EFFECTIVE MASS TENSOR TO IDENTIFY THE INERTIAL MASS AND PRINCIPAL AXIS OF A MULTIDIMENSIONAL MODE

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Abstract

In the classical modal analysis, it is well known that effective modal masses of one-dimensional multi-degrees-of-freedom systems (e.g., simple models for multistory buildings) are independent of how the modes are normalized. But those of multidimensional systems are dependent on the orientation of the coordinate system in which they are described.

In this paper, the concept of the effective mass tensor (EMT) is introduced. The linear momentum and kinetic energy of a multidimensional mode are keys to define EMT. EMT is one of the simplest second-order tensors in three-dimensional physical space with eigenvalues of one positive and two zeroes (or one zero in two-dimensional space). The one positive eigenvalue of EMT means the physical inertial mass, which is a scalar, of the equivalent single-degree-of-freedom (ESDF) system. This inertial mass is also an invariant of EMT. The principal axis of EMT corresponding to the positive eigenvalue represents the movable direction of the ESDF system. EMT is a complete representation of the inertial mass and direction of the ESDF system in a multidimensional space. A usual "effective modal mass" can be recognized as a component of EMT.

Through an example of a two-dimensional structure, characteristics of EMT are confirmed.

Keywords: Modal analysis, Equivalent single-degree-of-freedom system, Inertial mass, Principal axis, Invariant

1. Introduction

An eigenvalue problem for a linear multi-degree-of-freedom (MDF) system gives the natural frequencies as scalars. But the amplitude of the corresponding natural modes are not fixed, only the shapes (i.e., ratios among the degrees of freedom) of them are determined. On the other hand, the so-called effective masses (also referred to as the effective modal mass or equivalent mass) are fixed and independent of how the modes are normalized [1]. This is true as long as we are dealing with MDF systems with only one-dimensional vibration such as the shear-building idealization of structural frames. However, for a system with multi-dimensional vibrations, the values of effective masses vary according to the rotation of the coordinates. As shown in an example in this paper, even if we fix the direction of coordinates and change the orientation of the system considered, this variation remains essentially the same.

As one of the recent studies concerning effective mass and direction, Fujii [2] applied an equivalent linearization technique to nonlinear seismic responses of asymmetric RC-structures focusing on changes in the effective mass by coordinate rotation. He determined the modal principal direction in which the effective mass is maximized. He also said that the method to identify the principal direction is the same as that by Gonzalez [3]. The author et al. studied the vibration characteristics of uplift motions of buildings as a piecewise linear system [4]. Two-dimensional motions of a shear-beam model in an uplift phase were analyzed by the classical modal analysis. The equivalent mass and direction of the corresponding single-degree-of-freedom (SDF) system to each mode was defined based on the direction of its momentum and the amount of its kinetic energy.

Although the models studied by Fujii [2] are different from those of the author et al. [4], common features about effective masses and modal directions are seen : (a) the sum of the effective masses for two axes orthogonal to each other is independent of the orientation of coordinate system in which models are described, and (b) the change of the value of an effective mass is a function of the square of a direction cosine. From these features, should the effective mass be regarded as a tensor?

In this study, while checking the idea of the previous study [4], the concept of the effective mass tensor is introduced as a way of showing the relationship between the effective mass and the direction as a general rule. In fact, usual effective masses are just components corresponding to arbitrarily selected coordinates. Moreover, a modal mass should be recognized as a scalar. It is one of the principal invariants which is independent of the rotation of the coordinates as well as a usual mass. Although the method of determining the modal principal axis by Fujii [2] is different from that by the author et al. [4], it is shown that both are the same based on a tensor property. Through an example of a two-dimensional structure, characteristics of the effective mass tensor are confirmed.

Note that the contents of this paper have already been published in Japanese [5].

2. Effective mass tensor

This section discusses how effective mass tensor is defined. In this study, based on the linear analysis conditions, large displacements and finite rotations are not addressed.

