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STATISTICS FOR EXTREME VALUES OF ONE DEGREE-OF-FREEDOM SYSTEM EXCITED BY NON-STATIONARY WHITE NOISE

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SUMMARY

The objective of this study is to estimate the statistics of the maximum values of the structural response when the system is excited by a non-stationary process. To provide basic information for the design of structures, the response of a one degree-of-freedom (1DOF) system subjected to a non-stationary Gaussian white noise is considered. For this purpose, we derive the probabilistic properties of the maximum values of non-stationary and zero-mean white noise, whose standard deviation varies temporally under some limited conditions. Then, using the obtained properties, a method is proposed to estimate the stochastic properties of the maximum response of a given 1DOF system excited by a non-stationary white noise whose time-varying standard deviation are known. Furthermore, the response spectra can be estimated easily using the proposed method and the appropriateness of the obtained results is confirmed through Monte Carlo simulation.

INTRODUCTION

The statistics for extreme values of non-stationary processes are critical to designing structures in some engineering fields, such as earthquake engineering, coastal engineering, and so on. Our objective is to estimate the stochastic properties of the extreme values of the structural response when the system is excited by a non-stationary process. In order to provide basic information for the design of structures, the response of a one degree-of-freedom (1DOF) system subjected to a non-stationary Gaussian white noise is considered. Although a significant amount of research has been conducted with respect to this type of problem (for example, Vanmarcke [1], Kiureghian [2], Lin and Cai [3]), few studies address the probabilistic properties of the maximum response values under non-stationary excitation.

In the first half of this paper, we deal with the stochastic properties of the maximum values of non-stationary and zero-mean white noise, whose standard deviation varies temporally under some limited conditions. While we may deal with this problem in time domain, most research on this

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type of problem has been limited to stationary processes. Especially, in a case where a time series is stationary Gaussian white noise with zero mean, we can directly apply the asymptotic representation for the extreme values of i.i.d. (independent identically distributed) Gaussian variables. Thus, the closed form solutions are easily obtained. This asymptotic representation of extreme values was introduced to the engineering fields by Gumbel and most classic and basic formulation as known as Gumbel's distribution (for example, Gumbel [4] and Galambos [5]).

However, it is not easy to estimate the extreme values of non-stationary processes whose stochastic properties depend on time, because we have to deal with the i.n.n.i.d. (independent not necessarily identically distributed) random variables in a case of the simplest problem such as white noise. Although general representations for this type of problems can be obtained (Reiss [6], Ahsanullah and Nevzorov [7]), it is difficult to derive the closed form or asymptotic solutions for any specific distributions such as Gaussian distribution, etc. If such solutions are derived, their representations will be complicated and it is not suitable to apply them to the problems in the engineering fields.

The asymptotic representation of extreme values for i.i.d. random variable were derived on the basis of the ingenious ideas. As following this way, it is important to find an approximate representation for the extreme values of i.n.n.i.d. random variables using simple formulations. Unfortunately, nobody can propose appropriate representations for this type of problems, even though the problem is described for i.n.n.i.d. Gaussian variables. However, since mathematical theory guarantees that the probabilistic properties of an i.n.n.i.d. random variable can be replaced by ones of a proper independent identically distributed (i.i.d.) one, we propose a method to find the parameters of this i.i.d. random variable [6].

In the latter half of this paper, we discuss the maximum response of the 1DOF system. In the case where the damping ratio is small and the system is excited by stationary random process, the local maxima are random variables following a Rayleigh distribution with statistical parameters that can be derived analytically using the results by Cartwright and Longuet-Higgins [8]. This can be combined with the results from the first half of this paper and applied to the local maxima. Then, the statistics of the extreme values of the response are estimated under the assumption that the local maxima are independent. Furthermore, we will apply this to estimate the response spectrum and discuss its accuracy. The appropriateness of the analytical results will be confirmed by the Monte Carlo simulation (MCS).

Through this study, we will treat the stochastic properties with respect to the non-stationary Gaussian white noise as the simplest and most primary problem: i.n.n.i.d. Gaussian are used as the input motion to a 1DOF system. We also limit the property of non-stationarity to the simple case keeping the application to the earthquake ground motion in mind: specifically, the input motion is the discrete white noise whose mean is zero, and standard deviation depends on time. The standard deviation has one peak and predominates the peak value over the time. Hereafter, we call "white noise" instead of the "discrete band-limited white noise" for the simplicity.

PROBLEM SETTING AND OUTLINE OF THE ANALYSIS

We will deal with the approximate distribution, $F_Y(y)$ for maximum value of i.n.n.i.d. Gaussian variables X_i $(i = 1, 2, \ldots)$, as the simplest non-stationary process: that is,

$$X_i \equiv X(t_i) = \eta(t_i) \cdot W(t_i) \tag{1}$$

where, t_i stands for i-th discrete time, $W(t_i)$ for Gaussian white noise with zero mean and unit variance, and $\eta(t_i)$ for standard deviation which depends on time and varies smoothly with one

