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A SEISMIC WAVE PROPAGATION SIMULATION USING A MODIFIED WAVE EQUATION WITH DIFFUSION EFFECTS

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Abstract

A modified wave equation with diffusion effects is employed to develop a smoothing scheme for seismic wave propagation simulations. The modified wave equation is newly derived in Imai et al. [1] and there is still much room to investigate. With mathematical rigor we show that the solution of the new equation, which is derived as an analog of the advection-diffusion equation, can be obtained by the spatial convolution between a solution of the wave equation and the heat kernel and has a finite propagation speed and a diffusion effect. As a first step, we provide a smoothing scheme with the finite difference method for seismic wave propagation simulations. We show that the smoothing scheme based on the modified wave equation has some advantages. Firstly, it preserves the characteristics of the wave equation such as wave propagation speed. Secondly, it selectively removes the short-wavelength components of the solution. Since our smoothing scheme can be implemented by adding simple correction terms to usual scheme, it can easily be applied to the seismic wave equation. As a second step, we provide a smoothing scheme with the finite element method for seismic wave propagation simulations. The modified wave equation includes fourth order derivatives and higher regularity is required in the usual weak formulation. We propose a mixed FEM of the modified wave equation to ease the regularity. The mixed formulation reduces the modified wave equation to a coupled system of inhomogeneous diffusion equations. We mathematically provide some properties of a solution of an initial boundary value problem of the equation. In particular, we provide a representation of the solution by eigenfunction expansion of the Laplacian. This representation suggests that the solution possesses properties of both the wave equation and the heat equation and that the modified wave equation has a damping effect corresponding to the introduction of a Q factor that is inversely proportional to the frequency for each eigenmode. Furthermore, we derive a discretization for a mixed finite element method of the modified wave equation and construct a semi-implicit time integration scheme. By considering eigenvalue problems of the discretized equation, we show that a semi-implicit time integration scheme for a mixed FEM of the equation has essential properties for numerical stability and dispersion. Finally, we conduct wave propagation simulations by applying the semi-implicit time integration scheme to the elastic wave equation. Our scheme can easily be applied to not only the wave equation but also the elastic wave equation. This numerical experiments reveal that the proposed scheme is effective for filtering short-wavelength components in seismic wave propagation simulations.

Keywords: modified wave equation with diffusion effects; smoothing scheme; mixed finite element method

1. Introduction

For seismic hazard evaluations, a lot of simulations for seismic wave propagation have been conducted so far. Simulation results, which are obtained for many scenarios varying source data, underground model, are used to make probabilistic seismic estimation maps. Each scenario is consist of combinatorial sets of hypocenter location, fault parameters, source time function, underground physical property such as P-wave velocity, S-wave velocity, Q-factor, and density [2]. The underground model reflects the complex profile such as irregular layer structure and local fluctuation of physical property values. So, a stable long-term integration method is required in seismic wave propagation simulations for a shallow-and-deep integrated underground model. In practice, however, numerical instabilities often causes the calculations to diverge from actual values. From experience so far, numerical instability often seems to occur when the spatial distribution of the underground model data has locally severe contrast [3]. To reduce such numerical instability, much effort has been devoted to ad-hoc preprocessing to tune the underground velocity structure [2]. Now, a smoothing scheme for the seismic wave propagation simulation is desirable as a way of mitigating numerical instability.

In the field of computational fluid dynamics, several smoothing schemes have been studied for the advection equation [4,5]. There has also been some research concerning the constrained interpolation profile (CIP) method for the advection equation [6,7]. The CIP method is a higher order finite difference method which transports not only the value but also its gradient to maintain the profile. These schemes are developed to produce accurate and stable solutions for phenomena with discontinuous singularity propagation. In this paper, we, however, focus on the wave equation. While both the wave equation and the advection equation are hyperbolic partial differential equations, the former is of the second order and the latter is of the first order. In particular the wave equation does not have a specific direction along which phenomena propagate. This is why we cannot apply the previously mentioned schemes of the advection equation to the seismic wave propagation simulation.

