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AN EFFICIENT STRUCTRAL SEISMIC RELIABILITY ESTIMATION THROUGH A RECURSIVE CALCULATION OF SYMBOLIC GRADIENTS

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Abstract

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The reliability of both new and old structures subjected to earthquakes is of significant interest to the earthquake engineering community. An estimate of the reliability of a structure allows engineers to examine if a designed structural system is safe and if existing structural systems require reinforcement. Both are pivotal questions for earthquake-prone regions. Although selecting and scaling existing earthquakes records to generate the structural responses is a common method of seismic assessment, adopting synthetic earthquake models that simulate earthquake time histories from random processes is becoming more common. The probabilistic structural responses, which are indicative of the seismic risk, can then be generated through sampling methods, such as Monte Carlo. The computational cost of simulating the response of a system for multiple generated time histories, as often required by a Monte Carlo approach, may be significant for complex nonlinear systems. A relatively recent method, the Design Point Excitation Method, has been suggested as a potentially more computationally effective alternative. The method utilizes the First Order Reliability Method, paired with stochastic earthquake generation models. A failure curve is approximated by a linear approximation at the Most Probable Point. The latter is located by solving an optimization problem, where by default gradient based methods such as line search are used. By default, such methods require a numerical calculation of the gradient using finite differences. This could potentially be insufficiently precise and too expensive as the structural system becomes either more complex or its dimension increases.

This paper introduces a recursive estimation of the gradient using an analytic formula based on the Euler integration rule. The gradient of the failure function is calculated iteratively at each iteration step. This is then used in the Design Point Excitation Method to calculate the reliability of a structure over time. An example of a multi degree-of-freedom nonlinear shear building is elaborated to demonstrate the performance of the method for studying the reliability of structures under earthquake excitations. A realistic synthetic earthquake model, which simulates the seismic non-stationarities both in frequency and time, suggested in (Rezaeian and Der Kiureghian, 2008) is adopted in this paper. Results show that the suggested recursive calculation of the gradient increases the efficiency of the method over the traditional use of finite differences and the commonly employed alternative of Monte Carlo Sampling.

1. Introduction

Predicting the probabilistic dynamic response of structures under earthquakes is an essential part of developing strategies for reducing or mitigating earthquake risk. Incremental Dynamic Analysis (IDA) [1, 2] is a well-known method of estimating seismic risk through structural analysis. However, the accuracy of the method may be affected by the ground motions selected for analysis. Though earthquake selection methods such as the Epsilon method [3] may improve the analysis, finding a significant number of earthquakes that would allow a probabilistic analysis remains a challenge.

To tackle the previous challenge, synthetic earthquake models are often employed for the probabilistic assessment of seismic demand as they can produce large numbers of samples. The spectral representation method [4], is widely used with different spectral density models proposed. Accounting for local site properties and a dominant frequency in the ground motion, stationary non-white process models were suggested by Kanai [5], Tajimi [6], Housner and Jennings [7], and Liu and Jhaveri [8]. Faravelli [9] formulated a stationary ground motion model with mu1timodal spectral density. To realize the time-varying intensity in real accelerograms, a series of time-modulating functions were then introduced to produce non-stationary ground motion models. These include time-modulated harmonics [10], the modulated filtered white noise process [11], and the filtered modulated stationary process [12]. Rezaeian and Der Kiureghian [13] developed a stochastic earthquake model constructed with parameters that have seismological properties (i.e. the type of fault mechanism, the seismic magnitude, the source to site distance, and the soil condition) for which regional seismological authorities have further information. The model also realizes the accelerograms properly using modulated filtered white noise processes.

Seismic structural response estimates can be achieved by incorporating synthetic ground motion models into a probabilistic framework. A simple and common approach is the Monte Carlo method [14], in which probabilistic response estimates are obtained by repeatedly drawing stochastic time histories and applying structural dynamic analysis. The distribution of a quantity of interest is obtained as the size of the sample set becomes adequately large. To reduce computational cost, one may also adopt reliability methods to achieve a similar objective. For example, the Design Point Excitation (DPE) method [15], which uses the fundamental theory of the First Order Reliability Method [16], estimates the probability of failure of structural systems due to earthquake ground motions.