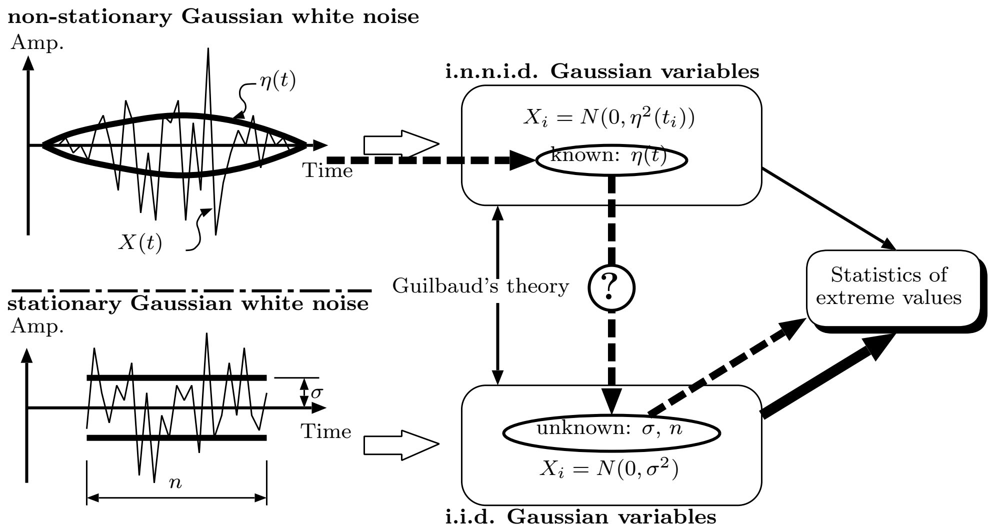

Figure 1 Schematic diagram to show the relationship between i.n.n.i.d. and i.i.d. Gaussian variables and concept to derive the statistics of extreme values of non-stationary Gaussian white noise.

extreme peak and $\eta(t; t \leq 0) = \eta(\infty) = 0$. The variations of $\eta(t_i)$ will be set much smaller than the time increments Δt. It is noted that $\eta(t_i)$ will play the role of a kind of envelop function of $X(t_i)$.

Generally speaking, the order statistics of i.n.n.i.d. random variables can be represented by using that of i.i.d. random variables because of the Guilbaud's theory (Reiss [6]). However, this theory does not give any information how we can find the appropriate i.i.d. random variable corresponding with i.n.n.i.d. random variables with a specific probability distribution of X_i. As shown as question mark, ⓘ in Figure 1, although the asymptotic distribution should be replaced by one of i.i.d. random variables in a case where the function $\eta(t)$, which is the probabilistic characteristics of i.n.n.i.d. random variable X_i, is given, there is no way to find parameters for the corresponding i.i.d. random variables.

To find the appropriate parameters for i.i.d. random variables are easier than to derive directly any asymptotic representation for the extreme values of i.n.n.i.d. random variables, because the asymptotic representation are already obtained for the extreme values for i.i.d. random variables.

From the above discussion, we will consider the approximate representations for the maximum values of non-stationary Gaussian white noise following the thick broken arrows in Figure 1. In this procedure, most significant problem is to represent the relationships between the parameters of i.i.d. and i.n.n.i.d. random variables. Thus, we will concentrate our concern into this problem, that is, to find the relationships as shown as ⓘ in Figure 1.

Next, we will discuss the stochastic characteristics of maximum response of a 1DOF system which are excited by Eq.(1). For this discussion, we deal with the relative displacement of the response given by Eq.(1) which is considered to be the acceleration. In a case where the damping factor, h is small, namely $h \ll 1$, the local maxima of the response are random variables following a Rayleigh distribution. Using this properties and the results from the discussion about the statistics of extreme for i.n.n.i.d. variables, we will derive the approximate representation for the probability density function (PDF) of maximum response of 1DOF system.

Some numerical examples are shown to confirm the analytical results. The values of parameters are not important in the calculations because our objective is to show the statistical representation of the extreme values. However, for easy understanding, the readers may imagine that the units for the time and frequency are "second" and "Hz," respectively, and that the response of the system may be "cm," if the unit of the left-hand side of Eq.(1) is "Gal."

The procedure of the analysis are summarized as follows:

(i) It is confirmed that we can replace i.n.n.i.d. variables with i.i.d. variables as the Guilbaud's theory says.

(ii) The asymptotic representation is derived analytically for the maximum values of i.n.n.i.d. Gaussian variables with two distributions: we consider a case where $\eta(t_i)$ takes only two values.

(iii) From the above analysis, we discuss the relationships between the parameters for two types of variables and determine the qualitative properties to examine the possibility of the approximate representation for maximum values of i.n.n.i.d. variables.

(iv) An approximate representation is derived for the statistics of extreme values of Eq.(1) using the properties obtained from the above discussion.

(v) It is confirmed that the PDF for local maxima of the response can be represented by the Rayleigh distribution.

(vi) Finally, we propose an approximate distribution of maximum response of 1DOF system and a method to estimate the response spectra.

To confirm the availability of the proposed method, we carry out some simple numerical calculations using the Monte Carlo simulation (MCS). The pseudo-random numbers are generated from the Mersenne Twister proposed by Matsumoto and Nishimura [9] and the Fortran code based on this method [10] are used. Since the generated random values follow the uniform distribution, they are transformed to Gaussian distribution if we need [11]. In the calculations, the time increments of the discrete time, $\Delta t = 0.01$ are used as small value regarding to the variance of $\eta(t)$ of Eq.(1).

