system with complex eigenvalues when the natural frequencies are separated far apart from each other. However, unlike the proposed complex mode superposition method in time domain, both CCQC and CSRSS methods are approximate computation formulas which come into existence only in probabilistic meaning under certain premises and inevitably comprise errors. The numerical example given in this paper show that CCQC are better than CSRSS, and normally gave conservative results.

3) It is interesting to point out that the results, calculated by forced uncoupling approach via neglecting all the off-diagonal elements in damping matrix in modal coordinate system, are fairly good in this example with strong non-proportional damping. The main reason is that the frequencies and contributions coming from various modes are turned to different levels, then keep away from such kind of unfavorable situation and thus get relatively good results in this example. It seems to illustrate that the forced uncoupling method proposed by many seismic codes is adoptive for relatively simple structures. Furthermore, the modified forced uncoupling method also gives good result. In order to improve accurate level for all the compared methods, doing further research seems to be necessary.

At the end of this paper it should be noted that the example discussed in this paper is relatively simple and only the storey displacements are analyzed and compared. For more complex structures, there are many generalized displacements and internal forces which would become important parameters. These parameters should be concerned in evaluating seismic safety and reliability. Further studies need to be done to analyze the accuracy and its sensibility to these generalized displacements and forces of complex structures subjected to earthquake excitation. Furthermore, the accuracy of all the response spectrum mode superposition algorithms including CQC, SRSS, CCQC and CSRSS depends on the types of input motion. This paper only considers EW component of El-Centro acceleration records being as excitation, so it should be careful when the conclusions getting from this paper are popularized into other types of input motion. In addition further theoretical study and examlpary analyses companying with different kind of input ground motion are valuable to check and verify the accuracy of the mentioned approximate analysis approaches in this paper.

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