Filtering approach and absorbing boundary technique, which mitigate long-term instabilities, are discussed in the field of seismic wave propagation simulations. For example, a time-filtering approach, which averages the solution at different time levels of the same spatial location, is proposed in [8]. It is demonstrated that the time-filtering approach is effective for mitigating the long-time instability when boundaries or interfaces are involved. Further, it is suspected that the high-frequency noises in numerical simulations regarding the boundary or interface treatments are responsible for the long-time instability. In [3], some issues of the instability in high-contrast media, which also motivates our research, are discussed. On the other hand, it is known that absorbing boundary technique is essential for long-time seismic modeling. A non-reflecting boundary condition for the elastic wave equation is proposed by [9] and adopted in some seismic wave propagation simulations [10]. Since the method proposed in this paper is based on a modified wave equation and independent of the boundary treatment, it can be combined with a non-reflecting boundary condition.

In this paper, we employ a modified wave equation with diffusion effects to develop a smoothing scheme for seismic wave propagation simulations. The modified wave equation is newly derived in Imai et al. [1] and there is still much room to investigate. We give mathematical discussions for some properties of a solution of the modified wave equation with diffusion effects and propose a smoothing scheme with the FDM and the FEM for seismic wave propagation simulations. Finally, we conduct wave propagation simulations by applying our method to the elastic wave equation. The numerical experiments reveal that the proposed scheme is effective for filtering short-wavelength components in seismic wave propagation simulations.

2. Method

2.1 Mathematical preliminaries

2.1.1 A modified wave equation with diffusion effects

The d'Alambertian in the one-dimensional case can be rewritten as a product of two differential operators of the advection equation as follows [11,12,13,14]:

$$\left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right)u = 0 \xrightarrow{\text{equivalent}} \left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)u = 0. \quad (1)$$

In the field of computational fluid dynamics, upwind difference methods are often used as accurate and stable schemes for solving the advection equation numerically [4,5]. It is well-known that the upwind difference method for the advection equation is substantially the same as adding a numerical diffusion term [5]. This suggests that constructing a numerical solution of the following advection-diffusion equation, instead of the advection equation, will lead us to a stable method:

$$\frac{\partial u}{\partial t} \pm a\frac{\partial u}{\partial x} \xrightarrow{\text{modify}} \frac{\partial u}{\partial t} \pm a\frac{\partial u}{\partial x} - b\frac{\partial^2 u}{\partial x^2} = 0. \quad (2)$$

Now, we formally apply this principle to the wave equation (1), given by the product of two differential operators of the advection equation. That is, we consider the product of two differential operators of the advection-diffusion equation, as follows:

$$\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)u = 0 \xrightarrow{\text{modify}} \left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x} - b\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x} - b\frac{\partial^2 u}{\partial x^2}\right)u = 0. \quad (3)$$

Equation (3) is an analog of the advection-diffusion equation for the wave equation and consists of the normal wave equation with simple correction terms. This equation can also be obtained by replacing the time derivative operator in the normal wave equation with a heat equation operator. The former expression is more convenient for implementing the smoothing scheme with the FDM, but the latter is more useful for discussing the theoretical solution.

Expression for implementing the smoothing schemes:

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + 2b\left(\frac{\partial v}{\partial t} - \frac{b}{2}\frac{\partial^2 v}{\partial x^2}\right), \quad v = \frac{\partial^2 u}{\partial x^2}. \quad (4)$$

Expression for discussing the theoretical solution:

$$\left(\frac{\partial}{\partial t} - b\frac{\partial^2}{\partial x^2}\right)^2 u = a^2\frac{\partial^2 u}{\partial x^2}. \quad (5)$$

The modified wave equation with diffusion effects is an artificial, heuristically derived equation, and the properties of its theoretical solution are not trivial. Therefore, we now present some of its properties, including the existence and the uniqueness of the solution.

First, we consider the initial value problem for the modified equation in the one-dimensional case. By using the Fourier transform, we can construct a concrete solution formula for the initial value problem in the Schwartz space $S(\mathbb{R})$. See [1] for details.

Theorem 1: *Let f be in the Schwartz space* $S(\mathbb{R})$*. Suppose that* $u(t,x)$ *is the solution of initial value problem for the modified wave equation with diffusion effects:*

$$\left(\frac{\partial}{\partial t} - b\frac{\partial^2}{\partial x^2}\right)^2 u(t,x) = a^2\frac{\partial^2 u}{\partial x^2}(t,x) \quad \text{in } \mathbb{R}_+ \times \mathbb{R}, \quad (6)$$

$$u(0,x) = f(x) \quad \text{in } \mathbb{R}, \quad (7)$$

$$\frac{\partial u}{\partial t}(0,x) = 0, \quad \text{in } \mathbb{R}. \quad (8)$$

Then the solution $u(t,x)$ *is obtained as follows:*

$$u(t,x) = \frac{1}{2}f * F^t_{a,b}(x) + \frac{1}{2}f * F^t_{-a,b}(x), \quad (9)$$

where

$$F^t_{\pm a,b}(x) = \left(1 \pm \frac{x \pm at}{2at}\right)\frac{1}{\sqrt{2bt}}\exp\left(\frac{-(x \pm at)^2}{4bt}\right), \tag{10}$$

and the symbol $*$ stands for the spatial convolution.