DISTRIBUTION FOR EXTREME VALUES OF GAUSSIAN VARIABLES WITH TWO DIFFERENT PROPERTIES

Let us consider X_i $(i = 1, 2, \ldots, n_1 + n_2)$ which consists of n_j independent Gaussian variables, X_{jk} with zero mean and variance σ_j^2 $(j = 1, 2,\ k = 1, 2, \ldots, n_j)$: that is, X_{jk} is $N(0, \sigma_j^2)$ and X_i should be X_{1k} or X_{2k}. Since X_i $(i = 1, 2, \ldots, n_1 + n_2)$ are independent mutually, we can renumber X_i without loss of generality. Thus, let us set X_{1k} $(k = 1, \ldots, n_1)$ for X_i $(i = 1, \ldots, n_1)$ and X_{2k} $(k = 1, \ldots, n_2)$ for X_i $(i = n_1 + 1, \ldots, n_2)$.

Then, the probability distributions for maximum value Y_j of X_{jk} $(j = 1, 2)$ can be approximately written for large n_j as follows [7]:

$$F_{Y_j}(y) = P(X_{jk} < y) \approx \exp\left[-\exp[-\alpha_j(y - u_j)]\right], \tag{2}$$

where $P(A)$ denotes the probability of A, and

$$\alpha_j = \sqrt{2\ln n_j}/\sigma_j \tag{3a}$$

$$u_j = \left\{ \sqrt{2\ln n_j} - \frac{\ln(\ln n_j) + \ln(4\pi)}{2\sqrt{2\ln n_j}} \right\} \sigma_j. \tag{3b}$$

Thus, the probability distribution for maximum values of X_i is represented as

$$F_Y(y) = P(X_i < y) = \prod_{j=1}^{2} P(X_{jk} < y) = \prod_{j=1}^{2} F_{Y_j}(y) \approx \exp\left[-\sum_{j=1}^{2} \exp[-\alpha_j(y - u_j)] \right]. \tag{4}$$

Eq.(4) gives an approximate representation from the meaning of the asymptotic distribution for a special case of i.n.n.i.d. random variable with large n_j.

According to Guilbaud's theory [6], Eq.(4) can be represented by an asymptotic distribution for the maximum values of i.i.d. random variable: namely, Eq.(4) can be replaced by

$$F_Y(y) \approx \exp[-\exp[-\alpha(y - u)]]. \tag{5}$$

As pointed out already in the previous sections, this theory does not give any information about the relationships between α and u of Eq.(5), and α_j and u_j $(j = 1, 2)$ of Eq.(2). Thus, we will discuss how α and u can be represented by α_j and u_j $(j = 1, 2)$ in this section.

Calculating the Eq.(4) with various values of the parameters n_j and σ_j $(j = 1, 2)$ of Eqs.(3a) and (3b), we obtained the following relationships as $\sigma_1 \approx \sigma_2$ and $n_1 \approx n_2$ by trial and error:

$$\alpha \approx \sqrt{2\ln n}/\sigma \tag{6a}$$

$$u \approx \left\{ \sqrt{2\ln n} - \frac{\ln(\ln n) + \ln(4\pi)}{2\sqrt{2\ln n}} \right\} \sigma, \tag{6b}$$

where

$$n = \sum_{j=1}^{2} n_j \tag{7a}$$

$$\sigma = \frac{\sum_{j=1}^{2} n_j \sigma_j}{n}. \tag{7b}$$

Although this is not the mathematical consequence, these results may be expected instinctively under the above condition of σ_j and n_j. σ of Eq.(7b) is given by the weighted mean of σ_j with respect to n_j. Considering the general characteristics of the standard deviation, σ^2 should be represented by the weighted mean of σ_j^2 $(j = 1, 2)$ with respect to n_j^2, though Eq.(7b) gives good approximation as $\sigma_1 \approx \sigma_2$ as $n_1 \approx n_2$.

In other cases such as $\sigma_\ell \ll \sigma_j$ $(\ell, j = 1, 2,\ \ell \neq j)$ and $n_1 \approx n_2$, $\alpha \approx \alpha_j$ and $u \approx u_j$ can be applied. This means that the distribution function, $F_Y(y)$ for the maximum values of i.n.n.i.d. Gaussian variables, X_i is approximately rewritten by the distribution for the maximum values of i.i.d. Gaussian variables with larger values of σ_j. Furthermore, in a case where σ_j is more than

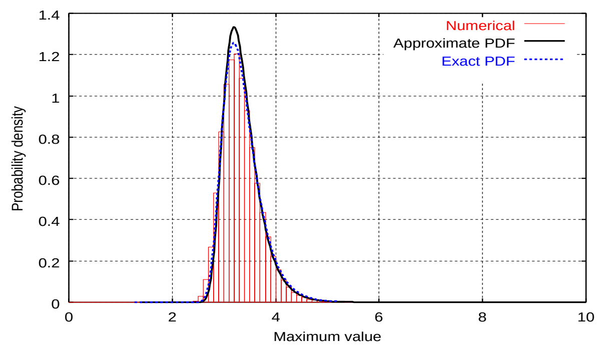

Figure 2 An example of distribution for extreme values of Gaussian white noise ($n_1 = n_2 = 500$, $\sigma_1 = 1.0$, $\sigma_2 = 1.05$).