2.1 Notations and symbols

With the goals of not being dependent on a particular coordinate system, promoting intuitive understanding, and clearly distinguishing between sums and products, direct notation of vectors (geometric vectors) is used mainly in this paper instead of index/components notation. For the fundamental theories and arithmetic rules, please refer to a textbook [6]. The geometric vectors are written in bold and lowercase letters, while the second-order tensors are displayed in bold and capital letters. A bold **e** is a unit vector. The dots over the symbols indicate differentials with respect to time *t*. The subscript numbers are added to variables of elements with inertia (referred to as inertia elements) by Roman capital letters (*I*, etc.); mode numbers are indicated by Roman lowercase letters, such as *j*, etc., and Greek lowercase letters, such as α, etc. indicate components of Cartesian coordinates corresponding to the dimensions ($\alpha = X, Y, Z$). $\mathbf{e}_\alpha$ is the base vector for

Cartesian coordinates. Matrix form indicating coordinate components of a tensor is displayed in parentheses. The dot product of the same vector is expressed as $\mathbf{a}\cdot\mathbf{a} = \mathbf{a}^2$, and the symbol "≡" means definition. The norm of a vector $\mathbf{a}$ is defined by $\|\mathbf{a}\| \equiv \sqrt{\mathbf{a}^2}$. The tensor product of $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a}\otimes\mathbf{b}$.

2.2 Effective mass tensor

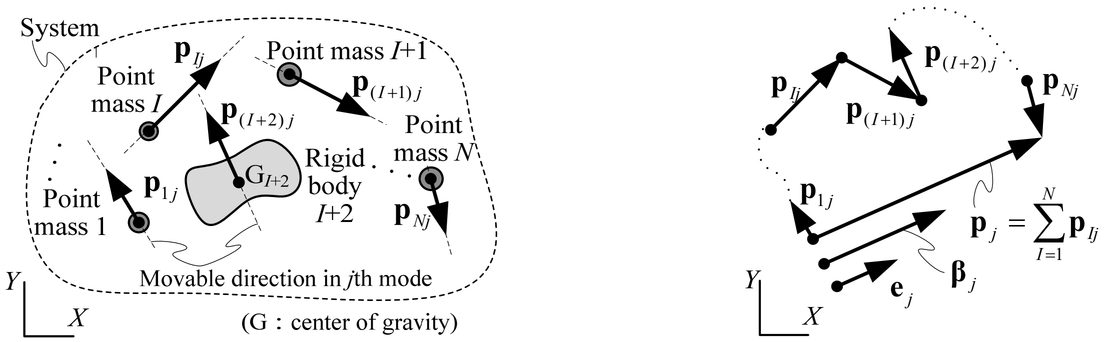
In the seismic engineering field, as the equation of motion is often explained as (mass)×(acceleration)=(force), little attention is paid to momentum; however, the left side of this equation shows the rate of temporal variation in momentum. By being conscious of momentum as a single physical quantity rather than separating it into mass and velocity, it becomes easier to interpret the effective mass in a physical sense.

In jth mode by the classical modal analysis (also in a generalized SDF system [1]), the linear momentum vector $\mathbf{p}_j$ (see Fig.1) and the kinetic energy T_j as a whole can be expressed as follows:

$$\mathbf{p}_j = \sum_I \mathbf{p}_{Ij} = \sum_I \mathbf{s}_{Ij}\dot{q}_j = \mathbf{s}_j\dot{q}_j = \boldsymbol{\beta}_j M_j \dot{q}_j = \boldsymbol{\beta}_j p_j \tag{1}$$

$$2T_j = 2\sum_I T_{Ij} = \sum_I M_{Ij}\dot{q}_j^{\,2} = M_j\dot{q}_j^{\,2} = p_j^{\,2}/M_j \tag{2}$$