It should also be noted that $F^t_{\pm a,b}(x) \to \delta(x \pm at)$ in the sense of the distribution when $b \to 0$ and that this solution formula generalizes d'Alembert's formula.

Next, we consider the initial value problem for the modified wave equation in the general-dimensional case. See [1] for details.

Theorem 2: *Let φ and ψ be in the Schwartz space $S(\mathbb{R})$. Suppose that $G_b(t,x) = (2bt)^{-n/2}\exp(-\|x\|^2/4bt)$ is the heat kernel and that $w(t,x)$ is the solution of the initial value problem for the wave equation:*

$$\frac{\partial^2 w}{\partial t^2}(t,x) = a^2 \Delta w(t,x) \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \tag{11}$$

$$w(0,x) = \varphi(x) \quad \text{in } \mathbb{R}^n, \tag{12}$$

$$\frac{\partial w}{\partial t}(0,x) = \psi(x) - b\Delta\varphi(x), \quad \text{in } \mathbb{R}^n. \tag{13}$$

Then the spatial convolution

$$u(t,x) = \int_{\mathbb{R}^n} w(t,y) G_b(t, x-y) dy \tag{14}$$

of $w(t,x)$ and $G_b(t,x)$ is the solution of the initial value problem for the modified wave equation with diffusion effects:

$$\left(\frac{\partial}{\partial t} - b\Delta\right)^2 u(t,x) = a^2 \Delta u(t,x) \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \tag{15}$$

$$u(0,x) = \varphi(x) \quad \text{in } \mathbb{R}^n, \tag{16}$$

$$\frac{\partial u}{\partial t}(0,x) = \psi(x), \quad \text{in } \mathbb{R}^n. \tag{17}$$

Moreover, we can show that the solution, which tends to zero rapidly at infinity, is unique. Therefore it is sufficient, when considering the solution of the initial value problem of the modified wave equation, to discuss only this spatial convolution solution. Considering the solution formula for the one-dimensional case and the solution construction for the general-dimensional case, it turns out that the solution of the modified wave equation has the following features:

- It has, in a sense, a finite propagation speed (i.e. it has the same propagation speed as in the wave equation).
- It gets smoother over time (i.e. it demonstrates the same diffusion effects as in the heat equation).

2.1.2 A mixed formulation of the modified wave equation

As shown in [1], a modified wave equation with diffusion effects is as follows.

$$\left(\frac{\partial}{\partial t} - b\Delta\right)^2 u(t,x) = a^2 \Delta u(t,x) \quad \text{in } \mathbb{R}_+ \times \Omega, \tag{18}$$

where $\mathbb{R}_+ = \{t \in \mathbb{R}; t > 0\}$ and Ω is a domain of $\mathbb{R}^n$. Eq.(18) includes the fourth order derivatives $b^2\Delta(\Delta u)$ with respect to spatial variables. A usual weak formulation of Eq.(18) leads us to the following:

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}\varphi dV + a^2\int_\Omega \nabla u\cdot\nabla\varphi dV + 2b\int_\Omega \nabla\left(\frac{\partial u}{\partial t}\right)\cdot\nabla\varphi dV + b^2\int_\Omega (\Delta u)(\Delta\varphi)dV$$
$$= a^2\int_{\partial\Omega}\frac{\partial u}{\partial n}\varphi dS + 2b\int_{\partial\Omega}\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial t}\right)\varphi dS + b^2\int_{\partial\Omega}\Delta u\frac{\partial\varphi}{\partial n}dS - b^2\int_{\partial\Omega}\frac{\partial(\Delta u)}{\partial n}\varphi dS, \quad (19)$$

where φ is any test function. Therefore, in the usual weak formulation, the H^2 regularity for a solution is required; here H^k denotes the Sobolev space of order k. In this case, adequate smooth shape function of C^1 class defined on the whole computational domain are needed. However, in practice, it is quite challenging to find such a smooth shape function and unknown nodal variables as well as to formulate boundary conditions. In this paper, we adopt a mixed formulation of weak solution to ease the H^2 regularity into the H^1 regularity. So, we introduce not only an intrinsic variable $u(t,x)$ but also an intermediate variable $v(t,x) = \frac{\partial u}{\partial t}(t,x) - b\Delta u(t,x)$ as unknowns. By considering both unknowns $u(t,x)$ and $v(t,x)$, we reduce the modified wave equation with diffusion effects to a coupled system of inhomogeneous diffusion equations:

$$\begin{aligned}\frac{\partial u}{\partial t} - b\Delta u &= v\\ \frac{\partial v}{\partial t} - b\Delta v &= a^2\Delta u\end{aligned} \quad (20)$$