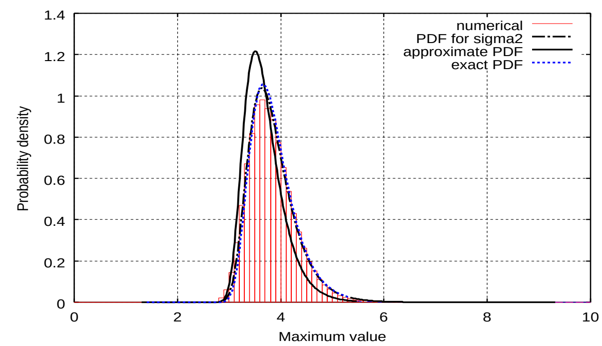

Figure 3 An example of distribution for extreme values of Gaussian white noise ($n_1 = n_2 = 500$, $\sigma_1 = 1.0$, $\sigma_2 = 1.25$).

only 1.1 to 1.2 times of σ_ℓ, the effect from the maximum values of X_i with σ_ℓ is negligible. Thus, it is enough to treat the two cases of $\sigma_1 \approx \sigma_2$ and $\sigma_\ell \ll \sigma_j$ as $n_1 \approx n_2$.

Figures 2 and 3 show the numerical examples of the approximation of probability distribution for maximum value, Y, of X_i as $n_1 = n_2$. In this figures, the histogram of Y, which are obtained from the MCS of 10000 times, is also shown. Figure 2 gives the result as $\sigma_1 \approx \sigma_2$ and it is observed that the shapes of Eq.(4) and Eq.(5) obtained by using Eqs.(6a) and (6b) coincide. On the other hand, Figure 3 shows the case for $\sigma_1 \ll \sigma_2$. In this case, Eq.(4) coincides with the distribution for the maximum value of X_i with σ_2: $F_{Y_2}(y)$. However, the distribution estimated by the weighted mean of σ_j $(j = 1, 2)$ fails to represent the histogram.

From the above numerical calculations, we can confirm the Guilbaud's theory: the probability distribution for maximum values of i.n.n.i.d Gaussian variables are replaced by one of i.i.d. Gaussian distribution. Furthermore, the results give an instructions how to determine the values of α and u of Eq.(5).

As a result, we can conclude the method to determine the parameters for substitute i.i.d. distribution as follows. In a case of $\sigma_1 \approx \sigma_2$, we can choose the value of σ to satisfy the equation

$$\sigma \left(\sum_{j=1}^{2} n_j \right) = \sum_{j=1}^{2} (n_j \sigma_j) . \tag{8}$$

This means that the area obtained by a rectangle σ long and $n = \sum_{j=1}^{2} n_j$ wide should be same as the total area from the rectangles n_j and σ_j $(j = 1, 2)$ as shown in Figure 4. On the other hand, in a case of $\sigma_\ell \ll \sigma_j$ $(\ell, j = 1, 2;\ \ell \neq j)$, we can use the probability distribution for the maximum value, Y_j, of i.i.d. Gaussian variable with $N(0, \sigma_j^2)$.

APPROXIMATE DISTRIBUTION FOR EXTREME VALUES OF NON-STATIONARY GAUSSIAN WHITE NOISE

Analytical Procedure

We discussed the relationships between the parameters of i.n.n.i.d. and i.i.d. Gaussian variable for a special case in the previous section. Then, we will apply the obtained properties to approximate the

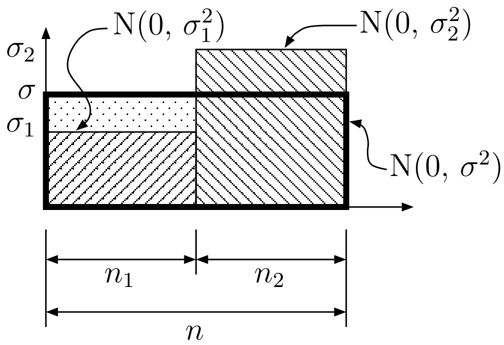

Figure 4 Concept to replace i.n.n.i.d. Gaussian variable with i.i.d. Gaussian variable.

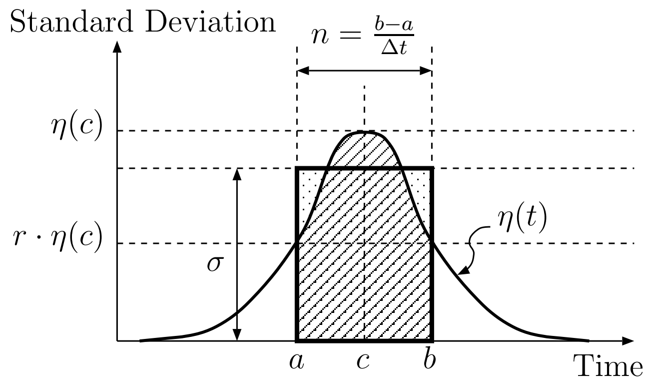

Figure 5 Concept to calculate the distribution of extreme value of non-stationary Gaussian white noise using the i.i.d. Gaussian variable.

probabilistic characteristics for maximum values of Eq.(1). Since we assume the standard deviation, $\eta(t)$ varies smoothly over the time as shown in Figure 5, we can approximate $\eta(t_i) \approx \eta(t_{i+1})$ at t_i and $t_{i+1} = t_i + \Delta t$, respectively, where Δt stands for the small increment of the discrete time. This suggests the possibility that the probability distribution for the maximum value, $F_Y(y)$, of X_i of Eq.(1) will be replaced with the asymptotic distribution for an i.i.d. Gaussian variable of Eq.(5) with parameters given by Eqs.(6a) and (6b).