In the DPE method, the reliability is approximated at one realization of the random input that has the highest joint probability. Finding this realization requires solving an optimization problem, which in turn involves gradient calculations. Thus, an efficient derivative calculation method is critical. Most of the existing research on the gradient computation of dynamical models are in the application of system control. A series of gradient computation methods are proposed in machine learning for data fitting using differential equations. The Finite Difference method [17] is the most general method. It is simple for applications but is time-consuming, especially for complex systems. Variations of the Finite Difference method can be found in [18]. There are also other methods that offer an estimate of the derivatives. For example, the adjoint method, first introduced in [19,20,26], approximates the gradient using the Lagrangian of the constraints implied by the dynamics. The stability of the adjoint equation for the underlying dynamical system may affect the application of the method. The Forward Sensitivity method [26], proposed in the context of modelling fMRI time series [19,21], is also widely used in derivative estimations. It employs the chain rule to reformulate the differential equations in terms of implicit derivatives.

In this paper, the Design Point Excitation (DPE) method is briefly introduced. A line search algorithm [22] is demonstrated to solve the optimization problem in the DPE method. The gradient calculations involved adopt the idea of the Forward Sensitivity method and apply together with an Euler approximation for the conversion of the continuous in time system to a discrete. The gradient is approximated by propagating the derivative term recursively. The method is illustrated using an example of assessing the reliability of a multi-degree of freedom system under an earthquake excitation. The reliability, as well as the derivative approximation computed using the method, are compared with the Monte Carlo method and a Numerical Derivative (ND) calculation method to show the accuracy and the efficiency.

2. The Design Point Excitation method (DPE)

The Design Point Excitation method (DPE) [15] is a reliability estimation method based on the First Order Reliability Method (FORM) [16]. The method estimates the probability that the system response does not exceed a given threshold when subjected to a random process excitation.

In seismic structural reliability problems, it is necessary to define the ground motion involved. The random process $F(t)$ is defined by $F(t) = s(U,t) = \int_0^t s(\tau)U(\tau)d\tau$, where U is a standard Gaussian process. After discretising $U(t)$ to independent components u_i, $i = 1,..,n$ the previous integral, the random process $F(t)$ is written as:

$$F(t) = \sum_{i}^{n} s_i(t)\, u_i \tag{1}$$

where $s_i(t)$ is an impulse response function associating u_i to $F(t)$.

Consider a nonlinear structural system defined by the state-space and measurement equations:

$$\dot{\boldsymbol{x}} = \boldsymbol{l}(\boldsymbol{x}, F) \tag{2}$$

$$y = \boldsymbol{h}(\boldsymbol{x}, F) \tag{3}$$

where $\boldsymbol{x}$ is a state variable, $\dot{\boldsymbol{x}}$ is the derivative of the state with respect to time t, $\boldsymbol{l}(\cdot,\cdot)$ is state-space function, which describe the dynamics of the structural system, and $\boldsymbol{h}(\cdot,\cdot)$ is the observation function that defines the output response of interest. In this paper, the output is assumed to be written as a linear term of the state variable $\boldsymbol{x}$.

Considering the problem of a structural response exceeding a threshold y_0 at the time τ, the Limit State Function (LSF), $G(\boldsymbol{u}, y_0, \tau)$, that separates the safety and the failure regions, is constructed as:

$$G(\boldsymbol{u}, y_0, \tau) = y_0 - y(\boldsymbol{u}, \tau) = 0 \tag{4}$$

where $\boldsymbol{u} = [u_1, \dots, u_n]$. $G(\boldsymbol{u}, y_0, \tau) > 0$ indicates the safety region.

