MATRIX METHOD OF LIMIT ANALYSIS AND GENERALIZED NORM OF STRESS FIELD OF BRACED FRAMES SUBJECTED TO SEISMIC LOADS

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Abstract

One of the popular limit design methods on framed structure is to utilize the lower bound theorem in limit analysis. In past studies, the first author has suggested the limit analysis of minimum-norm stress analysis for frames, which is a technique to choose a design stress field in equilibrium with ultimate seismic design loads. Furthermore, the designers' dissatisfaction to a stress field chosen is presented as the generalized norm based on the sum of weighted square stress.

The braced frames is usually adopted on steel building structure with high seismic resistant performance. During the seismic design procedures, the structural brace frame type is selected with some purpose. And the distribution of each resistant element is decided as a sharing ratio of horizontal resistant force. The horizontal force sharing ratio is defined on steel framed structure with braces, for example, the ratio is almost considered as 0.3-0.7 (the strength of steel brace vs whole frame). Also, the guarantee of failure mode formation is one of the purposes of structural seismic design goal.

This study investigates the relation of minimum-norm stress field of brace frames and design strategy, such as target sharing ratio and failure mode. From the previous studies, the generalized norm can accept the designers' strategies with the dissatisfaction weight. First, the relation of the generalized norm vs failure mode of braced frames are explained. Also, the relation of the sharing ratio is discussed.

To investigate the minimum norm stress analysis on braced frames, the single-story brace model is analyzed. From the analytical studies, to realize the target failure mode of frame, it is confirmed that the generalized norm is assumed with small weight on plastic hinge of failure mode. Also, in case of K-type brace frame, to prevent the unexpected failure mode, the dissatisfaction weight of brace member becomes important. That is, it is necessary to consider that the relation of strength of beam and brace member are considered. And the generalized norm with large weight at the middle of beam is needed.

Furthermore, to study the applicability of minimum norm stress analysis, the 4-story 4-span braced frame model is analyzed. Here, the K-type brace frame is considered. From the analytical studies, to realize the target failure mode of frame, the generalized norm with small weight on plastic hinge is considered.

Keywords: Braced frame, Limit analysis, Minimum-norm stress analysis, Horizontal force sharing ratio

1. Introduction

When a seismic design load or a required load carrying capacity on framed building structures are given, the building structural engineers' efforts are devoted to two kinds of procedures. One is the analysis procedure for getting a load carrying capacity of the frame and members which have already been proportioned. Another is a design procedure to proportion the members of the frame so that its load carrying capacity exceeds the required load. The former is often called "limit analysis", and the latter is called "limit design", and the structural engineers often make dual use of these two procedures.

One of the popular limit design methods is to utilize the lower bound theorem in limit analysis. In past studies, the first author has suggested the minimum-norm stress analysis for framed structure [1], which is a technique to choose a design stress field in equilibrium with ultimate seismic design loads. Furthermore, the designers' dissatisfaction to a stress field chosen is presented as the generalized norm based on the sum of weighted square stress.

Also, this optimization method of building frame is different from the ordinary optimization problem, such as minimum weight design and minimum cost design. A designer can draw any desirable situation about the resistance and the deformation rate of plastic portion. This method has a potential to accept an arbitrary designers' strategy in the ultimate state design of frames. Furthermore, the result of design stress field can be analyzed as an inverse-problem using minimum-norm stress analysis. Herein, the formulization of the minimum-norm stress analysis is explained. Also, the braced frame model is analyzed with minimum-norm stress analysis.

2. Theoretical Description of the Minimum-norm Stress Analysis

2.1 Minimum-norm Stress Analysis

From the standpoint of lower bound theorem, the following minimization problem is solved:

$$\text{to minimize } \{m\}^T[D]\{m\} \ \ (\text{generalized norm of member force}) \tag{1}$$

$$\text{subject to Equilibrium: } \{p\} = [C]\{m\} \tag{2}$$

Where, $\{m\}$ is the member force vector, $\{p\}$ is the load vector, $[C]$ is the connectivity matrix, $[C]^T$ is the transverse matrix of matrix $[C]$. $[D]$ is the square non-singular matrix called "dissatisfaction matrix", and a simple form of the matrix $[D]$ may be:

$$[D] = [diag.\ (D_i)] = \begin{bmatrix} \ddots & \cdots & 0 \\ \vdots & D_i & \vdots \\ 0 & \cdots & \ddots \end{bmatrix} \tag{3}$$

If a structural engineer wishes to suppress a member force m_i for some reason, a large value may be assigned to D_i. (i=1, 2…, n, n is the number of member force)