On the basis of this daring (and mathematically baseless) assumption, we will determine the parameters for an i.i.d. Gaussian variable substituting the i.n.n.i.d. Gaussian variable. The parameters to determine are σ and n of Eqs.(6a) and (6b). Since we considered the area formed by the number of variables and standard deviation to determine σ as shown in Figure 4, the same concept will be introduced as shown in Figure 5. The remaining part of this section is devoted to explain the procedure to obtain the probability distribution approximately using the Figure 5.

Let us consider $\eta(t)$ takes maximum value $\eta(c)$ at $t = c$. Then, introducing a real number r $(0 < r < 1)$, we will determine parameters a and b which satisfy $r \cdot \eta(c) = \eta(a) = \eta(b)$, where $a < c < b$. The area of $\eta(t)$, S_r, are obtained as a function of r at $[a, b]$ (see Eq.(10)). To replace the probability distribution for the maximum value of i.n.n.i.d. Gaussian variable with one of i.i.d. Gaussian variable, we will consider a Gaussian variable with constant standard deviation at $[a, b]$. For the standard deviation of this i.i.d. Gaussian variable, we adopt the height σ of the rectangle whose area and length of the base are S_r and $b - a$, respectively. Applying the obtained σ and $n = (b - a)/\Delta t$ to Eqs.(6a) and (6b) and using Eq.(5), we can obtain the approximate probability distribution for the maximum values of i.n.n.i.d. Gaussian variable X_i.

The above procedure is rewritten mathematically as follows: the parameters a and b are determined by

$$a = \sup_{t<c}\{t; \eta(t) = r\eta(c)\} \tag{9a}$$

$$b = \inf_{t>c}\{t; \eta(t) = r\eta(c)\}, \tag{9b}$$

where $\eta(a) = \eta(b)$. Then, the area surrounded by $\eta(t)$ at $[a, b]$ is

$$S_r = \int_a^b \eta(t)dt. \tag{10}$$

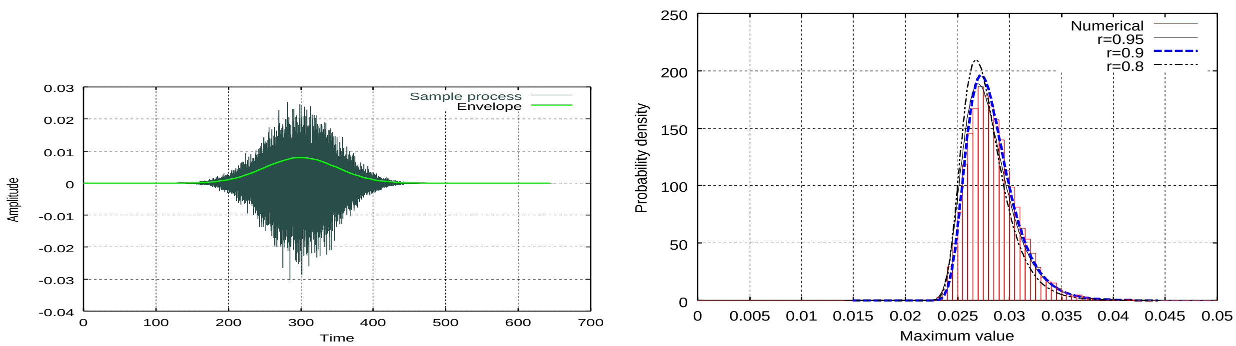

Figure 6 An example of the approximate distribution for extreme values of non-stationary Gaussian white noise. The left panel shows a sample process and $\eta(t) = \frac{1}{\sqrt{2\pi}50}\exp\left[-\frac{1}{2}\left(\frac{t-300}{50}\right)^2\right]$. The approximate distributions for $r = 0.95$, 0.9, and 0.8 are compared with the result of numerical simulation in the right panel.

Table 1 Comparison of the parameters for Gumbel's distribution for Eq.(5)

	Numerical	$r = 0.95$	$r = 0.9$	$r = 0.8$
u	0.02733	0.02699	0.027270	0.02678
α	530.74	512.65	532.90	569.92

n is obtained from $n = (b-a)/\Delta t$, and σ is determined by

$$\sigma = \frac{S_r}{n}. \tag{11}$$

Substituting n and σ for Eq.(5) derives the approximate probability distribution for the maximum value of X_i.

In this procedure, we did not mention the value of r. Generally speaking, since any function can be used for $\eta(t)$, we cannot examine the sensitivity of Eq.(5) with respect to r, exhaustively. We can say, in our calculations, Eq.(5) is not sensitive toward r. However, in a case where r is too small, the approximation of $\eta(t_i) \approx \eta(t_{i+1})$ is not assured. Thus, we may obtain better results by using the value of 0.8 to 0.9 as r. Since this suggests that the optimal value of r may depend on the time increment, Δt, Δt should be small with respect to the variance of $\eta(t)$ to satisfy the approximation $\eta(t_i) \approx \eta(t_{i+1})$.