where $\mathbf{p}_{Ij}$ is the momentum vector of the inertia element I, $\mathbf{s}_{Ij} \equiv \mathbf{p}_{Ij}/\dot{q}_j$ is a vector showing the relationship between $\mathbf{p}_{Ij}$ and the generalized velocity $\dot{q}_j$, $\mathbf{s}_j = \sum_I \mathbf{s}_{Ij} = \mathbf{p}_j/\dot{q}_j = \boldsymbol{\beta}_j M_j$, $\boldsymbol{\beta}_j$ is the vector which has components equal to the usual participation factors along the coordinate system used, $M_j = \sum_I M_{Ij}$ is the generalized mass of the jth mode, q_j represents the generalized coordinates (generalized displacement), and $p_j \equiv \partial T_j/\partial\dot{q}_j = M_j\dot{q}_j$ is the generalized momentum, T_{Ij} is the kinetic energy of inertia element I, and $M_{Ij} \equiv 2T_{Ij}/\dot{q}_j^{\,2}$. The functions of time t are $\mathbf{p}_j$, $\mathbf{p}_{Ij}$, q_j, p_j, T_j, T_{Ij} and the other variables are constants with no variations. In distributed mass systems (continuums), one can substitute infinitesimal small portions for inertia elements, and their sums can be substituted for integrals.



(a) A system with point masses (particles) and rigid bodies (b) Momentum vectors

Fig. 1 – Schematic diagram of a system and momentum vectors

From Eq. (1), usual participation factors can be recognized as coefficients representing the proportional relationship between the momentum vector $\mathbf{p}_j$ (or its coordinate components) and the generalized momentum p_j. Usually, when considering seismic responses, it is common to implement a participation factor as a coefficient for multiplying ground acceleration; however, note that Eq. (1) is not limited to seismic responses. Eq. (1) is true not only for lumped mass systems but also for systems with rigid bodies and/or finite elements.

$\mathbf{p}_j$ and $\boldsymbol{\beta}_j$ are in parallel as shown in Fig. 1(b). With a parallel unit vector $\mathbf{e}_j$ for both, new generalized coordinates q_{j0} can be defined as follows:

$$q_j \equiv \boldsymbol{\beta}_j \cdot q_{j0}\mathbf{e}_j = \|\boldsymbol{\beta}_j\| q_{j0} \tag{3}$$

Substituting Eq. (3) into Eq. (1) and(2),

$$\mathbf{p}_j = \overline{\mathbf{M}}_j(\dot{q}_{j0}\mathbf{e}_j) = \overline{M}_j(\dot{q}_{j0}\mathbf{e}_j) = p_{j0}\mathbf{e}_j \tag{4}$$

$$2T_j = (\dot{q}_{j0}\mathbf{e}_j)\cdot\overline{\mathbf{M}}_j(\dot{q}_{j0}\mathbf{e}_j) = \overline{M}_j\dot{q}_{j0}^{\ 2} = p_{j0}^{\ 2}\big/\overline{M}_j \tag{5}$$

where $p_{j0} \equiv \partial T_j \big/ \partial \dot{q}_{j0} = \overline{M}_j \dot{q}_{j0}$ is the conjugate generalized momentum with the new generalized coordinates q_{j0}. Eq. (4) shows that changing the scale of the generalized coordinates by Eq. (3) matches the size of the generalized momentum p_{j0} with the norm of the momentum vector $\mathbf{p}_j$. Thus, we can see from Eq. (4) that $\|\mathbf{p}_{j0}\| = |p_{j0}|$. $\overline{\mathbf{M}}_j$ in Eq. (4) and (5) has been defined as follows:

$$\overline{\mathbf{M}}_j \equiv M_j\boldsymbol{\beta}_j \otimes \boldsymbol{\beta}_j = \overline{M}_j\mathbf{e}_j \otimes \mathbf{e}_j = \frac{\mathbf{p}_j \otimes \mathbf{p}_j}{2T_j} \tag{6}$$