2.2 Minimum-norm solution of linear equation

The least squares member force vectors by $\{m^*\}$ for a given external force vector $\{p\}$ is denoted. It is assumed that the connectivity matrix $[C]$ (n rows by k columns) has following properties: $n \leqq k$, Rank $[C] = k$

To obtain $\{m^*\}$ is to minimize the generalized member force norm $\{m\}^T[D]\{m\}$ under the equilibrium condition ($\{p\}=[C]\{m\}$). According to Lagrange's method of indeterminate coefficients, this is identical to minimizing the following function G.

$$G = \frac{1}{2}\{m\}^T[D]\{m\} + \{\lambda\}^T\{\{p\} - [C]\{m\}\} \tag{4}$$

Where, $\{\lambda\}$ is the Lagrange's indeterminate coefficient vector.

Partial derivative of G with respect to m_i shall be zero at $m_i = m_i^*$. Then the following equation can be obtained:

$$\left.\frac{\partial G}{\partial M_i}\right|_{M_i^*} = 0 \ (i = 1, 2, \cdots, m) \tag{5}$$

$$[diag.\ (D_i)]\{M_i^*\} = [C]^T\{\lambda\} \tag{6}$$

Substituting this equation in the equilibrium equation, the following equation is obtained:

$$\{p\} = [C][diag.\ (1/D_i)][C]^T\{\lambda\} \tag{7}$$

Finally, the following equation is obtained:

$$\{m^*\} = [C^*]\{p\} \tag{8}$$

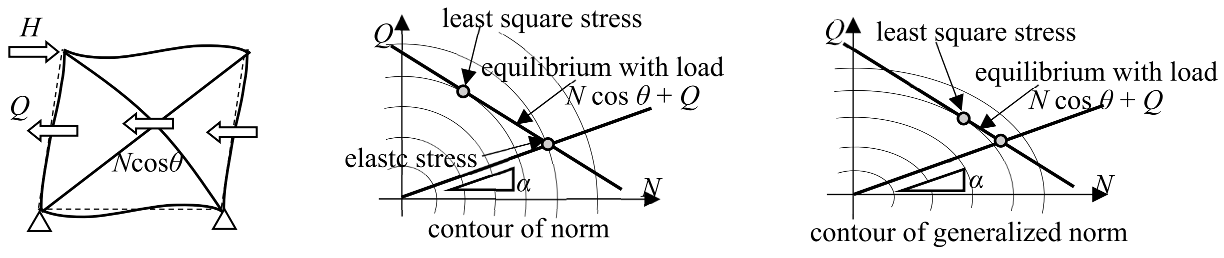
Where, $[C^*] = [diag.\ (1/D_i)][C]^T\left[[C][diag.\ (1/D_i)][C]^T\right]^{-1}$, this is the minimum-norm pseudo-inverse matrix.

2.3 Geometric expression of the minimum-norm stress field on braced frame

For example, considering the design problem of a single-story braced frame as shown in Fig.1 (a), where the restoring force is given by the restoring force of the frame Q and the axial force of brace N. First, to simplify the expression, the equilibrium equation is expressed as $H = N \cos\theta + Q$ (where, H is horizontal force, θ is angle of brace).

There are two stress fields for this problem, the elastic stress field and the least square stress field as shown in Fig.1 (b). The least squares stress field only depends on the frame size, brace angle and the design load, whereas the elastic stress field additionally depends on the stiffness parameter α.

Furthermore, to consider the generalized norm as $D_1 N^2 + D_2 Q^2$, the contour of the norm can be presented by an ellipse as shown in Fig.1 (c). Moreover, by using the generalized norm, the least square stress is obtained as shown in Fig.1 (c). It means that the combination of the shear force of the frame and the axial force of brace can be adjusted with the dissatisfaction weight D.



(a) simple braced frame model (b) relation of stress field and norm (c) expression of generalized norm

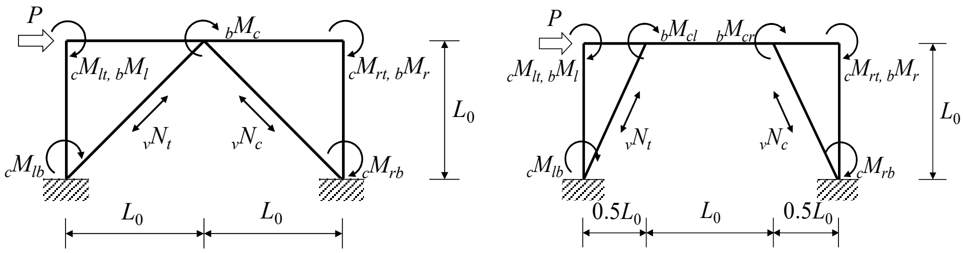
Fig. 1 – Geometric Expression of Minimum-norm Stress Analysis on Single Story Braced Frame