A Numerical Example

Figure 6 shows an example of the probability distribution for maximum value of X_i with $\eta(t)$ which is Gaussian type function. The left panel of this figure compares a realization with the shape of $\eta(t)$. In the right panel, the estimated distribution are shown with the histogram obtained from MCS of 10000 times. The lines show the results from the different value of r of Eqs.(9a) and (9b): 0.8, 0.9, 0.95. It is noted that these lines approximate the histogram well. This means that the proposed method gives good approximation to represent the probability distribution for maximum value of i.n.n.i.d. Gaussian value.

Furthermore, to examine the accuracy of the approximation, the values of parameters α and u of Eq.(5) are listed in Table 1. From this, it is observed that the statistical parameters are consistent values with the results from MCS. This means that the we can roughly set the value of r around 0.9, because the approximation is not so sensitive to r.

STOCHASTIC PROPERTIES FOR LOCAL MAXIMA OF 1DOF-SYSTEM RESPONSE

Before discussing the stochastic properties for the maximum response of 1DOF system, we have to describe analytically the statistics for the amplitude of the response. Let us introduce $X_R(t)$ which is the random response of 1DOF system excited by non-stationary Gaussian white noise:

$$X_R(t_i) = \zeta(t_i) * X(t_i) \quad (i = 1, 2, \ldots, N), \tag{12}$$

where $*$ stands for the convolution operation, N for the number of discrete time, and $\zeta(t_i)$ for the impulse response function of 1DOF system whose natural period and damping factor are T_0 and h, respectively. Since $X(t_i)$ is non-stationary, it is difficult to describe the stochastic properties of $X_R(t_i)$ at whole the time t_i $(i = 1, 2, \ldots, N)$. Thus, we limit our aim of discussion to the stochastic properties of local maxima of $X_R(t_i)$.

Cartwright and Longuet-Higgins [8] have discussed the statistics for the local maxima of the stationary processes and derived their probability distribution, considering the statistical properties of the processes over time because of their stationarity. Especially, they have also shown that the probability distribution of the local maxima can be described by Rayleigh distribution in a case where the process is narrow band. Although our problem satisfies the properties of the narrow band process because of the small damping factor of 1DOF system, $X_R(t_i)$ is not stationary. This means that we cannot use directly the description by Cartwright and Longuet-Higgins.

Thus, we will deal with the stochastic properties for the local maxima at each time t_i in the meaning of ensembles instead of the statistics over the time. A time, which $X_R(t_i)$ takes the local maximum, can vary for every sample, though the time intervals of the local maxima must be nearly equal to the natural period T_0 of the system. Therefore, we introduce the local maxima process defined as follows:

$$\hat{X}_R(s_\ell) = X_R(\hat{t}_i) \quad (s_\ell < \hat{t}_i \leq s_\ell + T_0;\ \ell = 1, 2, \ldots, N_{T_0}), \tag{13}$$

where $s_{\ell+1} - s_\ell = T_0$, $N_{T_0} = N\Delta t / T_0$, and

$$\dot{X}_R(t_i) = \left.\frac{dX_R(t)}{dt}\right|_{t=t_i} \begin{cases} > 0 & (s_\ell \leq t_i < \hat{t}_i) \\ = 0 & (t_i = \hat{t}_i) \\ < 0 & (\hat{t}_i < t_i \leq s_\ell). \end{cases} \tag{14}$$

In a case where $\eta(t_i)$ of Eq.(1) is constant over time, the probability distribution for $\hat{X}_R(s_\ell)$ coincides with Rayleigh distribution and its parameter is determined by the root mean squares (rms), namely $\eta(t_i)$ [8]. From this, we can expect that $\hat{X}_R(s_\ell)$ follows the Rayleigh distribution at every time s_ℓ in the meaning of the ensembles.

Actually, $\hat{X}_R(s_\ell)$ follows the Rayleigh distribution whose standard deviation $\sigma_{\hat{X}_R}(s_\ell)$ depends on time. Then, $\sigma_{\hat{X}_R}(s_\ell)$ can be written as

$$\sigma_{\hat{X}_R}(s_\ell) = \sqrt{\frac{G_0 T_0^3}{32\pi^2 h}} \cdot \eta(s_\ell), \tag{15}$$

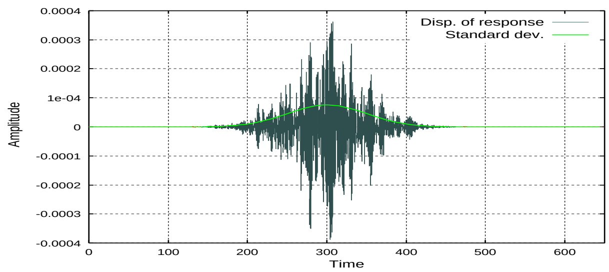

Figure 7 An example of response function of 1DOF system ($T_0 = 1$ and $h = 0.05$) excited by the non-stationary Gaussian white noise shown in Figure 6. The smooth line shows the shape of Eq.(15) witch is standard deviation of Rayleigh distribution estimated as the PDF for the local maxima.

