

METHODS FOR CALCULATING THE EARTHQUAKE RESPONSE OF "SHEAR" BUILDINGS

by

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ABSTRACT. This paper reviews briefly the history of experimental and analytical investigations of the dynamic response of multi-story buildings to ground motion, and presents a method for the use of the engineer in determining the response of "shear" buildings. The method involves the use of normal coordinates and the step-wise representation of the forcing function. Detailed applications are shown. Friction and some effects of asymmetry of plan have been included.

INTRODUCTION. In the past fifty years the subject matter of publications on the effects of earthquakes on multi-story buildings has passed from the description of structural damage, to the theory and calculation of the transient response of linear and non-linear multi-degree-of-freedom systems. Important parts of this evolution include the early recognition, now universally accepted, that the problem is one in dynamics; the establishing of a pattern of behavior through observation of full-size structures, experiments with models, and the dynamical analysis of simplified systems; and finally the extension of the analysis and of computing methods to include systems that are not basically simple.

The Bibliography includes references to experiments, dynamical analysis and computing methods applied to multi-story buildings, and some related general references. Prior to the appearance of Dr. Tachu Naito's treatise on Earthquake-Resisting Construction (1)** in 1923 little was to be found. However, considerable interest was generated by the Tokyo earthquake of 1923 and the Santa Barbara 'quake of 1925, and about 1926 L. S. Jacobsen at Stanford University, K. Suyehiro in Japan, and R. R. Martel at California Institute of Technology started experimenting with small-scale mechanical models. The importance of these experiments as a start in establishing the dynamical pattern of behavior can scarcely be overestimated.

In 1931-33 Jacobsen built a "shear" model of the then proposed new building for the Olympic Club, subjected it to ground motions of a simple type as well as of a type similar to those recorded for the Tokyo earthquake, and recorded the complete response with motion-picture photography. Unfortunately the results were never fully published. The only published references that can be found are a brief note in the book by John R. Freeman (2), and a "discussion" published in 1937 (13). The discussion gives the results of an investigation employing only the part of the model representing the 17-story tower of the building. In 1933-37 he directed a study of an idealized prototype approximating the Alexander Building in San Francisco, designed and constructed a 16-story "shear and flexure" model, and determined its response to simple transient ground motions (7), (16). This work indicated the impor-

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**Numbers in parenthesis refer to the Bibliography at the end of the paper.

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tance of the higher modes and the possibility of resonances.

At about the same time as the major activity in the use of models, the United States Coast and Geodetic Survey was actively engaged in determining the natural frequencies and modes of vibration of existing full-size structures. Many of the results have been reported by D. S. Carder (10) and later by others (24).

The earliest general analysis of the transient vibration of multi-story buildings appears to have been published by M. A. Biot (5), (20).

Starting about 1945, with the advent of government sponsored research, automatic computers, and the impact of the A-bomb, interest in the theory and calculation of dynamic response was greatly increased. Contributions have been made by G. W. Housner (21), (34), (37), N. M. Newmark (26), (31), (38), and M. G. Salvadori (22), (32) and others (25), (30), (35), (36), (39). It is interesting to note that about one-half of the references found have been published in the last ten years.

There is not room to give a complete set of references. The reader is referred to the Bibliography on Engineering Seismology, compiled by E. P. Hollis (33) for the Earthquake Engineering Research Institute, for further details.

The main purpose is to present analytical methods usable by the engineer for the calculation of the transient dynamic response of "shear" buildings of a few stories in height. The methods theoretically are applicable to systems having any finite number of degrees of freedom, but if more than a few stories are involved the calculations, by any of the available methods, become so laborious that automatic computers are indicated. However, in spite of the practical limitations, the writer feels that the methods have usefulness, first, because they provide insight into the general problem of transient dynamic response, and second, because a large proportion of buildings have no more than a few stories. Furthermore, automatic computing facilities are not readily available to all engineering offices.

