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WEAK-FORM PERFECTLY MATCHED LAYER FOR TIME-DOMAIN WAVE SIMULATION IN INFINITE-DOMAIN

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Abstract

For seismic wave simulation in regional scale, it is necessary and important to introduce the high accurate and stable artificial boundary condition to truncate out the computation domain and simulate the process of seismic wave radiating outside the computation domain. Perfectly Matched Layer (PML) is one kind of artificial boundary conditions formulated as absorbing layer. The governing motion equations in PML are routinely derived by complex coordinate stretching of the seismic wave equation. Since the wave impedances of PML are equal to their counterparts in computational domain, the radiating seismic waves outside the computational domain can enter theoretically into PML without any reflection. Moreover, those waves will damp out exponentially along their propagation inside PML. It has been shown in large-scale 3D seismic wave simulation, the absorbing efficiency of PML is excellent for both incident body waves and interface waves and the required thickness of PML is only several times of the interested shortest wavelength. In this paper, we summarize and rederive consistently several time-domain second order PML formulations in seismic wave simulation based upon the second order seismic wave equation and the continuous finite/spectral element method. Due to the consistence in their rederivation, we show clearly the special treatments which lead to the deviation in their final formulations.

Keywords: Perfectly matched layer; spectral element; seismic wave simulation; second order wave equations

1. Introduction

Compared with the full earth, the damage area of given earthquake is finite. Thus, it is reasonable to treat the seismic wave simulation as simulation of wave propagation in infinite domain. For interested damage area, we take into account fully the effect of source geometry, source dynamic process, topography, the underground distribution of medium property and *et al*., while treat the propagation of seismic wave outside the interested area as wave propagation in infinite domain. It is implicated that the wave radiated outside the interested damage area will propagate without any reflection into the infinite domain and damped out inside to a static equilibrium state.

For wave simulation in infinite domain based upon finite difference, finite element, spectral element method, artificial boundary conditions are needed for truncate out a finite computation region while simulate the effect of infinite domain. Since the pioneer work of Lysmer[1], various artificial boundary conditions have been developed. Among them, PML is one kind of artificial boundary conditions formulated as absorbing layer, which is initially proposed by Bérenger in electromagnet wave simulation [2]. Late, the interpretation of governing motion equations in PML as an analytic continuation of wave equations in real spatial domain to a complex coordinate spatial domain given by Chew *et al*. [3] have greatly simplify the derivation of the governing motion equations in PML, which are the complex-coordinate-stretched wave equations in infinite domain.

The developed PML formulation in seismic wave simulation can be divided into two basic categories, the first-order PML and the second-order PML. The former is typically derived by complex-coordinate-stretching the first-order seismic wave equation in velocity-stress formulation, while the latter is derived by complex-coordinate-stretching the second-order seismic wave equation in displacement formulation. Since the pioneer work of Dimitri and Tromp [4], several second-order PML formulations have been derived. In apparent look, they are quite different. Thus, in this paper, we given them a small summary and rederive them in a consistent way. Due to the consistence in their rederivation, we show clearly the special treatments which lead to the deviation in their final formulations.

2. The second-order wave equation in infinite domain

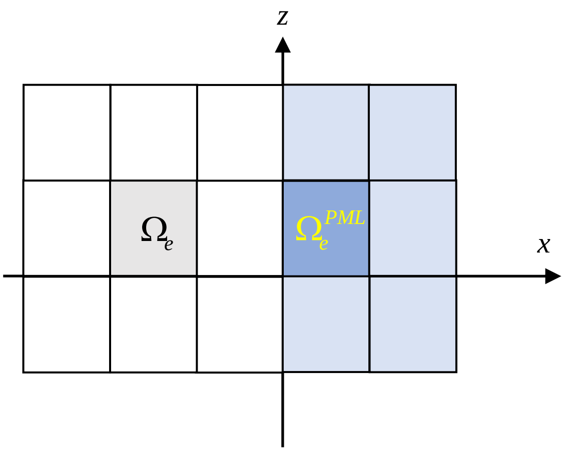

Fig. 1 The sketch of finite element mesh in computational domain and in region of PML