In this paper, the second-order symmetric tensor defined by Eq. (6) is referred to as the jth "effective mass tensor" (EMT). The scalar $\overline{M}_j$ in Eqs. (4)(5)(6) has been defined as follows:

$$\overline{M}_j \equiv M_j\boldsymbol{\beta}_j^{\ 2} = \frac{\mathbf{p}_j^{\ 2}}{2T_j} = \mathrm{tr}\overline{\mathbf{M}}_j \tag{7}$$

The scalar $\overline{M}_j$ is one of the principal invariants (i.e., trace) of the tensor $\overline{\mathbf{M}}_j$. From Eq. (7), we can understand the reason why the sum of three (or two) components of usual effective masses is independent of the rotation of the coordinate system, while each value of components is dependent. Because the sum is equal to the trace of $\overline{\mathbf{M}}_j$ and it is a principal invariant. A physical explanation of $\overline{M}_j$ can be given as follows:

$$\overline{M}_j = \frac{(\text{norm of the momentum vector of the } j\text{-th mode})^2}{2\times(\text{kinetic energy of the } j\text{-th mode})} \tag{8}$$

As shown in Eq. (6), the EMT can be expressed as the product of the scalar $\overline{M}_j$ and the operator $\mathbf{e}_j \otimes \mathbf{e}_j$. The action of the latter operator is a projection. The scalar $\overline{M}_j$ means the inertial mass and the operator indicates the axis of the movable direction of the point mass (particle). The reason for adding the term "inertial" is that the effective mass is related to the momentum and kinetic energy as shown in Eq. (6). In other words, effective mass corresponds only to movable masses in the system. If an arbitrary vector $\mathbf{f}$ acts on the jth mode, the vector projection of $\mathbf{f}$ is extracted as $(\mathbf{e}_j \otimes \mathbf{e}_j)\mathbf{f} = (\mathbf{f}\cdot\mathbf{e}_j)\mathbf{e}_j$ by the definition of the tensor product; the operator can be interpreted as geometric constraint conditions (boundary conditions) for the effective mass of the jth mode.

Fig.2 shows the equivalent single-degree-of-freedom (ESDF) system of the jth mode. ESDF in three (or two) dimensional space can be expressed by the corresponding EMT completely. Also, note that the new generalized coordinate q_{j0} is equal to the physical displacement of the ESDF.

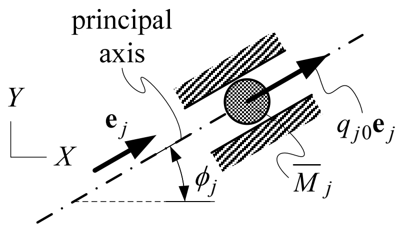


Fig. 2 – Equivalent single-degree-of-freedom (ESDF) system

2.3 Principal value and principal axis

With the arbitrary unit vector **e**, $\overline{M}_j^{\,e}$ is defined as follows:

$$\overline{M}_j^{\,e} \equiv \mathbf{e}\cdot\overline{\mathbf{M}}_j\mathbf{e} = \overline{M}_j(\mathbf{e}\cdot\mathbf{e}_j)^2 = \frac{(\mathbf{e}\cdot\mathbf{p}_j)^2}{2T_j} \tag{9}$$

$\overline{M}_j^{\,e}$ is proportional to the square of direction cosine $\mathbf{e}\cdot\mathbf{e}_j$. A physical explanation of $\overline{M}_j^{\,e}$ is given as follows:

$$\overline{M}_j^{\,e} = \frac{(\text{projection(or component) of the momentum vector of the } j\text{-th mode})^2}{2\times(\text{kinetic energy of the } j\text{-th mode})} \tag{10}$$

Let $\mathbf{e}_\alpha$ be a base vector expressing specific coordinates. In case $\mathbf{e}$ is substituted for $\mathbf{e}_\alpha$, $\overline{M}_j^{\,e}$ is usually called the effective mass. The extremal value of $\overline{M}_j^{\,e}$ is referred to as the principal value of $\overline{\mathbf{M}}_j$, and the direction of **e** at that time is referred to as the principal axis (principal direction). The principal values and principal axes of a second-order symmetric tensor are the same as the eigenvalues and eigenvectors.