3. Analytical Study and Discussions on Single-story Braced Frames

3.1 Outline of the analytical study

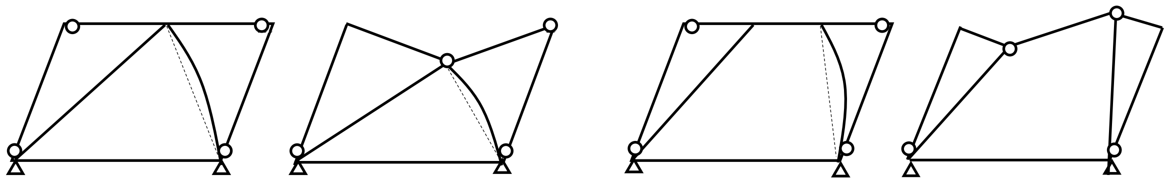
The single-story braced frame model as shown in Fig.2 is analyzed by minimum-norm stress analysis. Herein, the brace type (K-type and eccentrically type) is considered as a parameter.

Different failure modes of these braced frames as shown in Fig.3 are expected. During the ultimate seismic design procedure, predicting the formation of the desired failure mode is one of the purposes of seismic design. Notably, the restoring force characteristics of the brace during cyclic loading presents unstable behavior, such as buckling and fractures. After buckling occurs on the brace member, the restoring force of the frame is deteriorated significantly. Furthermore, when there is a difference in the restoring force of the tensile brace and compressive brace on the K-type brace frame, an additional shear force is generated on the beam. As a result, the beam at the joint of beam-brace yields, and the failure mode as shown in Fig.3 (a-2) is formed. Generally, once the restoring force is reduced, it is recovered to some extent during cyclic behavior because the buckled member is stretched. However, the failure mode (Fig.3 (a-2)) prevents the restoration of the compressive deformation of brace. Therefore, the structural designers make decisions to ensure the desired failure mode. Herein, the generalized norm that achieves the target stress field is investigated analytically, focusing on the expected failure mode.



(a) K-type brace (b) Eccentrically brace type

Fig. 2 – Analytical Model of Single-Story Braced Frame subjected to Horizontal Loads



(a-1) beam end (a-2) joint of brace- beam (b-1) beam end (b-2) joint of brace- beam

(a) K-type brace (b) Eccentrically brace type

Fig. 3 – Failure Mode of Single-story Braced Frame

3.2 Analytical results of the K-type braced frame model

The minimum-norm stress analysis on the K-type braced model is performed considering the following generalized norms:

a) A uniformly weight of the dissatisfaction matrix D_i=1.0 (i=1, 2, …, n) is assumed.

b) To consider the difference in the brace strength in the tensile and compressive direction, the generalized norm with a large dissatisfaction weight for the brace of the compression side (Dv_c =10) is assumed.

c) Also, to consider the difference in the brace strength in the tensile and compressive direction, the generalized norm with a small dissatisfaction weight for the brace of tensile side (Dv_t =0.1) is assumed.

d) To consider the plastic hinge formation on the center of beam, a large weight at the end of beam (D_{be} =10) is assumed, concurrently, the small weight (Dv_t =0.1) and large weight (Dv_c =10) are assumed.

e) To prevent the beam-center failure mode, a large weight at the joint of the beam and the brace (D_{bc} =10) is assumed, concurrently, the small weight (Dv_t =0.1) and large weight (Dv_c =10) are assumed.

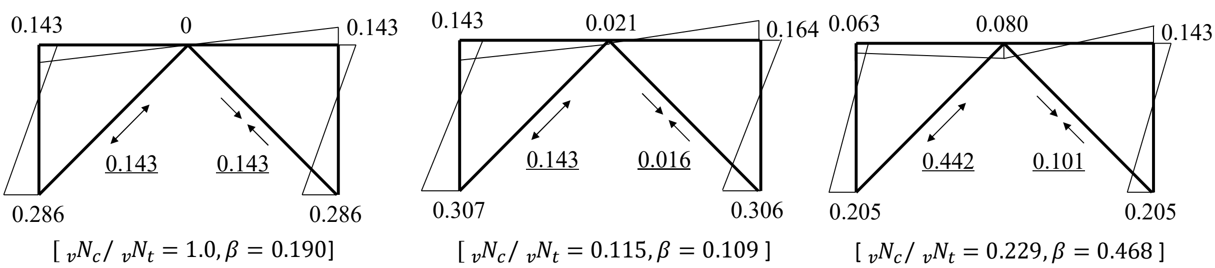
f) To consider the exposed-type column base with low strength, a large weight at the column base (D_b =10) is assumed, concurrently, the small weight (Dv_t =0.1) and large weight (Dv_c =10) are assumed. This is the combination of any design strategy.