The reliability is evaluated using a linearization of G at a specially chosen point, named the design point $\boldsymbol{u}^*$. This is the point on the limit state surface with the highest probability density in the $\boldsymbol{u}$ space. Since u_i are independent standard Gaussian variables, the design point is also found to be the point that minimizes the distance of the LSF to the origin. [15,16] Thus, the following optimization problem is formulated:

$$\boldsymbol{u}^*(y_0, \tau) = min\,\{\|u\|^2 | G(\boldsymbol{u}, y_0, \tau) = 0\} \tag{5}$$

The reliability at time instant τ is then approximated by $\mathrm{P}(y \le y_0) = \boldsymbol{\phi}(-\beta)$, where $\boldsymbol{\phi}(\cdot)$ denotes the standard normal cumulative probability function, the reliability index (β) defined as:

$$\beta = \alpha \cdot \boldsymbol{u}^* \tag{6}$$

$$\boldsymbol{\alpha} = -\frac{\boldsymbol{\nabla_u G}}{\|\boldsymbol{\nabla_u G}\|} \tag{7}$$

in which $\boldsymbol{u}^*$ is the design point, α is its direction vector, and $\boldsymbol{\nabla_u G}$ is the derivative of the limit state function with respect to $\boldsymbol{u}$ at the design point.

3. Line Search Optimization

Finding the design point, $\boldsymbol{u}^*$, is the critical step in this method. The design point is the solution to the optimization problem of eq. (5). Regarding to the starting guess of $\boldsymbol{u}$, existing research [15] has shown that for Single Degree of Freedom (SDOF) systems, the Mirror Image Excitation [15] can help in locating a warm starting guess of $\boldsymbol{u}$. The same idea can also be used for first mode dominated Multi Degree of Freedom (MDOF) systems [28], while for other MDOF systems, this idea is not applicable thus the starting guess of the optimization problem is arbitrarily chosen. Optimization methods such as the line search method [22, 23] or the Improved HL-RF algorithm [24] can be applied to find the exact solution. Here, the line search method is adopted due to its ease of use and wide applicability.

The design point excitation method (DPE) with line search iterations is conducted as follows:

Algorithm 1

1. Select a starting guess of $\boldsymbol{u}$.
2. Calculate the gradient $\boldsymbol{\nabla_u G}$ and the corresponding direction vector using $\widehat{\boldsymbol{\alpha}} = -\boldsymbol{\nabla_u G}/\|\boldsymbol{\nabla_u G}\|$.
3. Find β by solving $G(\beta\widehat{\boldsymbol{\alpha}}, \tau, x_0) = y_0 - y(\beta\widehat{\boldsymbol{\alpha}}, \tau) = 0$ using a root-finding algorithm.
4. Update the design point $\boldsymbol{u} = \beta \cdot \widehat{\boldsymbol{\alpha}}$.
5. Define the errors $e_1 = G(\boldsymbol{u}, \tau, y_0)/G(0, \tau, y_0)$ and $e_2 = 1 - \widehat{\boldsymbol{\alpha}} \cdot \boldsymbol{u}/\|\boldsymbol{u}\|$.
6. If $e_1 < 0.001$ and $e_2 < 0.001$, the algorithm ends, return β, and the reliability $\boldsymbol{\phi}(-\beta)$. Otherwise, continue to step 7.
7. Go back to step 2.

4. The Euler-Analytical estimate of the Derivative (AD)

In the line search method, derivative calculations of $G(\boldsymbol{u}, y_0, \tau)$ are needed repeatedly in each iteration step. Calculating the derivatives, $\boldsymbol{\nabla_u G}$, numerically using the Finite Difference method is computationally expensive, especially for higher-order nonlinear systems. In this section, a recursive method is introduced to calculate the derivatives symbolically using and Euler integration as an approximation of the discrete state-space equations of the state-space system from t_k to t_{k+1}.

In the optimization problem described by eq. (5), the derivative calculation of $\boldsymbol{\nabla_u G}$ is of interest. From eq. (4) this is:

$$\boldsymbol{\nabla_u G} = -\frac{d\, y(\boldsymbol{u}, \tau)}{d\boldsymbol{u}} \tag{8}$$