The methods involve the use of normal coordinates, the step-wise representation of the forcing function, the use of phase-plane graphical construction for evaluating the time-response in each natural mode, and, for "symmetrical" structures, the Holzer tabulation for determining the frequencies and amplitude ratios of the natural modes.

"First guesses" for use in the Holzer method have been obtained through the Rayleigh approximation for the fundamental natural frequency and by means of interpolation charts for the higher natural frequencies.

Applications have also been shown for "unsymmetrical" structures, i.e. those in which there is coupling among the horizontal plane coordinates. Coulomb (constant) friction is readily included in the phase-plane construction. The methods theoretically may be extended, by step-wise variation in the constants, to include the plastic range of deformation, but this is very laborious in the multi-story case if many variations are involved. Other methods are more useful for this purpose.

A "shear" building is one in which the floor systems are assumed to move only as rigid parallel planes. A vertical cross-section has been shown in Fig. 1, in which m_i and k_i are the mass and horizontal stiffness, respectively, of the i th story. The displacement and excitation symbols will be introduced later.

SYMMETRICAL, SINGLE-STORY STRUCTURE. We represent the structure as an undamped system having one degree of freedom, and indicate three types of excitation that may be applied, as shown in Fig. 2. The excitations, which are known functions of time, include the following: a) force F applied to the mass, b) ground displacement x_g , or c) ground acceleration \ddot{x}_g . In cases (a) and (b) the variable x is referred to a fixed reference, and in case (c) to a reference on the moving ground. The differential equations of motion are as follows:

$$\begin{array}{lll} \text{a) } m\ddot{x} + kx = F; & \text{b) } m\ddot{x} + kx = kx_g; & \text{c) } m\ddot{x} + kx = m\ddot{x}_g \end{array}$$

Letting $X(t) = F/k$ or x_g or $m\ddot{x}_g/k$, the equations may be written in the general form,

$$m\ddot{x} + kx = k \cdot X(t). \quad (1)$$

Step-wise forcing function. If $X(t)$ is represented by a sequence of steps of height $X_1, X_2, \dots, X_n, \dots$, Fig. 3, the right-hand side of equation (1) is constant for each step interval of time, so that for the n th step the differential equation and its solution are given by

$$m\ddot{x} + kx = k \cdot X_n, \quad \text{for } t_{n-1} \leq t \leq t_n \quad (2a)$$

$$x - X_n = (x_{n-1} - X_n) \cos p (t - t_{n-1}) + \frac{\dot{x}_{n-1}}{p} \sin p (t - t_{n-1}), \quad (2b)$$

where x_{n-1} and \dot{x}_{n-1} are the displacement and velocity at the time t_{n-1} (starting conditions for the n th step), and where $p = \sqrt{k/m}$ = natural frequency in radians per second. The velocity equation is

$$\frac{\dot{x}}{p} = -(x_{n-1} - X_n) \sin p (t - t_{n-1}) + \frac{\dot{x}_{n-1}}{p} \cos p (t - t_{n-1}) \quad (2c)$$

Phase-plane graphical method. Squaring equations (2b) and (2c) and adding, we find

$$\left(\frac{\dot{x}}{p}\right)^2 + (x - X_n)^2 = \left(\frac{\dot{x}_{n-1}}{p}\right)^2 + (x_{n-1} - X_n)^2, \quad (2d)$$

which is the equation of a circle with center at $\dot{x}/p = 0$ and $x = X_n$, and having the radius $R_n = \sqrt{(\dot{x}_{n-1}/p)^2 + (x_{n-1} - X_n)^2}$. The solution for equation (2a) may therefore be represented graphically as the arc of the circle of equation (2d), subtended by the angle $p(t_n - t_{n-1})$; Fig. 4. Time is implicit in the arc, and is positive in the counter-clockwise direction. The \dot{x}/p , x coordi-

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nate system is a form of phase-plane.