The results of minimum-norm stress analysis on K-type brace model are presented in Fig.4. Also, the ratio of the compression and the tension of brace ${}_vN_c/{}_vN_t$ is presented. And the sharing ratio of the restoring force of the frame and the horizontal element of axial force of brace β is shown. In later discussions, the failure mode is assessed by referring to the moment diagram of the beam, that is, by assuming the use of the beam of the same cross section, it can be determined by comparing the moment at the beam end and the joint.

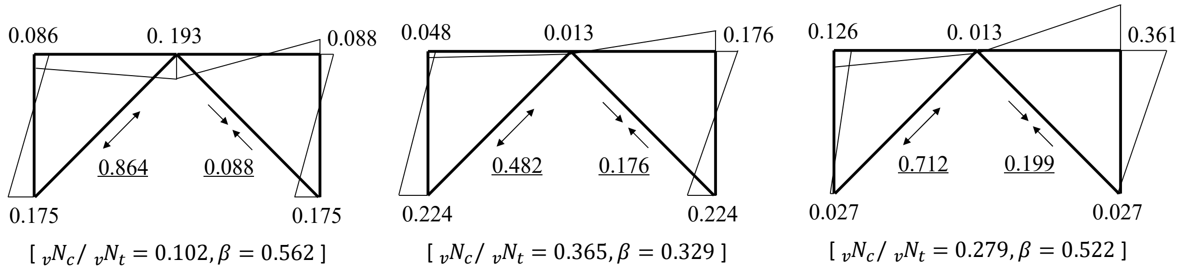
From the analytical results of Fig.4 (a), in case of uniformly weight of dissatisfaction, the most horizontal force is shared by frame (β=0.19). And the beam end failure mode is expected. Also, the same axial force is occurred on both brace member, it means that the tensile and compressive brace member are needed to keep the same strength.

From the results of Figs.4 (b), (c), in case of different dissatisfaction weight on each brace member, the moment at the joint of the beam-brace is occurred, because an additional shear force on the beam is derived from the difference between the restoring force of each brace. Furthermore, in case of large difference between the axial force of each brace member (case (c)), the moment at the joint of the beam-brace becomes larger than the moment at the beam end. Therefore, it is expected that the beam at the joint of beam-brace is yield. By comparing Figs.4 (b) and (c), the horizontal sharing ratio of the brace in case of (c) is larger than that in case (b), because the magnitude of the axial force of the brace increases because of the effect of the small dissatisfaction weight. However, in case (b), the magnitude of the axial force of the brace reduces and is restrained by the large dissatisfaction weight.

From the result of Fig.4 (d), in case of small dissatisfaction weight on the joint of beam-brace, the moment at the joint of beam-brace increases, and the moment at beam end slightly reduces. That is, it is expected that the beam at joint of beam-brace is yield (Fig.3 (b-2)). Furthermore, by comparing cases (b) and (c), the horizontal sharing ratio of the brace increases, which means that the magnitude of the axial force of the brace increases because of the effect of the small dissatisfaction weight.



(a) uniformly weight (D_i=1.0) (b) large D at comp. brace (Dv_c=10) (c) small D at tensile brace (Dv_t=0.1)



(d) small D at joint (D_{bc}=0.1) (e) large D at beam end (D_{be}=10) (f) large D at column base (D_{bs}=10)

Fig.4 Results of Minimum-norm Stress Analysis Considering Generalized Norms on K-type Brace Model

(unit: moment=PL_0, axial force (under bar) =P)

From the result of Fig.4 (e), in case of the large dissatisfaction weight on the joint of beam-brace, the moment at joint reduces more than the moment at both beam end, which means that the beam end failure mode is expected. Moreover, the horizontal force sharing ratio is balanced from the seismic design perspective.

From the result of Fig.4 (f), in case of the large dissatisfaction weight on column base, the moment significantly reduces, which shows that the magnitude of moment can be supported by the exposed type column base. However, this results in an increase of the moment at another member.