If the state space system is affine in the inputs and the measurement equations are linear with respect to the states x, the structural system defined by eq. (2)-(3) can be rewritten as

$$\dot{\boldsymbol{x}} = \boldsymbol{l}_a(\boldsymbol{x}) + \boldsymbol{l}_b(\boldsymbol{x}) \cdot F \tag{9a}$$

$$y = C \cdot \boldsymbol{x} \tag{9b}$$

If the sampling time step Δt is the selected, the structural responses are integrated using Euler integration at a sequence of equally spaced time steps where $t_k = k \cdot \Delta t$, $k = 0,1,\ldots,n, n = \frac{\tau}{\Delta t}$. For the purposes of approximating the gradients in eq. (8), the Euler integration is used to discretise eq. (9) in time [24, 19]:

$$\boldsymbol{x_{k+1}} \approx \boldsymbol{x_k} + \dot{\boldsymbol{x}}_{\boldsymbol{k}} \cdot \Delta t \tag{10a}$$

$$y_{k+1} \approx C \cdot (\boldsymbol{x_{k+1}}) \tag{10b}$$

where $\boldsymbol{x_k}, \dot{\boldsymbol{x}}_{\boldsymbol{k}}$ and y_k denote $\boldsymbol{x}(t_k)$, $\dot{\boldsymbol{x}}(t_k)$ and $y(t_k)$ respectively. $\boldsymbol{\nabla_u G}$ defined by eq. (8) can be obtained by evaluating the following equations, which differentiate eq. (10) with respect to u_is, recursively:

$$\frac{d\,\boldsymbol{x_{k+1}}}{du_i} = \frac{d\boldsymbol{x_k}}{du_i} + \frac{d\dot{\boldsymbol{x}}_{\boldsymbol{k}}}{du_i} \cdot \Delta t \tag{11a}$$

$$\frac{d\,y_{k+1}}{du_i} = C \cdot \left(\frac{d\,\boldsymbol{x_{k+1}}}{du_i}\right) \tag{11b}$$

eq.(9a) is substitute into eq.(11a) to obtain:

$$\begin{aligned} \frac{d\boldsymbol{x_{k+1}}}{du_i} &= \frac{d\boldsymbol{x_k}}{du_i} + \frac{d\left(\boldsymbol{l_a}(\boldsymbol{x_k}) + \boldsymbol{l_b}(\boldsymbol{x_k}) \cdot f_{t_k}\right)}{du_i} \cdot \Delta t \\ &= \frac{d\boldsymbol{x_k}}{du_i} + \left[\boldsymbol{\nabla}_{\boldsymbol{x_k}} \boldsymbol{l_a}(\boldsymbol{x_k}) \cdot \frac{d\boldsymbol{x_k}}{du_i} + \left[\boldsymbol{\nabla}_{\boldsymbol{x_k}} \boldsymbol{l_b}(\boldsymbol{x_k}) \cdot \frac{d\boldsymbol{x_k}}{du_i}\right] \cdot f_{t_k} + \boldsymbol{l_b}(\boldsymbol{x_k}) \cdot \frac{df_{t_k}}{du_i}\right] \Delta t \end{aligned} \tag{12}$$

where $\boldsymbol{\nabla}_{\boldsymbol{x_k}} \boldsymbol{l_a}(\boldsymbol{x_k})$, $\boldsymbol{\nabla}_{\boldsymbol{x_k}} \boldsymbol{l_b}(\boldsymbol{x_k})$ are calculated symbolically; $\frac{df_{t_k}}{du_i}$ can be obtained from the deterministic function eq.(1) that describes the excitation; and $\frac{d\boldsymbol{x_k}}{du_i}$ is the term calculated in the previous time step k. Thus, $\frac{d\boldsymbol{x_{k+1}}}{du_i}$ as well as $\frac{d\,y_{k+1}}{du_i}$ can be calculated recursively until the time step $t_{k+1} = \tau$. The vector $\frac{dy(\boldsymbol{u},\tau)}{du} = [\frac{dy(\boldsymbol{u},\tau)}{du_1}, \ldots, \frac{dy(\boldsymbol{u},\tau)}{du_n}]$ can be finally evaluated through recursion. Note that computing $\boldsymbol{\nabla}_{\boldsymbol{x_k}} \boldsymbol{l_a}(\boldsymbol{x_k})$, $\boldsymbol{\nabla}_{\boldsymbol{x_k}} \boldsymbol{l_b}(\boldsymbol{x_k})$ require the structural responses $\boldsymbol{x} = [\boldsymbol{x_1} \ \ldots \ \boldsymbol{x_\tau}]$ excited by $F(t)$ described in eq. (1). Euler integrations are not necessary in calculating $\boldsymbol{x}$ since good precisions are pursued. Generally, in this paper, the Runge-Kutta (2, 3) pair (ode23) [27] is adopted to calculate $\boldsymbol{x_k}$ which are then substituted in eq. (12) to obtain the gradients.