We focus on wave simulation based upon finite element or spectral element method. Thus, in this section, we set up the strong-form and weak-form problem of wave propagation in non-overlapping element Ω_e in infinite domain (Fig. 1). Though the wave equation in interested damage area can be set to be non-linear, we typically set the counterparts in infinite domain as the second order linear elastic wave equation. Without loss of generality, we consider here the isotropic P-SV elastic wave equations, which are

$$\rho \ddot{u}_x = \partial_x \sigma_{xx} + \partial_z \sigma_{xz}, \tag{1}$$

$$\rho \ddot{u}_z = \partial_x \sigma_{zx} + \partial_z \sigma_{zz}, \tag{2}$$

$$\sigma_{xx} = (\lambda + 2\mu)\partial_x u_x + \lambda \partial_z u_z,\ \sigma_{zz} = \lambda \partial_x u_x + (\lambda + 2\mu)\partial_z u_z,\quad \sigma_{xz} = \sigma_{zx} = \mu(\partial_x u_z + \partial_z u_x), \tag{3}$$

where ρ denotes the density, u_i the displacement in x_i direction, σ_{ij} the stress component, $\partial_j = \partial/\partial x_j$, λ and μ are the Lamé constants. Along the boundary of Ω_e, the traction and displacement are continuous

$$\left[\sigma_{ij} n_j\right]_-^+ = 0,\ \left[u_i\right]_-^+ = 0, \tag{4}$$

where n_j denotes the component of outward-pointing normal vector along the two sides of the interface. In case the boundary of Ω_e is aligned with the air-solid surface of earth, the tractions are equal to zero. Moreover, we set the initial state of displacement and velocity are equal to zero. The wave equation coupling the boundary condition and the initial state condition define the strong form wave propagation problem in Ω_e inside infinite domain, which can be extended for whole region of infinite domain after the introduction of the corresponding boundary and initial stable condition in the infinite domain. Following the principle of virtual work, that is the virtual work of a system of equilibrium forces vanishes when compatible virtual displacements w_x and w_z are imposed, we can write the weak form wave propagation problem in Ω_e as

$$\begin{aligned}&\int_{\Omega_e} (w_x \ddot{u}_x + w_z \ddot{u}_z) d\Omega_e + \int_{\Omega_e} (\partial_x w_x \sigma_{xx} + \partial_z w_x \sigma_{xz} + \partial_x w_z \sigma_{zx} + \partial_z w_z \sigma_{zz}) d\Omega_e = \\ &\int_{\Gamma_e} (w_x \sigma_{xx} + w_z \sigma_{zx}) dz + (w_x \sigma_{xz} + w_z \sigma_{zz}) dx\end{aligned} \tag{5}$$

Due to the compatibility of the virtual displacements and the continuity of the traction along the interface of different elements, the right-side integration in (5) vanished for interior elements. For exterior element subjected to traction or traction and displacement boundary conditions, the computation of integration can be found in Dimitri and Tromp [5]. Moreover, assuming that the virtual and real displacements share the same scheme of approximation inside elements, the solution of (5) can be achieved by Garlerkin continuous finite/spectral element method.

2. The second-order PML in frequency domain

Following the complex coordinate stretching approach, the derivation of PML can be can be divided into three steps. The first step is to introduced the complex-stretched coordinate in PML region. The second is that to mapping the Fourier-transformed second order wave equation into the complex stretched coordinate, which is transformed back into real coordinate to get the frequency-domain PML. The third is to transform the frequency-domain PML into time domain utilizing the inverse Fourier transform. The essential difference lies in third step, where different operations on the frequency-domain PML used in order to get an author's favorite time-domain PML in terms of their adopted numerical methods. Taking the element Ω_e^{PML} inside PML region as an example, let us firstly introduce the complex-stretched coordinate in x direction as

$$\tilde{x}(x) = \int_0^x s(x) dx, \tag{6}$$

where $s(x)$ is the coordinate stretching function. The commonly used $s(x)$ is the single-pole rational function