Eq.(20) is a coupled system of reaction-diffusion equations for two components, in which the right-hand sides v and $a^2\Delta u$ are interpreted as reaction terms. Before the formulation of the finite element method, we will now summarize the initial boundary value problem of the modified wave equation (20).

Theorem 3: *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. Then, there exists a unique solution of the initial boundary value problem for the system of inhomogeneous diffusion equations:*

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) - b\Delta u(t,x) &= v(t,x) \quad \text{in} \quad \mathbb{R}_+\times\Omega.\\ \frac{\partial v}{\partial t}(t,x) - b\Delta v(t,x) &= a^2\Delta u(t,x) \quad \text{in} \quad \mathbb{R}_+\times\Omega.\end{aligned} \quad (21)$$

Moreover the solution $u(t,x)$ and $v(t,x)$ have the eigenfunction expansion for the Laplacian Δ as follows:

$$\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix} = \sum_{n=1}^{\infty}\begin{pmatrix}u_n(t)\\ v_n(t)\end{pmatrix}e_n(x), \quad (22)$$

where $e_n(x)$ is the eigenfunction of the Laplacian Δ and the coefficients $\begin{pmatrix}u_n(t)\\ v_n(t)\end{pmatrix}$ are determined by the following ordinary differential equation.

$$\frac{d}{dt}\begin{pmatrix}u_n\\ v_n\end{pmatrix} = \begin{pmatrix}-b\lambda_n & 1\\ -a^2\lambda_n & -b\lambda_n\end{pmatrix}\begin{pmatrix}u_n\\ v_n\end{pmatrix}, \quad (23)$$

with λ_n as the eigenvalue for the Laplacian Δ. In particular, the coefficients $\begin{pmatrix}u_n(t)\\ v_n(t)\end{pmatrix}$ can be obtained by

$$\begin{pmatrix}u_n(t)\\ v_n(t)\end{pmatrix} = \exp(-b\lambda_n t)\begin{pmatrix}\cos(a\sqrt{\lambda_n}t) & \frac{\sin(a\sqrt{\lambda_n}t)}{a\sqrt{\lambda_n}}\\ \frac{-\sin(a\sqrt{\lambda_n}t)}{a\sqrt{\lambda_n}} & \cos(a\sqrt{\lambda_n}t)\end{pmatrix}\begin{pmatrix}u_n(0)\\ v_n(0)\end{pmatrix}. \quad (24)$$

Some discussions for Q-factor and the modified equation are left in [15]. Now we give a mixed formulation of the modified wave equation. To simplify some symbols, we introduce the following bilinear forms.

$$A(f,g) = a^2 \int_{\Omega} (\nabla f) \cdot (\nabla g) dV. \tag{25}$$

$$B(f,g) = b \int_{\Omega} (\nabla f) \cdot (\nabla g) dV. \tag{26}$$

$$M(f,g) = \int_{\Omega} f g dV. \tag{27}$$

Using these symmetric and positive definite bilinear forms, the mixed formulation can be written as follows:

$$M\left(\frac{\partial u}{\partial t}, \varphi\right) + B(u,\varphi) - M(v,\varphi) = b \int_{\partial\Omega} \frac{\partial u}{\partial n} \varphi dS, \quad (\forall\, \varphi : \text{test function}). \tag{28}$$

$$M\left(\frac{\partial v}{\partial t}, \psi\right) + A(u,\psi) + B(v,\psi) = b \int_{\partial\Omega} \frac{\partial v}{\partial n} \psi dS + a^2 \int_{\partial\Omega} \frac{\partial u}{\partial n} \psi dS, \quad (\forall\, \psi : \text{test function}). \tag{29}$$

2.2 A smoothing scheme with FDM

As a scheme for smoothing numerical solution of the wave equation, we propose a method for constructing numerical solutions for the modified wave equation with diffusion effects. The discretization of the modified equation scheme is as follows:

A smoothing scheme with FDM

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\delta t^2} = a^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\delta x^2} + 2b\left(\frac{v_i^n - v_i^{n-1}}{\delta t} - \frac{b}{2} \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\delta x^2}\right), \tag{30}$$

$$v_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\delta x^2}, \tag{31}$$

$$v_i^{n-1} = \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{\delta x^2}. \tag{32}$$

Although the modified wave equation is an artificial, heuristically derived equation, its theoretical solution was shown to have good mathematical properties in Section 2.1.1. By applying von Neumann stability analysis to this discretization, we can show that the modified equation scheme is able to remove short-wavelength components from the solution. In fact, for the modified equation scheme, high-wavenumber components have smaller amplitude ratio than low-wavenumber ones. This means that the modified equation scheme has a selective smoothing effect on short-wavelength components. Some discussions for a relationship between the amplitude ratio and the wavenumber are left in [1].

2.3 A smoothing scheme with mixed FEM

A discretization of Eq.(28) and Eq.(29) is given by the Galerkin method. As a shape function, we can choose the usual multilinear function on every element because only first order derivatives in spatial direction appear in Eq.(28) and Eq.(29). Let $N^p(x) \in H_0^1(\Omega)$ be a shape function for a nodal point p and N be the number of nodal points. We define a mass matrix $\mathbb{M} \in \mathbb{R}^{N\times N}$ and a diffusion matrix $\mathbb{D} \in \mathbb{R}^{N\times N}$ as follows.

$$\mathbb{M} = (m_{pq}) = \left(\int_{\Omega} N^p N^q dV\right). \tag{33}$$

(34)

$$\mathbb{D} = (d_{pq}) = \left(\int_{\Omega} \nabla N^p \cdot \nabla N^q dV \right).$$

Then, a mixed FEM for the modified wave equation is derived as follows:

$$\mathbb{M} \frac{d\mathbb{u}}{dt} + \mathbb{B}\mathbb{u} = \mathbb{M}\mathbb{v} + \mathbb{f}_1, \tag{35}$$

$$\mathbb{M} \frac{d\mathbb{v}}{dt} + \mathbb{B}\mathbb{v} = -\mathbb{A}\mathbb{u} + \mathbb{f}_2, \tag{36}$$

where $\mathbb{A} = a^2\mathbb{D}, \mathbb{B} = b\mathbb{D}$ and $\mathbb{f}_1$ and $\mathbb{f}_2$ are derived from the right-hand of Eq.(28) and Eq.(29). Our aim in this paper is to develop a stable scheme for seismic wave propagation simulations. In general, seismic wave propagation simulations require huge number of grids. Maeda et al., for example, conducted the wave propagation simulation by using a model with 22 billion grids [2]. So, the implicit time integration scheme, which uses a matrix solver, is not preferable. In this paper, we propose a semi-implicit time integration scheme which does not use a matrix solver.

$$\mathbb{M} \frac{\mathbb{u}^{n+1} - \mathbb{u}^n}{\Delta t} + \mathbb{B}\mathbb{u}^n = \mathbb{M}\mathbb{v}^{n+1} + \mathbb{f}_1^n. \tag{37}$$

$$\mathbb{M} \frac{\mathbb{v}^{n+1} - \mathbb{v}^n}{\Delta t} + \mathbb{B}\mathbb{v}^n = -\mathbb{A}\mathbb{u}^n + \mathbb{f}_2^n. \tag{38}$$

By considering eigenvalue problems of the discretized equation, we can show that the semi-implicit time integration scheme has essential properties for numerical stability and dispersion. For instance, it turns out that the semi-implicit scheme damps the amplitude of eigenmodes more rapidly for large eigenvalue. Detailed discussions are left to [15].