As we can see from Eq. (6), the eigenvalues of EMT $\overline{\mathbf{M}}_j$ are $\overline{M}_j$ and zero (in case of 3D, the latter is a duplicated eigenvalue). The corresponding eigenvectors are $\mathbf{e}_j$ (or $\mathbf{p}_j$ or $\boldsymbol{\beta}_j$) and its orthogonal vector (in case of 3D, two vectors orthogonal to $\mathbf{e}_j$ and each other) respectively.

The method for determining principal axis (direction) and effective mass in that direction by Fujii [2] corresponds to evaluating the principal values and principal axes of EMT. The author et al. have determined the effective mass corresponding to Eqs. (7) and (8) by focusing on the direction of the momentum vector of a mode. Although the methods are different, it can be said that the results are exactly the same due to the nature of symmetric tensor.

2.4 Matrix representation of components

With Cartesian rectangular coordinates having $\mathbf{e}_\alpha$ as base vectors, if we use the usual participation factors $\beta_{j\alpha} \equiv \boldsymbol{\beta}_j\cdot\mathbf{e}_\alpha$ and the usual effective masses $\overline{M}_{j\alpha} \equiv \mathbf{e}_\alpha\cdot\overline{\mathbf{M}}_j\mathbf{e}_\alpha$, the components of EMT are represented in matrix form as follows:

$$\left[\overline{\mathbf{M}}_j\right] = M_j\begin{bmatrix} \beta_{jX}^{\;2} & \beta_{jX}\beta_{jY} & \beta_{jX}\beta_{jZ} \\ & \beta_{jY}^{\;2} & \beta_{jY}\beta_{jZ} \\ \text{sym.} & & \beta_{jZ}^{\;2} \end{bmatrix} = \begin{bmatrix} \overline{M}_{jX} & \sigma_{jXY}\sqrt{\overline{M}_{jX}\overline{M}_{jY}} & \sigma_{jXZ}\sqrt{\overline{M}_{jX}\overline{M}_{jZ}} \\ & \overline{M}_{jY} & \sigma_{jYZ}\sqrt{\overline{M}_{jY}\overline{M}_{jZ}} \\ \text{sym.} & & \overline{M}_{jZ} \end{bmatrix} \tag{11}$$

Note that the diagonal terms are the usual effective masses $\overline{M}_{j\alpha}$. For non-diagonal terms, the three positive or negative signs of

$$\sigma_{j\alpha\gamma} \equiv \mathrm{sgn}\left(\beta_{j\alpha}\beta_{j\gamma}\right) \tag{12}$$

are related; however, from the relationship of $\sigma_{j\alpha\gamma} = \sigma_{j\alpha\beta}\sigma_{j\beta\gamma}$, there are two independent signs. In other words, in the three-dimensional case, EMT is represented by three non-negative values of $\overline{M}_{j\alpha}$ and two signs.

For two-dimensional, the components of EMT are represented as follows:

$$\left[\overline{\mathbf{M}}_j\right] = M_j \begin{bmatrix} \beta_{jX}^{\ 2} & \beta_{jX}\beta_{jY} \\ \text{sym.} & \beta_{jY}^{\ 2} \end{bmatrix} = \begin{bmatrix} \overline{M}_{jX} & \sigma_{jXY}\sqrt{\overline{M}_{jX}\overline{M}_{jY}} \\ \text{sym.} & \overline{M}_{jY} \end{bmatrix} \tag{13}$$

This comprises two non-negative values of $\overline{M}_{j\alpha}$ and one sign σ_{jXY}.