$$s(x) = \kappa(x) + \frac{d(x)}{\alpha(x) + \mathbf{i}\omega} = \kappa(x)\frac{\beta(x) + \mathbf{i}\omega}{\alpha(x) + \mathbf{i}\omega}, \tag{7}$$

where $d(x)$ is the attenuation factor that causes the amplitude of the wavefield to be damping exponentially inside the PML layer, $\alpha(x)$ is the frequency-shifted factor that makes the damping effect frequency dependent, and $\kappa(x)$ is the scaling factor. The latter has been found in numerical tests to be important for absorption of evanescent waves and near-grazing incident waves [6], ω is the angular frequency and $\mathbf{i}=\sqrt{-1}$, $\beta(x)=\alpha(x)+d(x)/\kappa(x)$. Due to the fact that a non-zero $\alpha(x)$ will reduce the absorbing efficiency of PML for low frequency incident wave, $s(x)$ of multi-poles rational function has also been introduced [7], such as

$$s(x)=\prod_j \kappa_j(x)\frac{\beta_j(x)+\mathbf{i}\omega}{\alpha_j(x)+\mathbf{i}\omega}. \tag{8}$$

Then, in second step, we firstly map the frequency-domain P-SV wave equation into the stretched coordinate,

$$-\rho\omega^2\hat{u}_x=\partial_{\tilde{x}}\left[(\lambda+2\mu)\partial_{\tilde{x}}\hat{u}_x+\lambda\partial_z\hat{u}_z\right]+\partial_z\left[\mu\left(\partial_{\tilde{x}}\hat{u}_z+\partial_z\hat{u}_x\right)\right], \tag{9}$$

$$-\rho\omega^2\hat{u}_z=\partial_{\tilde{x}}\left[\mu\left(\partial_{\tilde{x}}\hat{u}_z+\partial_z\hat{u}_x\right)\right]+\partial_z\left[\lambda\partial_{\tilde{x}}\hat{u}_x+(\lambda+2\mu)\partial_z\hat{u}_z\right], \tag{10}$$

where a caret "$\wedge$" denotes the Fourier transform of the subtended function. According to the relationship derived from (6),

$$\partial_{\tilde{x}}=s_x^{-1}\partial_x. \tag{11}$$

We transform (9)-(10) into real coordinate that are

$$-\rho\omega^2\hat{u}_x=s_x^{-1}\partial_x\left[(\lambda+2\mu)s_x^{-1}\partial_x\hat{u}_x+\lambda\partial_z\hat{u}_z\right]+\partial_z\left[\mu\left(s_x^{-1}\partial_x\hat{u}_z+\partial_z\hat{u}_x\right)\right], \tag{12}$$

$$-\rho\omega^2\hat{u}_z=s_x^{-1}\partial_x\left[\mu\left(s_x^{-1}\partial_x\hat{u}_z+\partial_z\hat{u}_x\right)\right]+\partial_z\left[\lambda s_x^{-1}\partial_x\hat{u}_x+(\lambda+2\mu)\partial_z\hat{u}_z\right]. \tag{13}$$

In order to get frequency-domain PML ready for discretized with frequency-domain finite/spectral element methods, we multiply two sides of (12) and (13) with s_x to get

$$-\rho\omega^2 s_x\hat{u}_x=\partial_x\left[(\lambda+2\mu)s_x^{-1}\partial_x\hat{u}_x+\lambda\partial_z\hat{u}_z\right]+\partial_z\left[s_x\left(\mu\left(s_x^{-1}\partial_x\hat{u}_z+\partial_z\hat{u}_x\right)\right)\right], \tag{14}$$

$$-\rho\omega^2 s_x\hat{u}_z=\partial_x\left[\mu\left(s_x^{-1}\partial_x\hat{u}_z+\partial_z\hat{u}_x\right)\right]+\partial_z\left[s_x\left(\lambda s_x^{-1}\partial_x\hat{u}_x+(\lambda+2\mu)\partial_z\hat{u}_z\right)\right]. \tag{15}$$

Eq. (14) and (15) define the frequency-domain PML, which server for derivation of the time-domain PML in [4,8-13] .