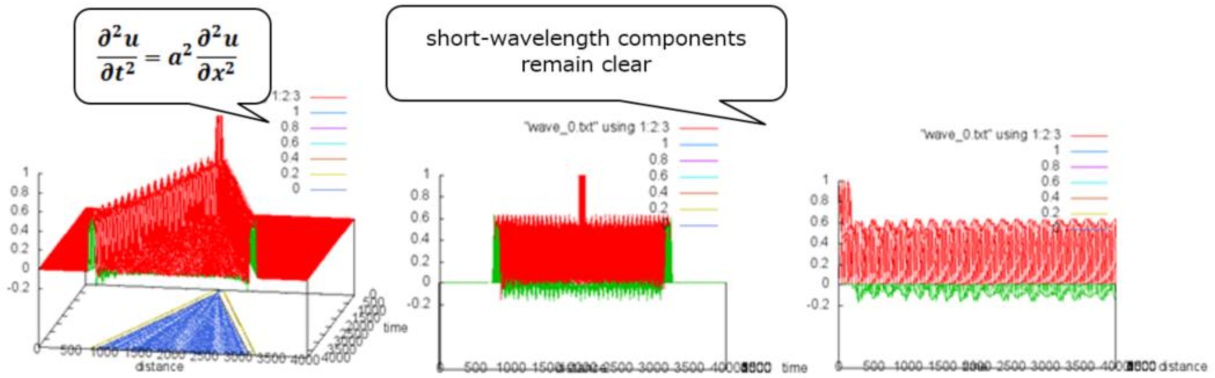
3. Simulations

3.1 Simulations with FDM

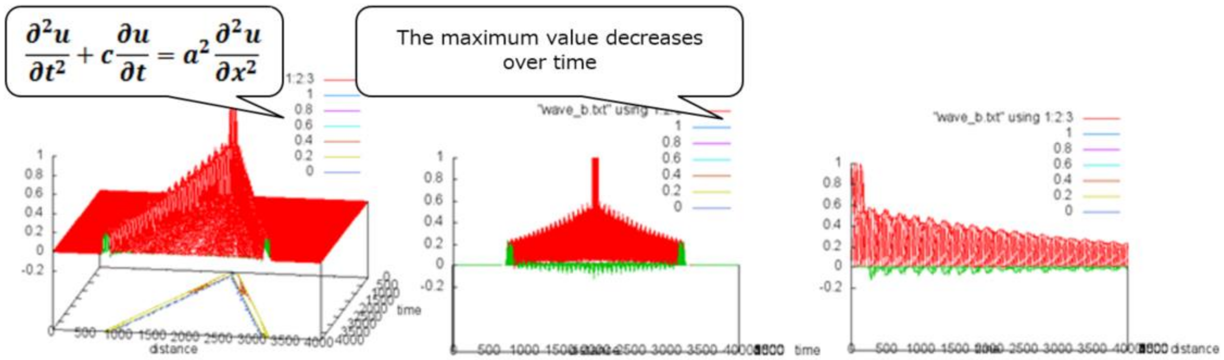
Simple numerical experiments are conducted on the initial value problem for the one-dimensional wave equation. To investigate numerical solutions that included a sufficient number of short-wavelength components, we take the initial value to be a rectangular distribution with a small support. The height of the rectangular distribution used for the initial value is 1.0 and the wave propagation speed was 0.3. Plots of the time evolution of the spatial distribution are shown in Fig.1. Here, the damping coefficient in the damping scheme was $c = 0.0005$ and the parameter b in the modified equation scheme was 0.04. From the results of these numerical experiments, we found the following.

- For the standard scheme, the short-wavelength components are clear and overshoot near discontinuities.
- For the damping scheme, the short-wavelength components remain clear and the maximum value decreases over time.
- For the modified equation scheme, the short-wavelength components are removed.

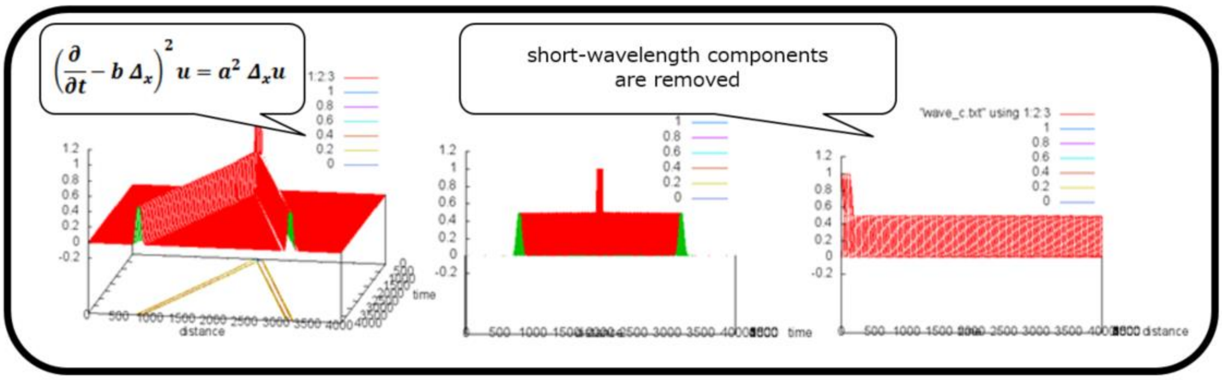
From these results, it is clear that the modified equation scheme contributes to the removal of short-wavelength components, but the damping scheme does not. The damping scheme may increasingly underestimate the maximum value over time and cannot reasonably estimate the characteristics of the actual phenomenon. Because the correction terms added in the modified equation scheme are simple, it can easily be applied to the two- or three-dimensional wave equation and also to general seismic wave equations. See [1] for simulations of the two-dimensional wave equation and a system of seismic wave equations.