For implementations, the algorithm is summarized as the following:

Algorithm 2

Initialization: Set Δt. The algorithm starts at step $k = 0$, where $t_0 = 0s$ and $\frac{dx_k}{du_i} = \frac{dx_0}{du_i} = 0$.

1. Calculate the structural responses $\boldsymbol{x} = [\boldsymbol{x_1} \ \ldots \ \boldsymbol{x_\tau}]$ excited by $F(t)$ described in eq. (1) with the updated $\boldsymbol{u}$ obtained from step 4 of *Algorithm 1*.

2. Derive $\nabla_x \boldsymbol{l_a}(\boldsymbol{x})$, $\nabla_x \boldsymbol{l_b}(\boldsymbol{x})$, $\frac{\partial F}{\partial u_i}$ from the state-space model and the excitation model.
3. At step $k+1$, time step $t_{k+1} = (k+1)\Delta t$, evaluate $\nabla_{x_k} \boldsymbol{l_a}(\boldsymbol{x_k})$, $\nabla_{x_k} \boldsymbol{l_b}(\boldsymbol{x_k})$, $\boldsymbol{l_b}(\boldsymbol{x_k})$ and $\frac{df_{t_k}}{du_i}$.
4. Calculate $\frac{dx_{k+1}}{du_i}$ and $\frac{d\,y_{k+1}}{du_i}$ using eq. (12) and eq. (11b).
5. If $t_{k+1} = \tau$, the algorithm ends, output $\frac{dy_\tau}{du_i}$. Otherwise consider $k = k+1$, replace the expression of $\frac{dx_{k+1}}{du_i}$ and $\frac{dy_{k+1}}{du_i}$ with $\frac{dx_k}{du_i}$ and $\frac{dy_k}{du_i}$, and return to step 3.

If a higher-order complex dynamical example is considered, the above algorithm can be further developed with replacing the Euler approximation in eq. (10) by more precise integration algorithms to improve the precision of the approximation. This will be further discussed in the corresponding journal version of this paper. The method presented in *Algorithms 1* and *2* will be referred to as the Design Point Excitation method with an Euler-Analytical estimate of the Derivative (DPE-AD).

5. Example – Multi-degree-of-freedom oscillator with Bouc-Wen nonlinearity subjected to earthquake excitation.

Three different approaches, the Design Point Excitation method with an Euler-Analytical estimate of the Derivative (DPE-AD), the DPE method with Numerical estimate of the Derivative (using the Finite Difference method) (DPE-ND) [25], and the Monte Carlo method (MC), are applied to estimate the seismic reliability of a building and ultimately in order to show the reliability estimation performance of the DPE-AD method. All three approaches use the Runge-Kutta (2, 3) pair (the ode23 implementation in Matlab) [27] to integrate the structural responses.

Consider the following 4-story shear building shown in Fig.1 is subjected to ground motions, and the reliability of the top story displacement is of interest. The masses, stiffnesses, and damping coefficients are specified in Fig. 1. Suppose the first story of the shear building has a Bouc-Wen nonlinearity, and the rest of the stories are linear elastic. The 4-story shear building model can be represented by the following differential equations:

$$\mathbf{M}\ddot{\boldsymbol{x}}(\boldsymbol{t}) + \boldsymbol{C}\dot{\boldsymbol{x}}(\boldsymbol{t}) + \boldsymbol{K}(\alpha\boldsymbol{x}(\boldsymbol{t}) + (1-\alpha)\boldsymbol{Z}(\boldsymbol{t})\,) = \boldsymbol{m}\,f(t) \tag{13a}$$

$$\dot{z}(t) = -\gamma|\dot{x}(t)||z(t)|^{n-1}z(t) - \eta|z(t)|^n\dot{x}(t) + A\dot{x}(t) \tag{13b}$$