3. Example

Fig. 3 shows two-dimensional lumped-mass models as a simple example. Each of them is composed of three lumped masses m and six linear truss elements with the same Young's modulus, cross-section, and length. The three models are the same, but each orientation is different from one another, with B being A rotated +30° in the clockwise direction and C being A rotated -60°.

Fig. 4 shows the mode shapes for model A.

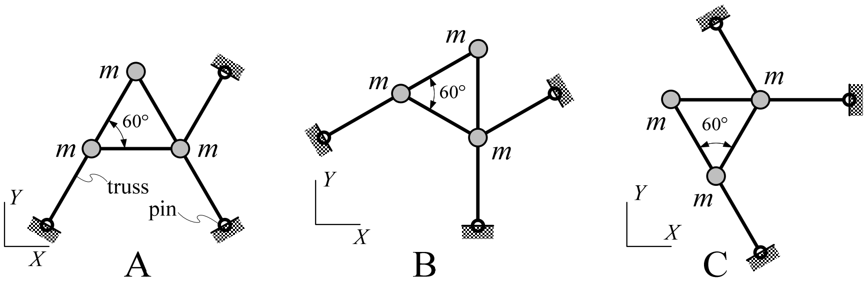
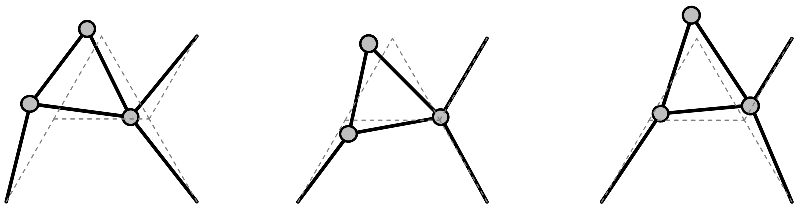


Fig. 3 – Two-dimensional lumped-mass models



(a) 1st mode (b) 2nd mode (c) 3rd mode

Fig. 4 – Mode shapes (model A)

From the results of the eigenvalue analysis, Table 1 gives the natural-period ratio ω_1/ω_j (ω_j is the jth natural circular frequency) and the usual effective masses $\overline{M}_{jX}$ and $\overline{M}_{jY}$ along the X and Y coordinate axis respectively (normalized by the total mass M=3m). The differences in $\overline{M}_{jX}$ or $\overline{M}_{jY}$ according to the model show those dependencies on the direction relative to the coordinate axis of X or Y. The inertial mass ratio $\overline{M}_j / M$ as the positive principal invariant of EMT is shown on the right side of the table. We can confirm the relationships in Eq. (7) for any model of A to C, i.e.,

$$\overline{M}_j / M = \left(\overline{M}_{jX} + \overline{M}_{jY} \right) / M \tag{14}$$

ϕ_j in the table is the angle between the X-axis and the principal axis of EMT (see also Fig. 2), which is defined as follows:

$$\tan\phi_j \equiv \beta_{Yj} / \beta_{Xj} = \sigma_{jXY} \sqrt{\overline{M}_{jY} / \overline{M}_{jX}} \tag{15}$$

From Eq.(9):

$$\begin{cases} \overline{M}_{jX} = \overline{M}_j (\mathbf{e}_X \cdot \mathbf{e}_j)^2 = \overline{M}_j \cos^2\phi_j \\ \overline{M}_{jY} = \overline{M}_j (\mathbf{e}_Y \cdot \mathbf{e}_j)^2 = \overline{M}_j \sin^2\phi_j \end{cases} \tag{16}$$