(a) Standard scheme (for comparison)



(b) Damping scheme (for comparison)



(c) Modified equation scheme (our proposed scheme)

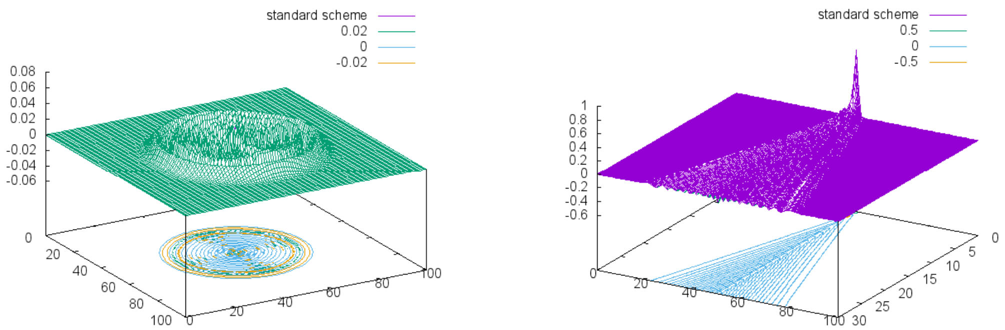
Fig. 1 – Time evolution of the numerical solutions in the one-dimensional case with FDM

3.2 Simulations with FEM

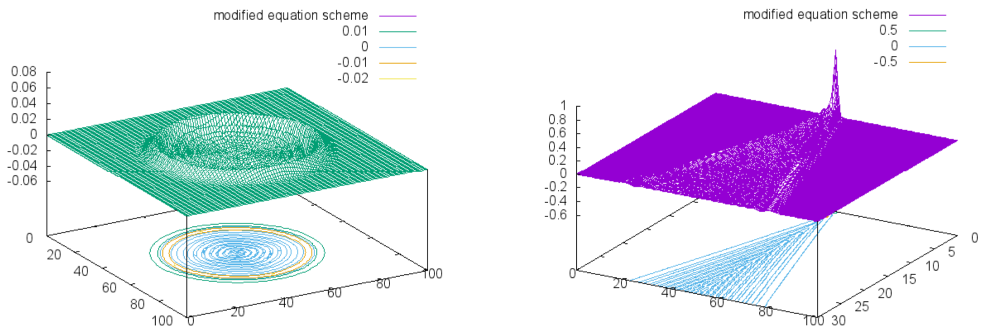
By applying the semi-implicit time integration scheme to the two-dimensional wave equation, we conduct the wave propagation simulation. To demonstrate that our proposed method exhibits selective filtering effects on high-wavenumber components, we compare the semi-implicit time integration scheme of our method with a central difference time integration scheme for the general wave equation. Here we note that our method with $b = 0$ reduces to the central difference time integration scheme for the general wave equation. In order to investigate numerical solutions including the sufficient number of short-wavelength components, the initial value distribution is set to be almost rectangular with small support. The computational domain is a square with length 100 and the support of the initial value distribution is arranged at the center of the square. The parameters used are shown in Table 1.

Table 1 – Simulation parameters

mesh size	$\Delta x = \Delta y = 1$
time step	$\Delta t = 0.1$
boundary condition	$u = 0$
propagation speed	$a = 1.0$



(a) standard scheme (for comparison)



(b) modified equation scheme (our proposed scheme)

Fig. 2 – Time evolution of the numerical solutions in the two-dimensional case with FEM