where **M**, **C** and **K** are the mass matrix, damping coefficient matrix and the stiffness matrix of corresponding linear shear building, respectively and $\boldsymbol{m} = [m_1, m_2,\ m_3, m_4]$. The state vectors $\boldsymbol{x}(\boldsymbol{t}) = [x_1(t), x_2(t), x_3(t), x_4(t)]$ and $\boldsymbol{Z}(\boldsymbol{t}) = [z(t), 0, 0, 0]$ are defined. α controls the degree of hysteresis. $\alpha = 0.2$ is assumed for the first story and the rest of the stories are assumed to be linear ($\alpha = 1$). $z(t)$ is the Bouc-Wen hysteretic term governed by eq. (13b), where the parameters $n = 3$, $A = 1$ and $\gamma = \eta = 1/(2\sigma_0^n)$ are selected, in which $\sigma_0^2 = 0.127$. The earthquake signal is the only excitation considered in this problem, and the probability of the top floor displacement exceeding $3mm$ at 15 seconds is of interest. Thus, we have the LSF $G(\boldsymbol{u}, 15\ s) = 3\ mm - x_4(\boldsymbol{u}, 15\ s) = 0$.

For the stochastic earthquake excitations, the seismic model developed by Der Kiureghian et al. [13, 14] is adopted. The earthquake excitation is simulated by a modulated filtered white noise:

$$f(t) = q(t) \cdot \sum_{i=1}^{n} h(t - t_i)w_i \tag{14}$$

in which the white noise $w_i = \sigma u_i$, $\sigma = \sqrt{2\pi S_0 \Delta t}$ describes the intensity of the white noise input and $S_0 = \frac{1}{m^2}$. A time step of $\Delta t = 0.01s$ is chosen and $n = \frac{t}{\Delta t}$. The modulating function is given by $q(t) = 0.03 \cdot t^2 \cdot \exp(-0.2t)$. And the unit-impulse response function of the filter is defined by

$$h(t) = -\left[\frac{\omega_f}{\sqrt{1-\zeta_f^2}} \sin\left(\omega_f t\sqrt{1-\zeta_f^2}\right) + 2\zeta_f \omega_f \cos\left(\omega_f t\sqrt{1-\zeta_f^2}\right)\right] \exp(-\zeta_f \omega_f t) \qquad (15)$$

where $\omega_f = 5\pi$ rad/s and $\zeta_f = 0.4$.

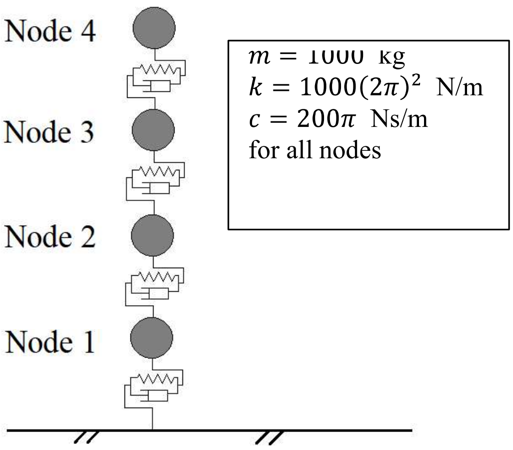

Fig. 1 - Four stories shear building model

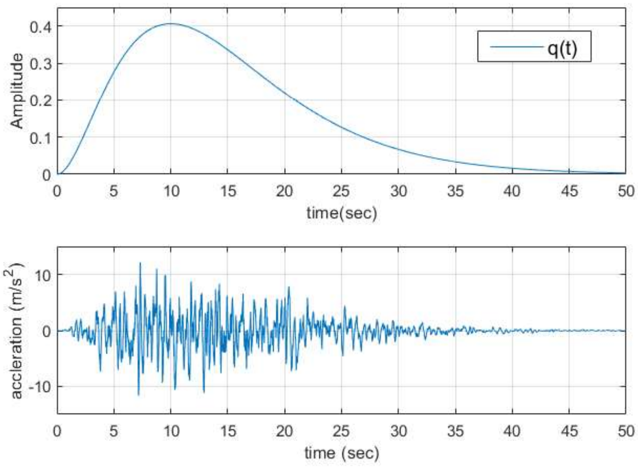

Fig. 2 - The Modulating function and a sample realization of the ground motion model