3. The second-order PML in time domain

3.1 The second-order split PML in Dimitri and Tromp [4]

It is worth to note here that the second-order split PML in time domain proposed in Dimitri and Tromp [4] is the first second-order PML ready for numerical discretization using finite/spectral element method. In order to get their formulation, we firstly restructure (14) and (15) as

$$-\rho\omega^2 s_x\hat{u}_x=s_x^{-1}\left\{\partial_x\left[(\lambda+2\mu)\partial_x\hat{u}_x\right]\right\}+s_x's_x^{-2}\left\{(\lambda+2\mu)\partial_x\hat{u}_x\right\}+\left\{\partial_x\left[\lambda\partial_z\hat{u}_z\right]+\partial_z\left[\mu\partial_x\hat{u}_z\right]\right\}+s_x\left\{\partial_z\left[\mu\partial_z\hat{u}_x\right]\right\}, \tag{16}$$

$$-\rho\omega^2 s_x\hat{u}_z=s_x^{-1}\left\{\partial_x\left[\mu\partial_x\hat{u}_z\right]\right\}+s_x's_x^{-2}\left\{\mu\partial_x\hat{u}_z\right\}+\left\{\partial_x\left[\mu\partial_z\hat{u}_x\right]+\partial_z\left[\lambda\partial_x\hat{u}_x\right]\right\}+s_x\left\{\partial_z\left[(\lambda+2\mu)\partial_z\hat{u}_z\right]\right\}. \tag{17}$$

Then, following the idea in deriving the first-order PML as done in [14], we split (16) and (17) as

$$-\rho\omega^2 (s_x)^2 \hat{u}_x^{(1)} = \partial_x\left[(\lambda+2\mu)\partial_x\hat{u}_x\right],\quad -\rho\omega^2 (s_x)^2 \hat{u}_z^{(1)} = \partial_x\left[\mu\partial_x\hat{u}_z\right], \tag{18}$$

$$-\rho\omega^2\left[(s_x)^3/s_x'\right]\hat{u}_x^{(2)} = (\lambda+2\mu)\partial_x\hat{u}_x,\quad -\rho\omega^2\left[(s_x)^3/s_x'\right]\hat{u}_z^{(2)} = \mu\partial_x\hat{u}_z, \tag{19}$$

$$-\rho\omega^2 s_x\hat{u}_x^{(3)} = \partial_x\left[\lambda\partial_z\hat{u}_z\right]+\partial_z\left[\mu\partial_x\hat{u}_z\right],\ -\rho\omega^2 s_x\hat{u}_z^{(3)} = \partial_x\left[\mu\partial_z\hat{u}_x\right]+\partial_z\left[\lambda\partial_x\hat{u}_x\right], \tag{20}$$

$$-\rho\omega^2\hat{u}_x^{(4)} = \partial_z\left[\mu\partial_z\hat{u}_x\right],\ -\rho\omega^2\hat{u}_z^{(4)} = \partial_z\left[(\lambda+2\mu)\partial_z\hat{u}_z\right], \tag{21}$$

$$\hat{u}_x = \hat{u}_x^{(1)}+\hat{u}_x^{(2)}+\hat{u}_x^{(3)}+\hat{u}_x^{(4)},\ \hat{u}_z = \hat{u}_z^{(1)}+\hat{u}_z^{(2)}+\hat{u}_z^{(3)}+\hat{u}_z^{(4)}. \tag{22}$$

Taking into account the stretching function used in [4], that is $s(x)=1+d(x)/\mathbf{i}\omega$, we can easily recover the time-domain second-order split PML in [4].