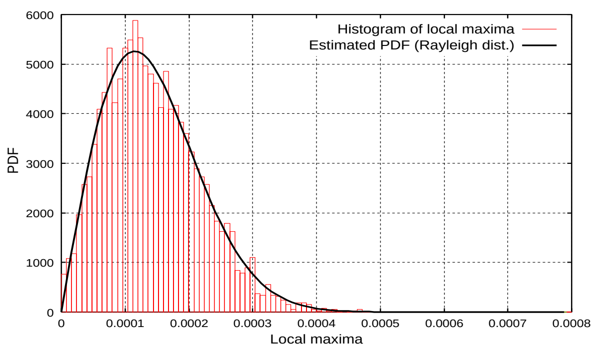

Figure 8 Comparison of probability distributions for local maxima of response of 1DOF system at a specific time $s_\ell = 300$. The histogram is obtained from 5000-time MCS and the thin solid line is Rayleigh distribution calculated analytically.

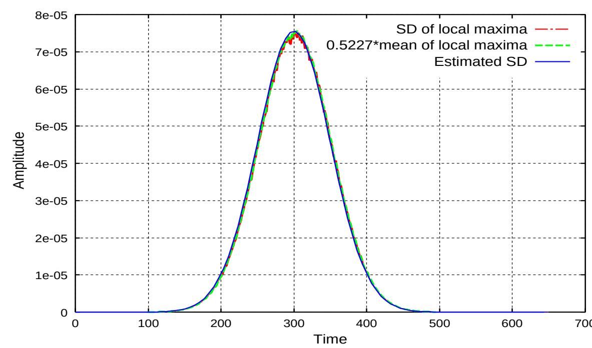

Figure 9 Comparison of standard deviations for local maxima of response of 1DOF. This figure includes three traces: values of left and right hand sides of Eq.(16) calculated from 5000 samples and Eq.(15).

where the parameter G_0 is a constant value and should be determined analytically. However, we determine empirically this value as $G_0 = \sqrt{2}\Delta t/10$ on the basis of many numerical calculations, because to derive analytically the value for G_0 is difficult by now. We confirmed the stability of this value within the scope of our calculations. This value for G_0 is used in the following numerical examples.

In Figures 7 to 9, we will present a simple numerical example in which the properties for the non-stationary Gaussian white noise of Figure 6 are used as the input motion into the 1DOF system with natural period $T_0 = 1.0$ and damping factor $h = 0.05$. Figure 7 shows a realization of response of 1DOF system. In this figure, smooth line denotes the shape of Eq.(15). $\hat{X}_R(s_\ell)$ is simulated 5000 times by MCS under the above condition and the histogram of $\hat{X}_R(s_\ell)$ at $s_\ell = 300$ is obtained as shown in Figure 8. The smooth line is the shape of Rayleigh distribution with the parameter determined by $\sigma_{\hat{X}_R}(300)$ of Eq.(15). Furthermore, to confirm that the local maxima follow the Rayleigh distribution at any time s_ℓ, the relationships between the standard deviation $\sigma_{\hat{X}_R}(s_\ell)$ and mean $\mu_{\hat{X}_R}(s_\ell)$ are examined. If $\hat{X}_R(s_\ell)$ is Rayleigh distributed variable, the following relationships should be satisfied:

$$\sigma_{\hat{X}_R}(s_\ell) = \sqrt{\frac{4}{\pi} - 1} \cdot \mu_{\hat{X}_R}(s_\ell). \tag{16}$$

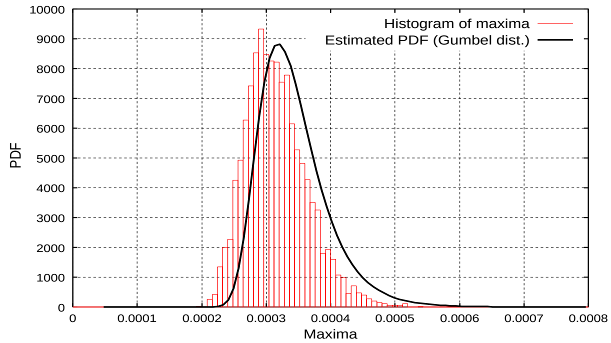

Figure 10 Comparison of the PDF for maximum response. The histogram is obtained from 5000-time MCS and the thin solid line is Gumbel's distribution calculated analytically.

Thus, we calculate the standard deviation and mean of the 5000 samples at whole discrete time s_ℓ and plot the values of left and right hand sides of Eq.(16) on Figure 9. This figure includes one more line, which is the shape of Eq.(15). It is noted that these three lines agree with each other. This suggests that the local maxima follow the Rayleigh distribution and the estimation of $\sigma_{\hat{X}_R}(s_\ell)$ by using Eq.(15) works well.

From this, we can describe analytically the stochastic properties for the local maxima of response. However, in a case where the band width of $X_R(t_i)$ is not enough narrow, namely h is not so small, there may be the negative local maxima. In the above example, the number of negative local maxima is about 5% of all the local maxima. Since the Rayleigh distribution is defined in the positive values, we truncate the negative local maxima to calculate the statistics in Figures 8 and 9.