Since the structural modes are closely spaced for this system, a warm starting guess is not simple to locate using the idea of the Mirror Image Excitation as indicated in [28]. In this example the starting guess for $\boldsymbol{u}$, is set as the Mirror Image Excitation [15] defined for a SDOF system with mass, stiffness and damping coefficient equal to m, k and c respectively. Note that this SDOF system is not equivalent to the properties of the MDOF system, thus this starting guess is not necessarily a good approximation of the design point $\boldsymbol{u}^*$.

Using the Design Point *Algorithms 1* and *2*, two iterations are required in the line search algorithm and the reliability index β is -2.10. The corresponding probability of exceedance is $\mathrm{P}(x_4 \geq 3mm) = 1 - (\phi(-\beta))$ =1.8 %. The same results are obtained by the DPE-AD and DPE-ND methods.

For the MC method, results with proper accuracy are obtained by using an adequately large sample set of generated time histories. The probability of exceedance is estimated by $\mathrm{P}(x_4 \geq 3mm) = \frac{No.of\ samples\ of\ x_4 \geq 3mm}{Total\ No.of\ the\ simulated\ samples}$. Fig. 3 shows the evolution of the estimated $\mathrm{P}(x_4 \geq 3mm)$ obtained from the MC method with an increasing number of time histories. After investigations, if a maximum accepted relative error of 3% is considered, 20000 time histories are sufficient for this example, and $\mathrm{P}(x_4 \geq 3mm)$ is found to be 1.815%. The relative estimation error of the DPE method to the estimate of the MC is about 1% which is acceptable. Thus, the reliability estimations are adequate for both the DPE-AD and DPE-ND methods.

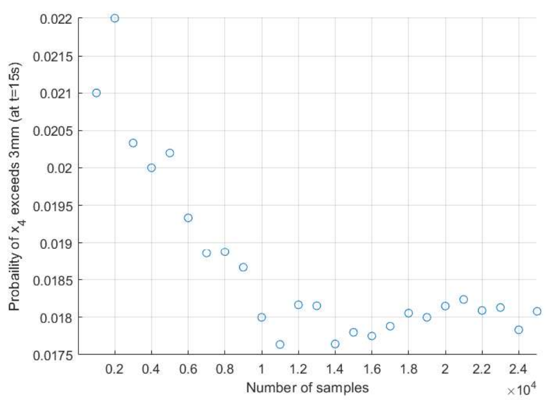

Fig. 3 – The evolution of $P(x_4 \geq 3mm)$ approximated by MC with an increased sample set

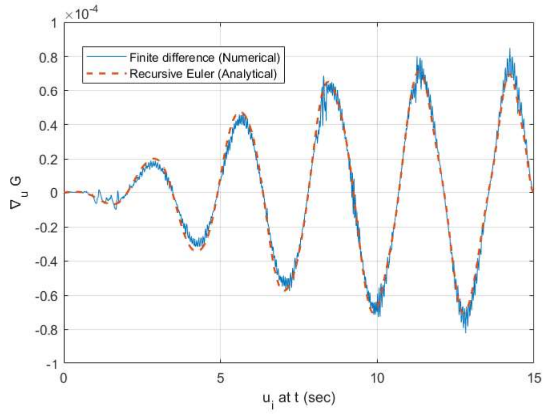

Fig. 4 - $\boldsymbol{\nabla_u G}$ approximations at the first iteration step

Regarding to the computational cost, the DPE-AD, the DPE-ND and the MC method takes about 17 seconds, 0.6 hours and 2.5 hours respectively. More specifically, if we focus on the derivative approximation process, Fig. 4 shows an example of the derivative the LSF with respect to u_i s, $\boldsymbol{\nabla_u G}$, calculated by the Euler-Analytical estimate of the Derivative (AD) and Numerical estimate of the Derivative (using the Finite Difference method) (ND) at the first iteration step. For the Finite Difference method, the central differences are considered and the following equation:

$$\boldsymbol{\nabla_u G} = \left[-\frac{dx_4(\boldsymbol{u}, 15s)}{du_i}\right] = \left[-\frac{x_4(\boldsymbol{u} + \Delta u_i,\ 15s) - x_4(\boldsymbol{u} - \Delta u_i, 15s)}{2 \cdot \Delta u_i}\right] \tag{16}$$

is adopted. Each calculation of $\boldsymbol{\nabla_u G}$ at different iterations of the line search *Algorithm 1* using the ND method requires about 9.8 minutes, while using the AD method takes only 4.6 seconds. As explained in section 2.3, since the calculation of $\boldsymbol{\nabla_u G}$ is required in each iteration of *Algorithm 1*, a more efficient gradient approximation method can substantially save the computational cost. The Numerical estimate of the Derivative (using the Finite Difference method) (ND) is time-consuming because the calculation of $x_4(\boldsymbol{u}, 15s)$, is demanded repeatedly as the inputs are changed according to the Finite Difference scheme in eq. (16). The number of integrations required increases proportionally with the number of terms considered in the vector $\boldsymbol{u}$, n**,** according to eq. (16). Table 1 is a comparison of the number of structural dynamic analysis required in each of the methods, which is an indication of the computational costs. N_1 and N_2 are the number of iterations needed in *Algorithm 1* to find the exact design points $\boldsymbol{u}^*$using DPE-AD and DPE-ND respectively. Note that N_1 and N_2 can be different because the derivative $\boldsymbol{\nabla_u G}$ affects the predictions of *Algorithm 1*.

Table 1 Comparison of the number of structural dynamic analysis required.

Method	Number of structural dynamic analyses required
Monte Carlo	The number of the generated time histories
DPE-ND	N_1* 2n
DPE-AD	N_2

Regarding to the accuracy of the approximated derivative $\boldsymbol{\nabla_u G}$, the one calculated by *Algorithm 2* is more precise than the one estimated using the Numerical estimate of the Derivative (the Finite Difference method) (ND). As seen in Fig. 4, it produces a smoother shape, while the approximated $\boldsymbol{\nabla_u G}$ according to ND (the Finite Difference method) is noisy. As a result, the root finding algorithm, step 3 in *Algorithm 1* (the line search procedure), may require longer computation time. It is also expected that more iteration steps may be required in *Algorithm 1* when the approximated derivative $\boldsymbol{\nabla_u G}$ lacks precision. Selecting a proper difference Δu_i in eq. (16) is an extra step and is important for approximating $\boldsymbol{\nabla_u G}$ in the Finite difference method. An improper Δu_i s may lead to an inaccurate approximation of $\boldsymbol{\nabla_u G}$, and this may lead to divergence in finding the exact design point. As shown in Fig. 4, even when the increment is chosen carefully, the numerical approximations can be noisy. Variations of the Finite Difference methods and other Numerical estimate of the Derivative (ND) algorithms [25] that produce $\boldsymbol{\nabla_u G}$ with higher precision can be adopted, but they will be computationally more expensive and consequently weaken the computational advantage of the DPE method compare to the Monte Carlo method.

6. Conclusion

The results in this paper demonstrate that the DPE-Analytical Derivative (DPE-AD) and DPE-Numerical Derivative (DPE-ND) methods both estimate the reliability accurately with good precision and are more efficient than the default MC method for the example studied. The Euler-Analytical estimate of the Derivative (AD) is more accurate and computationally cheaper compared to the Numerical estimate of the Derivative (ND) (using the Finite Difference method). Since the derivative approximations are critical in finding the exact solution and are required to compute repeatedly in *Algorithm 1* (the line search procedure) in the DPE method, a better derivative calculation method can save the computational cost in the reliability approximations. The DPE-AD method is demonstrated in the problem studied to be substantially more robust than the MC method for calculating the reliability at a specific time instance.

The method can be further developed in the direction of finding structural reliability under synthetic earthquake models that have parameters that are of random nature with known distribution. Such random earthquake model parameters can be included as extra terms of the design point $\boldsymbol{u}$ and the analytical/symbolic derivative can be derived. The computational cost to solve these problems will be extremely expensive using the Monte Carlo method, thus the computational advantages of the DPE method are expected to be more evident.

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