3.3 Analytical results of the eccentrically type braced frame model

The minimum-norm stress analysis on the eccentrically type braced model is performed while considering the following generalized norms:

a) A uniform weight of the dissatisfaction matrix D_i=1.0 (i=1, 2, …, n) is assumed.

b) To consider the difference of brace strength in the tensile and compressive directions, the generalized norm with a large dissatisfaction weight for the brace of the compression side (Dv_c =10) is assumed.

c) Also, to consider the difference in the brace strength in the tensile and compressive directions, the generalized norm with a small dissatisfaction weight for the brace of the tensile side (Dv_t =0.1) is assumed.

d) To consider the plastic hinge formation at the joint of the beam-brace, the generalized norm with a small dissatisfaction weight for the joint of the beam-brace (D_{bj} =0.01) is assumed.

e) Furthermore, to consider the plastic hinge formation at the joint of the beam-brace, the generalized norm with a large dissatisfaction weight at the beam end (D_{be} =10) is assumed. Concurrently, the generalized norm with a small dissatisfaction weight for the joint of beam-brace (D_{bj} =0.01) is assumed.

f) To consider the exposed-type column base with low strength, a large weight of the column base (D_b =50) is assumed, and the large weights of the brace (Dv_t = Dv_c =5) are assumed. Also, to consider the plastic hinge formation at the joint of the beam-brace, the generalized norm with a small dissatisfaction weight at the joint of beam-brace (D_{bj} =0.01) and with large weight for the beam end (D_{be} =10) are assumed. This is the combination of any design strategy.

The results of minimum-norm stress analysis on the eccentrically brace model are illustrated on Fig.5. The ratio of compression and tension of brace ${}_vN_c/{}_vN_t$ is presented too. And the sharing ratio of the restoring force of the frame and the horizontal element of the axial force of brace β is shown. In later discussions, the failure mode is assessed by referring to the moment diagram of the beam, that is, by assuming the use of a beam of the same cross section, it can be determined by comparing the moment at the beam end and that at the joint of the beam-brace.

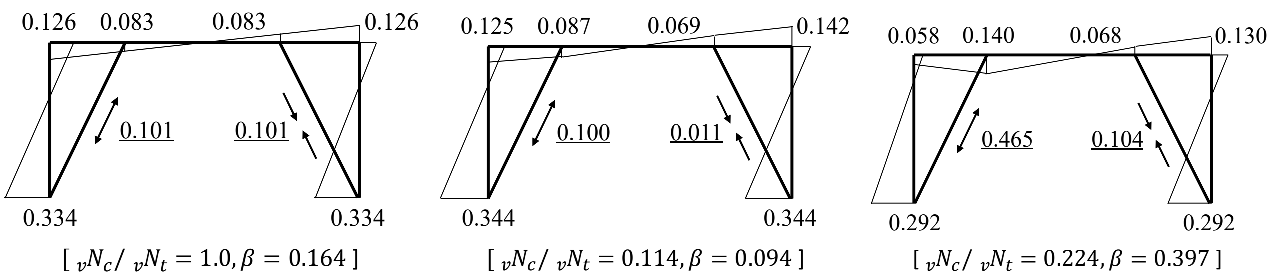
From the result of Fig.5 (a), in case of uniform weight of dissatisfaction, the most horizontal force is shared by the frame (β=0.16). And the beam end failure mode is expected such as Fig.3 (b-2). Moreover, the same axial force occurs on both brace members, which means that the tensile and compressive brace members must keep the same strength. However, if the deformation as shown Fig.3 (b-2) is expected, the compressive brace member should be significantly deformed. The compressive strength after buckling is deteriorated significantly, and the equilibrium is different from the result of Fig.5 (a) (small compressive strength is expected).

From the results of Figs.5 (b), (c), in case of different dissatisfaction weights on each brace member, the moment diagram of the beam is different form the Fig.5 (a), because an additional shear force on the beam is derived from the difference between the restoring forces of each brace. Furthermore, in case of a significant difference between the axial forces of each brace member (case (c)), the moment at the joint of the beam-brace on the tensile brace side increases more than the moment at the beam end. Therefore, it is expected that the beam at joint of beam-brace and beam end (right side) are yield. By comparing Figs.5 (b) and (c), the horizontal sharing ratio of the brace in case (c) is larger than the case of (b), because the magnitude of the axial force of the brace increases because of the effect of the small dissatisfaction weight. However, in case (b), the magnitude of the axial force of brace reduces, restrained by the large dissatisfaction weight. The expected deformation of the frame as shown in Fig.5 (c) satisfies the compatibility condition.