3.2 The second-order unsplit PML in Basu and Chopra [8]

In Basu and Chopra [8], the first second-order unsplit PML has been proposed by introducing element-defined memory variables associated with the strain and stress. In order to recover their formulation, we firstly restructure (14) and (15) as

$$-\rho\omega^2 s_x\hat{u}_x = \partial_x\left[\hat{\sigma}_{xx}\right]+\partial_z\left[s_x\hat{\sigma}_{xz}\right], \tag{23}$$

$$-\rho\omega^2 s_x\hat{u}_z = \partial_x\left[\hat{\sigma}_{zx}\right]+\partial_z\left[s_x\hat{\sigma}_{zz}\right], \tag{24}$$

$$\hat{\sigma}_{xx} = (\lambda+2\mu)\hat{\varepsilon}_{xx}+\lambda\hat{\varepsilon}_{zz},\ \hat{\sigma}_{zx}=\hat{\sigma}_{zx}=2\mu\hat{\varepsilon}_{xz},\ \hat{\sigma}_{zz}=\lambda\hat{\varepsilon}_{zz}+(\lambda+2\mu)\hat{\varepsilon}_{zz}, \tag{25}$$

$$\hat{\varepsilon}_{xx} = s_x^{-1}\partial_x\hat{u}_x \Rightarrow \mathbf{i}\omega(s_x)^2\hat{\varepsilon}_{xx} = \mathbf{i}\omega s_x\partial_x\hat{u}_x,\ \hat{\varepsilon}_{xz} = \left(s_x^{-1}\partial_x\hat{u}_z+\partial_z\hat{u}_x\right)/2 \Rightarrow \mathbf{i}\omega s_x\hat{\varepsilon}_{xz} = \mathbf{i}\omega\left(\partial_x\hat{u}_z+s_x\partial_z\hat{u}_x\right)/2. \tag{26}$$

Inserting $s(x)=\kappa(x)+d(x)/\mathbf{i}\omega$ into (23)-(26), we get the time-domain PML in Basu and Chopra [8], that are

$$\rho\left[\ddot{u}_x+d(x)\dot{u}_x\right] = \partial_x\left[\sigma_{xx}\right]+\partial_z\left[\kappa(x)\sigma_{xz}+d(x)\Sigma_{xz}\right], \tag{27}$$

$$\rho\left[\ddot{u}_z+d(x)\dot{u}_z\right] = \partial_x\left[\sigma_{zx}\right]+\partial_z\left[\kappa(x)\sigma_{zz}+d(x)\Sigma_{zz}\right], \tag{28}$$

$$\sigma_{xx} = (\lambda+2\mu)E_{xx}+\lambda\varepsilon_{zz},\ \sigma_{zx}=\sigma_{zx}=2\mu E_{xz},\ \sigma_{zz}=\lambda E_{xx}+(\lambda+2\mu)\varepsilon_{zz}, \tag{29}$$

$$\Sigma_{xz} = \int_0^t \sigma_{xz}dt,\ \Sigma_{zz} = \int_0^t \sigma_{zz}dt, \tag{30}$$

$$\kappa^2(x)\dot{E}_{xx}+2\kappa(x)d(x)E_{xx}+d^2(x)\int_0^t E_{xx}dt = \kappa(x)\partial_x\dot{u}_x+d(x)\partial_x u_x, \tag{31}$$

$$\kappa(x)\dot{E}_{xz}+d(x)E_{xz} = \partial_x\dot{u}_z/2+\kappa(x)\partial_z\dot{u}_x/2+d(x)\partial_z u_x/2. \tag{32}$$

On basis of the PML formulation given in Basu and Chopra [8], a different version of PML has been introduced by Fathi *et al* by introducing only the element-defined memory variables associated only with the stress, which in frequency domain can be written as

$$-\rho\omega^2 s_x\hat{u}_x = \partial_x\left[\hat{\sigma}_{xx}\right]+\partial_z\left[s_x\hat{\sigma}_{xz}\right], \tag{33}$$

$$-\rho\omega^2 s_x\hat{u}_z = \partial_x\left[\hat{\sigma}_{zx}\right]+\partial_z\left[s_x\hat{\sigma}_{zz}\right], \tag{34}$$