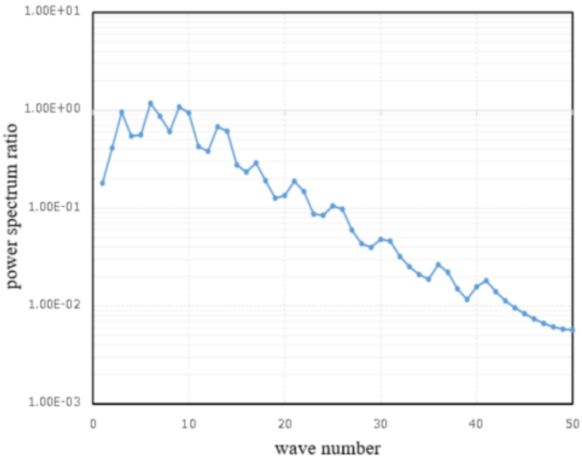


Fig. 3 – Power spectrum ratio between the proposed method and conventional method

Fig. 2 and Fig.3 show the simulation results. Fig.2 qualitatively shows that short-wavelength components are noticeable in the case of the conventional scheme. Fig.3 quantitatively shows that the proposed scheme filters high-wavenumber components more largely than the conventional scheme. This means that the proposed scheme selectively eliminates short-wavelength components.

For not only the wave equation but also the elastic wave equation, we can construct the same mixed FEM method with one in Section 2.3. The modified elastic wave equation with diffusion effects is as follows:

$$\left(\frac{\partial}{\partial t} - b\Delta\right)^2 u_i = \frac{\partial \sigma_{ij}}{\partial x_j} \quad (i = 1,2,3), \tag{39}$$

where the stress tensor σ_{ij} and strain tensor $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ satisfy the Hook's law $\sigma = \mathbb{C}^{el}: \varepsilon$ with fourth order elastic tensor $\mathbb{C}^{el} = 2G\mathbb{I}^{\mathrm{sym}} + \left(K - \frac{2}{3}G\right) I \otimes I$ (K is bulk modulus and G is shear modulus). We introduce intermediate variables $v_i = \frac{\partial u_i}{\partial t} - b\Delta u_i \;\; (i = 1,2,3)$ to construct a mixed finite element method of the modified elastic wave equation. Them, Eq.(39) can be rewritten as the following coupled system of inhomogeneous diffusion equations:

$$\begin{aligned} \frac{\partial u_i}{\partial t} - b\Delta u_i &= v_i \quad (i = 1,2,3). \\ \frac{\partial v_i}{\partial t} - b\Delta v_i &= \frac{\partial \sigma_{ij}}{\partial x_j} \quad (i = 1,2,3). \end{aligned} \tag{40}$$

When we let the bilinear forms $M(f,g) = \int f_i g_i dV$, $A(f,g) = \int \varepsilon(f): \mathbb{C}: \varepsilon(g) dV$,and $B(f,g) = \int (\nabla f_i) \cdot (\nabla g_i) dV$, then the weak formulation for Eq.(40) is derived as follows:

$$M\left(\frac{\partial u}{\partial t}, \varphi\right) + B(u, \varphi) - M(v, \varphi) = b \int_{\partial\Omega} \frac{\partial u_i}{\partial n} \varphi_i dS, \quad (\forall\, \varphi_i : \text{test function}). \tag{41}$$

$$M\left(\frac{\partial v}{\partial t}, \psi\right) + B(v, \psi) + A(u, \psi) = b \int_{\partial\Omega} \frac{\partial v_i}{\partial n} \psi_i dS + \int_{\partial\Omega} \sigma_{ij} n_j \psi_i dS, \quad (\forall\, \psi_i : \text{test function}). \tag{42}$$

Eq.(41) and Eq.(42) are the same form with Eq.(28) and Eq.(29). So, we can construct the same mixed FEM method with one in Section 2.3. Details for a derivation of the method and numerical experiments are left to [15].

4. Conclusion

We have mathematically described in detail some properties of a solution of the modified wave equation with diffusion effects. As a first step, we provide a smoothing scheme with the FDM for seismic wave propagation simulations. We show that the smoothing scheme based on the modified wave equation has some advantages. As a second step, we provide a smoothing scheme with the FEM for seismic wave propagation simulations. The modified wave equation includes fourth order derivatives and higher regularity is required in the usual weak formulation. We propose a mixed FEM of the modified wave equation to ease the regularity. The mixed formulation reduces the modified wave equation to a coupled system of inhomogeneous diffusion equations. Furthermore, we derive a discretization for a mixed FEM of the modified wave equation and construct a semi-implicit time integration scheme. Finally, we conduct wave propagation simulations by applying our method to the elastic wave equation. This numerical experiments reveal that the proposed scheme is effective for filtering short-wavelength components in seismic wave propagation simulations.

5. References

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