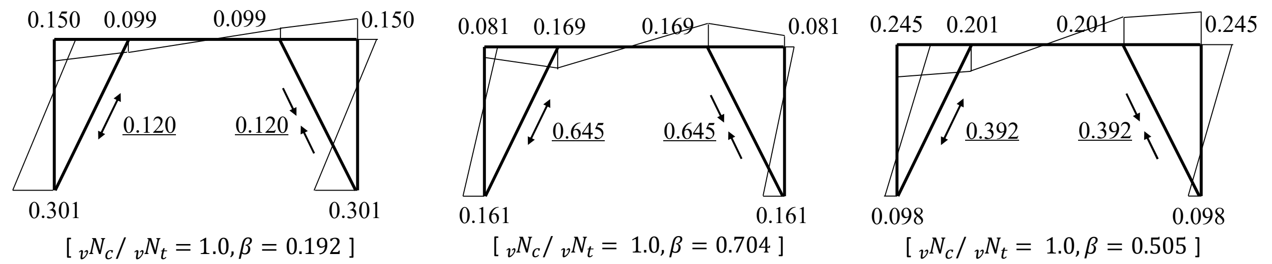
From the result of Fig.5 (d), in case of the small dissatisfaction weight on the joint of the beam-brace, the entire moment diagram shows almost same with Fig.5 (a). If the moment magnitude at joint of beam-brace, the additional shear is derived from the difference between tensile and compressive forces of the brace member.

From the result of Fig.5 (e), in case of the large dissatisfaction weight on the joint of the beam-brace, the moment at the joint reduces more than the moment at both beam ends, which means that the beam end failure mode is expected. Also, the horizontal force sharing ratio is balanced from the seismic design perspective.

From the result of Fig.5 (f), in case of the large dissatisfaction weight at the column base, the moment of the column base reduces, which shows that the magnitude of the moment can be supported by the exposed type column base. However, this results in an increase of the moment at another member is increasing. Moreover, the beam end failure mode is expected, however, the axial force of compressive brace member is the same as that of the tensile brace member, which means that the compatibility is not satisfied.



(a) uniformly weight (D_i=1.0) (b) large D at comp. brace (Dv_c=10) (c) small D at tensile brace (Dv_t=0.1)



(d) small D at joint (D_{bj}=0.01) (e) small D at joint (D_{bj}=0.01) (f) combined D

Fig.5 Results of Minimum-norm Stress Analysis Considering Generalized Norms on Eccentrically Type Brace Model (unit: moment=PL_0, axial force (under bar) =P)

3.4 Consideration and significant conclusions of the analytical study

From the above analytical studies, the minimum-norm stress analysis using the generalized norm can reflect the design strategy, and the structural designers assume their diverse ideas and strategies. Each stress field is satisfied by the equilibrium condition. Moreover, if the structural designer wants to reduce the magnitude of the moment and the axial force, a large dissatisfaction weight is considered on the member. Furthermore, the horizontal force sharing ratio of the frame vs. the brace can be adjusted with the generalized norm.

4. Analytical Study and Discussions on Multi-story Braced Frames

4.1 Outline of the analytical study

Herein, a seismic design problem of the middle-rise moment frame with a brace installed is analyzed as shown in Fig.6 (a). This frame is designed with the Japanese Seismic Design Code [2]. The moment strength M_p of column and beam are 723kN m, 592kN m respectively. The axial strengths of brace member are N_y=1,208kN (tensile) and N_c=246kN (compression). The ultimate state subjected to the seismic design load is calculated by the matrix method of limit analysis based on the lower bound theorem (Compact Procedure method [3]). Herein, two combinations of the brace strength are considered: 1) the same strength of brace in the direction of tensile and compressive ($N_c = N_y$), 2) the compressive strength of the post buckling strength ($N_c = 0.12N_y$). The results of limit analysis are presented in Fig.6. From the results of limit analysis (Figs.6 (b), (c)), two failure modes can be observed, such as beam end failure and beam center (joint of beam-brace) failure mode. Also, the load factors of ultimate horizontal strength are 0.143 (in case of $N_c = N_y$) and 0.096 (in case of $N_c = 0.12N_y$).