$$\mathbf{i}\omega\{s(x)\hat{\sigma}_{xx}\}=\mathbf{i}\omega\{(\lambda+2\mu)\partial_x\hat{u}_x+\lambda s(x)\partial_z\hat{u}_z\}, \quad (35)$$

$$\mathbf{i}\omega\{s(x)\hat{\sigma}_{zz}\}=\mathbf{i}\omega\{\lambda\partial_x\hat{u}_x+s(x)(\lambda+2\mu)\partial_z\hat{u}_z\}, \quad (36)$$

$$\mathbf{i}\omega\{s(x)\hat{\sigma}_{zx}\}=\mathbf{i}\omega\{s(x)\hat{\sigma}_{zx}\}=\mathbf{i}\omega\{\mu[\partial_x\hat{u}_z+s(x)\partial_z\hat{u}_x]\}, \quad (37)$$

Inserting $s(x)=\kappa(x)+d(x)/\mathbf{i}\omega$ into (33)-(37), we get the corresponding time-domain PML as

$$\rho[\kappa(x)\ddot{u}_x+d(x)\dot{u}_x]=\partial_x[\dot{\Sigma}_{xx}]+\partial_z[\kappa(x)\dot{\Sigma}_{xz}+d(x)\Sigma_{xz}], \quad (38)$$

$$\rho[\kappa(x)\ddot{u}_z+d(x)\dot{u}_z]=\partial_x[\dot{\Sigma}_{zx}]+\partial_z[\kappa(x)\dot{\Sigma}_{zz}+d(x)\Sigma_{zz}], \quad (39)$$

$$\kappa(x)\ddot{\Sigma}_{xx}+d(x)\dot{\Sigma}_{xx}=(\lambda+2\mu)\partial_x\dot{u}_x+\lambda[\kappa(x)\partial_z\dot{u}_z+d(x)\partial_z u_z], \quad (40)$$

$$\kappa(x)\ddot{\Sigma}_{zz}+d(x)\dot{\Sigma}_{zz}=\lambda\partial_x\dot{u}_x+(\lambda+2\mu)[\kappa(x)\partial_z\dot{u}_z+d(x)\partial_z u_z], \quad (41)$$

$$\kappa(x)\ddot{\Sigma}_{xz}+d(x)\dot{\Sigma}_{xz}=\kappa(x)\ddot{\Sigma}_{zx}+d(x)\dot{\Sigma}_{zx}=\mu[\partial_x\dot{u}_z+\kappa(x)\partial_z\dot{u}_x+d(x)\partial_z u_x]. \quad (42)$$

3.3 The second-order unsplit PML in time domain in Matzen [10]

Utilizing the commonly adopted assumption that the material properties are constant inside each element, Matzen [10] proposed by introducing only node-defined memory variables associated with displacement. Assuming that s_x is constant inside in each element, we can rewrite (14) and (15) as

$$-\rho\omega^2 s_x\hat{u}_x=\partial_x\left[(\lambda+2\mu)\partial_x\hat{U}_x^1+\lambda\partial_z\hat{u}_z\right]+\partial_z\left[\mu\left(\partial_x\hat{u}_z+\partial_z\hat{U}_x^2\right)\right], \quad (43)$$

$$-\rho\omega^2 s_x\hat{u}_z=\partial_x\left[\mu\left(\partial_x\hat{U}_z^1+\partial_z\hat{u}_x\right)\right]+\partial_z\left[\lambda\partial_x\hat{u}_x+(\lambda+2\mu)\partial_z\hat{U}_z^2\right], \quad (44)$$

$$\hat{U}_x^1=s_x^{-1}\hat{u}_x,\ \hat{U}_z^1=s_x^{-1}\hat{u}_z,\hat{U}_x^2=s_x\hat{u}_x,\ \hat{U}_z^2=s_x\hat{u}_z. \quad (45)$$