For example, with the first mode of model C, we can confirm

$$\overline{M}_{1X} / M = 0.904 \times \cos^2(36.3^\circ) = 0.587 \tag{17}$$

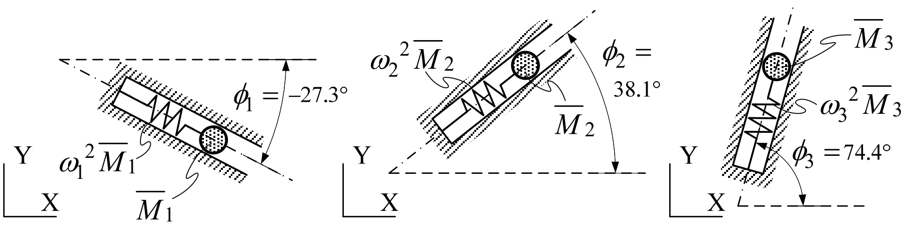
Moreover, with models B and C, because they are rotated by 90°, the values of $\overline{M}_{jX}$ and $\overline{M}_{jY}$ are substituted one for the other.

Table 1 – Effective mass, etc.

Mode	ω_1/ω_j	Model A				Model B				Model C				$\overline{M}_j/M$
		$\overline{M}_{jX}/M$	$\overline{M}_{jY}/M$	σ_{jXY}	ϕ_j (deg.)	$\overline{M}_{jX}/M$	$\overline{M}_{jY}/M$	σ_{jXY}	ϕ_j (deg.)	$\overline{M}_{jX}/M$	$\overline{M}_{jY}/M$	σ_{jXY}	ϕ_j (deg.)	
1	1.000	0.759	0.146	-1	-23.7	0.317	0.587	-1	-53.7	0.587	0.317	+1	36.3	0.904
2	0.809	0.159	0.098	+1	38.1	0.252	0.005	+1	8.1	0.005	0.252	-1	-81.9	0.257
3	0.466	0.052	0.670	+1	74.4	0.368	0.354	+1	44.4	0.354	0.368	-1	-45.6	0.722
4	0.293	0.025	0.034	+1	49.3	0.053	0.006	+1	19.3	0.006	0.053	-1	-70.7	0.059
5	0.249	0.005	0.019	-1	-63.4	0.000	0.024	+1	86.6	0.024	0.000	-1	-3.4	0.024
6	0.208	0.000	0.032	+1	87.8	0.009	0.023	+1	57.8	0.023	0.009	-1	-32.2	0.032
Σ		1.000	1.000			1.000	1.000			1.000	1.000			2.000

Fig. 5 shows the corresponding ESDF systems for model A.

Fig. 6 is another concise representation of the ESDF systems defined by EMT for model A. Each line segment of length $2\overline{M}_{jX} / M$ along the principal axis represents the ESDF system for the first to the third mode respectively.



(a) 1st mode (b) 2nd mode (c) 3rd mode

Fig. 5 – Equivalent single-degree-of-freedom systems (model A)

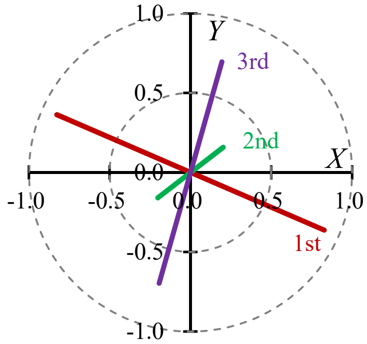


Fig. 6 – A diagram to depict the equivalent single-degree-of-freedom systems (model A)

4. Conclusion

In this paper, the concept of the effective mass tensor (EMT) for multidimensional modes in eigenvalue analysis (also for a generalized single-degree-of-freedom system) has been introduced. Coordinate components of the tensor are usually referred to as the effective modal masses. Although the values of the components are dependent on the direction of the coordinate system, note that representation as a tensor is independent of the coordinate system. The trace of EMT is the only one positive principal value (eigenvalue). Its physical meaning is the inertial mass of the ESDF system of the mode. The corresponding principal axis (eigenvector) indicates the movable direction of the ESDF system. This shows that a multidimensional vibration mode can be recognized as a simple inclined ESDF system in the multidimensional space.

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