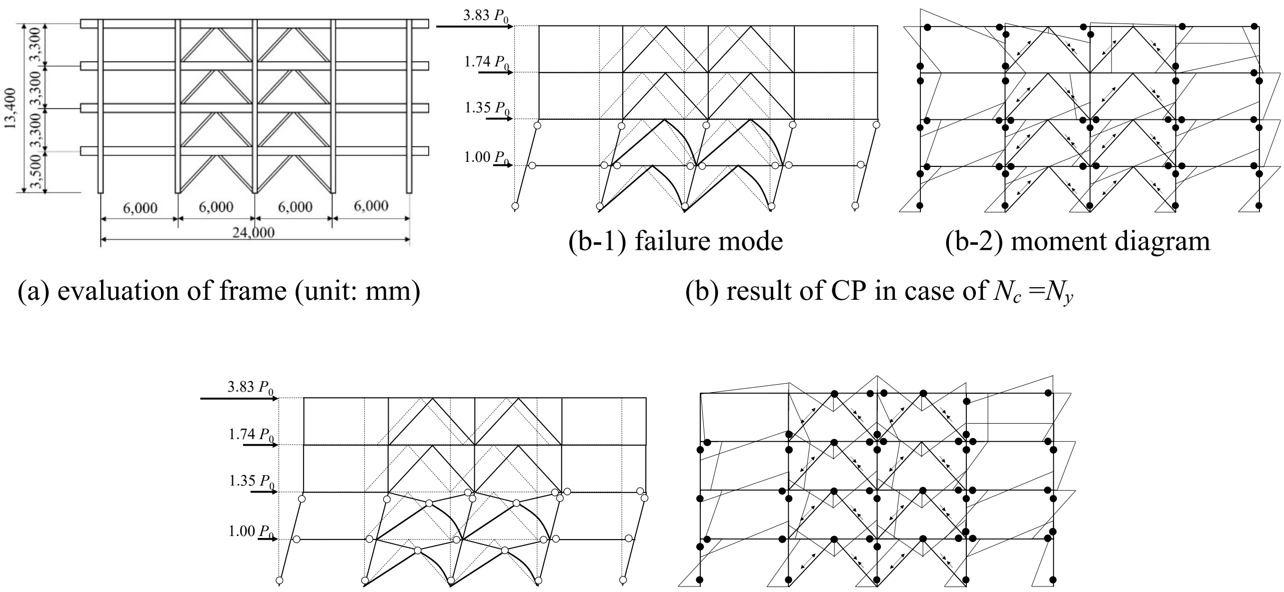
Herein, the minimum-norm stress analysis on a 4-story braced frame model is performed while considering the following generalized norms:

a) A uniform weight of the dissatisfaction matrix D_i=1.0 (i=1, 2, …, n) is assumed.

b) To consider the difference in the brace strength in the tensile and compressive directions, the generalized norm with a small weight for the brace of the tensile side (Dv_t =0.1), and a large weight for the brace of the compression side (Dv_c =10) are assumed.

c) To prevent the beam-center failure mode, a large weight on the joint of the beam and the brace (D_{bc} =10) are assumed, and the small weight (Dv_t =0.1) and large weight (Dv_c =10) are assumed.

d) To consider the exposed-type column base (strength is not large), a large weight of column base (D_b =10) is assumed, and the small weight (Dv_t =0.1) and large weight (Dv_c =10) are assumed.



(b-1) failure mode (b-2) moment diagram

(a) evaluation of frame (unit: mm) (b) result of CP in case of $N_c = N_y$

(c-1) failure mode (c-2) moment diagram

(c) result of CP in case of N_c =0.2N_y

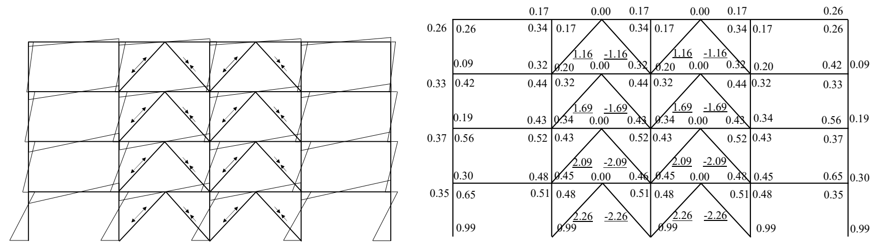
Fig.6 Analytical Model of 4-story Braced Frame and Results of Limit Analysis

(● presents the plastic moment, and all brace member yield)

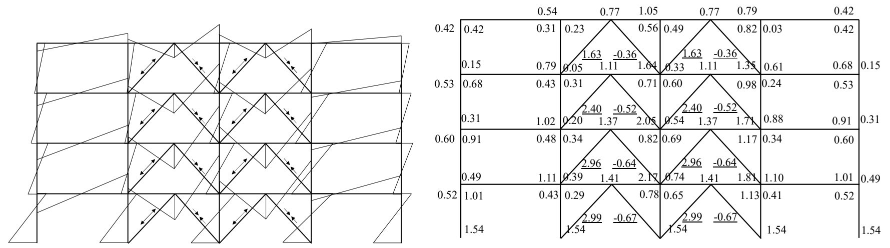
4.2 Analytical results and considerations of minimum-norm stress analysis with generalized norm

The minimum-norm stress analysis, considering the design strategy mentioned above, is performed, and the results of moment diagram is presented in Fig.7.