Inserting $s(x)=\kappa(x)+d(x)/\mathbf{i}\omega$ into (33)-(37), we get the corresponding time-domain PML as

$$\rho[\kappa(x)\ddot{u}_x+d(x)\dot{u}_x]=\partial_x[(\lambda+2\mu)\partial_x U_x^1+\lambda\partial_z u_z]+\partial_z\left[\mu\left(\partial_x u_z+\partial_z U_x^2\right)\right], \quad (46)$$

$$\rho[\kappa(x)\ddot{u}_z+d(x)\dot{u}_z]=\partial_x\left[\mu\left(\partial_x U_z^1+\partial_z u_x\right)\right]+\partial_z[\lambda\partial_x u_x+(\lambda+2\mu)\partial_z U_z^2], \quad (47)$$

$$U_x^1=\frac{1}{\kappa(x)}+\frac{d(x)-\kappa(x)}{\kappa^2(x)}\int_0^t e^{-d(x)/\kappa(x)(t-\tau)}u_x(\tau)d\tau,\ U_z^1=\frac{1}{\kappa(x)}+\frac{d(x)-\kappa(x)}{\kappa^2(x)}\int_0^t e^{-d(x)/\kappa(x)(t-\tau)}u_z(\tau)d\tau, \quad (48)$$

$$U_x^2=\kappa(x)u_x+d(x)\int_0^t u_x(\tau)d\tau,\ U_z^2=\kappa(x)u_z+d(x)\int_0^t u_z(\tau)d\tau. \quad (49)$$

However, as pointed out by Dan et al. [15] the removal of the requirement that parameter in PML element be constant is important when higher order basis functions are employed to improve the absorbing efficiency of PML. Thus, Xie *et al*. [12] rederive a time-domain PML as following

$$\rho[\kappa(x)\ddot{u}_x+d(x)\dot{u}_x]=\partial_x[(\lambda+2\mu)E_{xx}+\lambda\partial_z u_z]+\partial_z[\mu(\partial_x u_z+E_{zx})], \quad (50)$$

$$\rho[\kappa(x)\ddot{u}_z+d(x)\dot{u}_z]=\partial_x[\mu(E_{xz}+\partial_z u_x)]+\partial_z[\lambda\partial_x u_x+(\lambda+2\mu)E_{zz}], \quad (51)$$

$$E_{xx} = \frac{1}{\kappa(x)} + \frac{d(x) - \kappa(x)}{\kappa^2(x)} \int_0^t e^{-\frac{d(x)}{\kappa(x)}(t-\tau)} \partial_x u_x(\tau) d\tau \, , \; E_{xz} = \frac{1}{\kappa(x)} + \frac{d(x) - \kappa(x)}{\kappa^2(x)} \int_0^t e^{-\frac{d(x)}{\kappa(x)}(t-\tau)} \partial_x u_z(\tau) d\tau \, , \quad (52)$$

$$E_{zx} = \kappa(x) \partial_z u_x + d(x) \int_0^t \partial_z u_x(\tau) d\tau \, , \; E_{zz} = \kappa(x) \partial_z u_z + d(x) \int_0^t \partial_z u_z(\tau) d\tau \, . \quad (53)$$

Though compared with Matzen's formulation, the two are seeming to be the same in terms of the number of memory variables and the computational work. However, it is not the truth since that E_{ij} are element defined. Thus, the needed storage increases in our formulation. However, the computational work of Matzen's formulation is slightly bigger than ours due to the increase in work for computing $\partial_i U_j^1$, $\partial_i U_j^2$.

4. CONCLUSIONS

In this paper, we summarize and rederive consistently several time-domain second order PML formulations for infinite domain truncation in seismic wave simulation based upon the second order seismic wave equation and the continuous finite/spectral element method. We only show the numerically long-time stable formulations, which have been already validated with numerical tests. Though we only show consistently their derivation using the simplest single-pole complex coordinate stretching function, the extension to more complex functions are straightforward. We will finish that in our future work. It is worthy to carry out the comparison of the mentioned second-order PMLs using the same finite/spectral element method for space discretization together with the same time scheme for integrating the semi-discrete finite/spectral element equations.

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6. Copyrights

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