APPROXIMATE DISTRIBUTION FOR THE MAXIMUM VALUES OF 1DOF-SYSTEM RESPONSE

Analytical Derivation

This section is devoted to derive the approximate distribution for the maximum response of 1DOF system excited by non-stationary Gaussian white noise. For this, we will assume that the local maxima are mutually independent. Our problem can be rewritten as follows: we will approximate the PDF for maximum values of Rayleigh distributed variables whose standard deviation varies slowly over time.

This problem can be solved using the same way as the procedure to find the approximate distribution for the maximum value of i.n.n.i.d. Gaussian variable. However, in this case, we have to consider the Rayleigh distribution for the local maxima instead of the Gaussian distribution. Thus, we use the following parameters for Gumbel's distribution instead of Eqs.(6a) and (6b):

$$\alpha \approx \frac{\sqrt{(4-\pi)\ln n}}{\sigma} \tag{17a}$$

$$u \approx \frac{2\sigma\sqrt{\ln n}}{4-\pi}, \tag{17b}$$

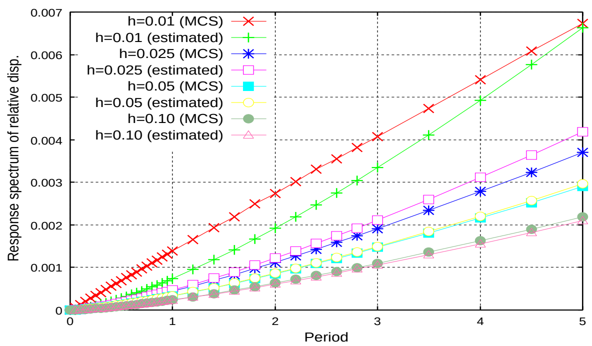

Figure 11 Means of response spectra with various damping factor h. The estimated response spectra are compared with the results of MCS.

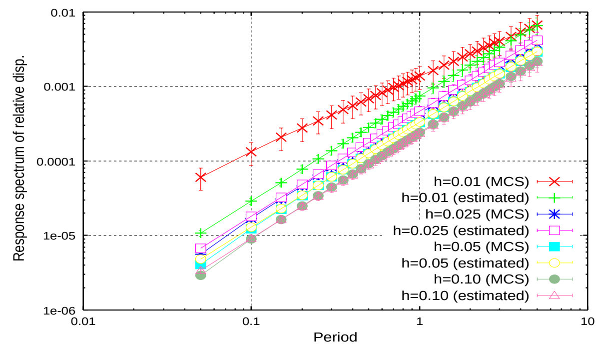

Figure 12 Means and standard deviations of response spectra with various damping factor h. The estimated response spectra are compared with the results of MCS in log-log scale to represent clearly the differences in the short period range.

where n and σ can be redefined as follows using a, b, and S_r of Eqs.(9a), (9b), and (10), respectively:

$$n = (b - a)/T_0 \tag{18a}$$

$$\sigma = S_r/n. \tag{18b}$$

Figure 10 shows the histogram of the maximum response obtained through 5000 times of the MCS. The numerical conditions are same as ones in the previous section. In this figure, the Gumbel's distribution determined by Eqs.(17a) and (17b) is also drawn, in which $r = 0.9$ is used to determine a and b of Eqs.(9a) and (9b). From this figure, while the shape of the histogram and estimated PDF are similar, the mode is slightly different. It is considered that this difference comes from the truncation of negative local maxima of the response.

After some other numerical calculations, it is observed that the estimation errors of maximum responses are large for very small h such as $h = 0.01$, because the shape of $\sigma_{\hat{X}_R}(s_\ell)$ is not similar to $\eta(t_i)$. For this case, we may give the delay of phase for the response as a cause of the large estimation errors. Obviously, we have to note that the accuracy of the estimation depends on the shape of $\eta(t_i)$. However, considering that the estimation errors are not so large except for the above special case, the proposed method may be acceptable as a simple estimation method for the maximum response.

Estimation of Response Spectra

Using the above result, we can estimate the response spectra stochastically. The response spectrum is defined as a diagram of the maximum response of 1DOF system versus its natural period T_0. Therefore, after estimating the maximum response for some T_0 and h, we can obtain the response spectra easily.

Under the same conditions as the previous examples, we calculate the response spectra and the results are shown in Figures 11 and 12. In these figures, we compare the results from the proposed method with the results from 5000-time MCS. As pointed out already, the estimation for the case $h = 0.01$ is not good, but other cases may be acceptable.

CONCLUSIONS

The conclusions derived from this study are summarized as follows:

- We have analytically derived the asymptotic representation for maximum values of Gaussian variable with two different properties and found qualitative properties.
- Using these properties, an approximate representation were proposed for maximum values of non-stationary Gaussian white noise.
- We have shown the probability distribution for the local maxima of response of 1DOF system excited by non-stationary Gaussian white noise can be described by Rayleigh distribution whose standard deviation varies with similar shape as the standard deviation of Gaussian white noise as the input motion.
- Using the above results, we have proposed a method to estimate the approximate distribution of maximum response.
- The response spectrum has been also estimated.
- The appropriateness of the proposed method has been confirmed through the Monte Carlo simulations.

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