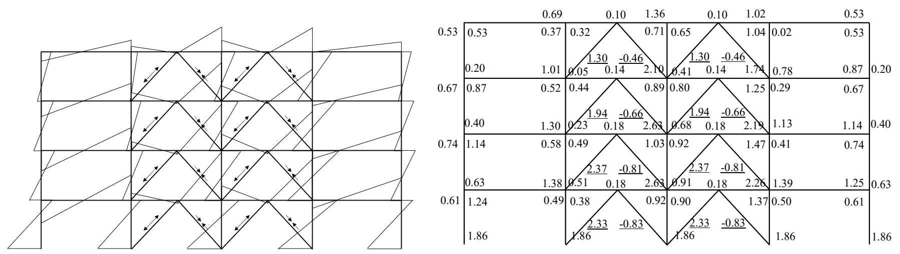
From the result of Fig.7 (a), in case of uniform weight, there is no moment at the joint of the beam-brace, which means that the beam end failure mode is expected. And from the result of Fig.7 (b), in case of different weight on the brace in each direction (case b), the moment at the joint of the beam-brace increases because of an additional shear force is generated on the beam, which means that the center of beam failure mode will be occurred. To prevent this mode, a large weight at the joint is considered (case c), the moment of the beam end becomes larger than the beam center, which means that the beam end failure mode will be occurred. To reduce the magnitude of moment, the large weight is considered.



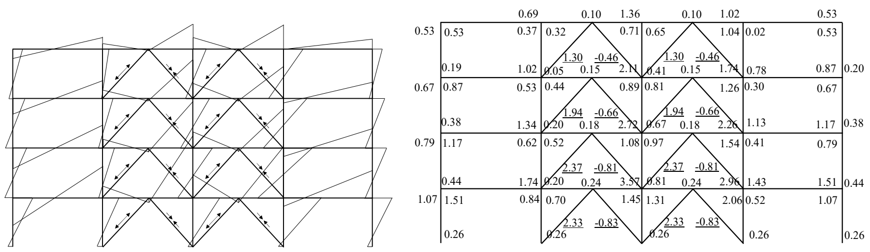
(a) case a, uniformly weight (D_i = 1.0)



(b) case b, different weight on brace (Dv_t =0.1, Dv_c =10)



(c) case c, large weight on joint (Dv_t =0.1, Dv_c =10, Db_c =10)



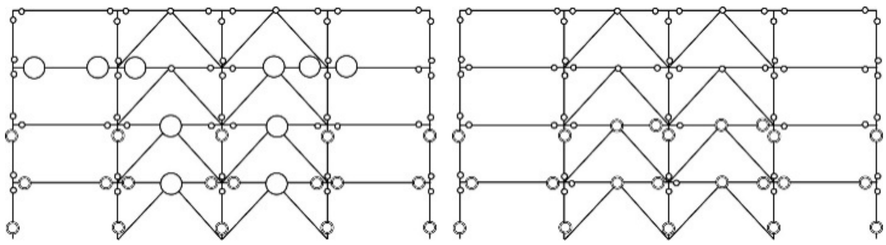
(d) case d, large weight on column base (Dv_t =0.1, Dv_c =10, D_b =10)

Fig.7 Results of Minimum-norm Stress Analysis with Design Strategy
(unit: moment P_0L_0, axial force (under bar) P_0)

4.3 Inverse-problem of the dissatisfaction weight

To investigate the dissatisfaction weight to reach the seismic design stress field, the inverse-problem is performed. Here, two compatibilities (failure mode) $\{\theta_p\}$ are considered as shown in Figs.6 (b), (c). Also, each moment diagrams of limit analysis are presented in Figs.6 (b), (c).

The distribution of the weight is calculated by Eq. $\{\theta_p\} = [C]^T\{\lambda\}$, and the results are presented in Fig.8. From the analytical results, it is confirmed that the large weight is obtained at the member of large moment on target stress field. Furthermore, the dissatisfaction weight at the plastic hinge becomes large.



(a) dissatisfaction weight 1 (b) dissatisfaction weight 2

Fig.8 Inverse-problem on Dissatisfaction Weight Related to Target Failure Mode
(the size of circle is proportion to dissatisfaction weight)

5. Conclusions

This paper studies the minimum-norm stress analysis, which is one of the limit analysis methods for a framed structure. Herein, the theoretical expression of the analytical method is explained. Also, this paper focuses on the limit analysis of braced frames.

First, a generalized norm is assumed, which is considered with various design strategies related to the seismic design procedure. And the moment considering the large dissatisfaction weight reduces because of the effect of equalization with a drawback.

Furthermore, this paper presents the inverse-problem for the dissatisfaction weight of the minimum-norm stress field on braced frames. Moreover, an analytical method is formulated. Herein, a 4-story 4-span K-type frame with a brace installed is analyzed. From the analytical results, the large dissatisfaction weight is considered at the portion of small moment. Also, the weight at plastic hinge becomes large.

6. Acknowledgement

We would like to thank Editage (www.editage.com) for English language editing.

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