The response of the system to a sequence of steps may be found by drawing connected circular arcs (a phase trajectory) in the manner shown in the example in Fig. 5. The assumed starting conditions for the example are, at $t = 0$, $x = 0$, $\dot{x} = 0$; consequently, the phase trajectory starts at the origin. Any other starting conditions could have been assumed. In order to obtain accurate results the time intervals usually should not be longer than about one-fourth the natural period. The step-wise graphical construction in the phase-plane was developed by J. Lamoen (9).

It is convenient, for later use in showing the extension of the graphical method to multi-degree-of-freedom systems, to show the complete step-wise analytical solution of equation (1). Assuming a start from rest, the solutions for the first few steps are as follows:

$$x = X_1(1 - \cos pt), \quad 0 \leq t \leq t_1 \quad (3a)$$

$$x = X_1(1 - \cos pt) + (X_2 - X_1) [1 - \cos p(t - t_1)], \quad t_1 \leq t \leq t_2 \quad (3b)$$

$$x = X_1(1 - \cos pt) + (X_2 - X_1) [1 - \cos p(t - t_1)] \\ + (X_3 - X_2) [1 - \cos p(t - t_2)], \quad t_2 \leq t \leq t_3, \text{ etc.} \quad (3c)$$

To satisfy general starting conditions, $x = x_0$ and $\dot{x} = \dot{x}_0$ at $t = 0$, it is necessary to add the terms, $x_0 \cos pt + (\dot{x}_0/p) \sin pt$, to the above solutions. The displacements could be obtained by numerical evaluation of the above expressions, but it is obvious that the work is greatly reduced by the use of the graphical method already shown.

Damping. The procedure may be extended to include damping of various types. However, Coulomb or constant damping is reasonably characteristic of the over-all damping in a composite structure and it is the type that is most readily included in the step-wise construction. Modifying equation (1) to take account of the constant damping force C , and writing the equation so that C appears on the right-hand side, we find

$$m\ddot{x} + kx = kX(t) \mp C, \quad (4)$$

where the \mp sign means that the sign of C is opposite to that of the velocity and must be changed whenever the velocity changes sign. This well known effect has been shown graphically in Fig. 6 for an example of free vibration starting from an initial static displacement. An example for transient forced vibration is given in Fig. 7.

Non-linear damping (11) and restoring forces may be handled by step-wise linearization. A general procedure for doing this, called the phase-plane-delta method, has been developed by L. S. Jacobsen (28). It is particularly useful in the case of structures that may be represented by single-degree-of-freedom systems. For example, the effect of dynamic loading causing displacement in the "plastic" range is readily determined (25).

SYMMETRICAL, MULTI-STORY STRUCTURES. The extension of the linear analysis to include structures that must be represented as having more than one degree of freedom, is as follows: Fig. 8 shows a symmetrical three-story structure acted upon by the three types of excitation. The displacement variables, x_1 , x_2 , and x_3 , are referred to a fixed reference in cases (a) and (b) and to a reference on the moving ground in case (c). The differential equations of motion are as follows:

$$\begin{aligned} m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 &= q_1 \\ m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 - k_3 x_3 &= q_2 \\ m_3 \ddot{x}_3 - k_3 x_2 + k_3 x_3 &= q_3 \end{aligned} \tag{5a}$$

where the forces q_1 , q_2 and q_3 are known functions of time, given by the following:

$$\begin{aligned} \text{for case (a): } q_1 &= F_1, \quad q_2 = F_2, \quad q_3 = F_3 \\ \text{(b): } q_1 &= k_1 x_g, \quad q_2 = 0, \quad q_3 = 0 \\ \text{(c): } q_1 &= m_1 \ddot{x}_g, \quad q_2 = m_2 \ddot{x}_g, \quad q_3 = m_3 \ddot{x}_g \end{aligned} \tag{5b}$$

In the case that the q functions are steps of constant height Q_1, \dots , Fig. 9a, the particular solutions s of equations (5a) are found by solution of the following equations:

$$\begin{aligned} +(k_1 + k_2) s_1 & & - k_2 s_2 & & & = Q_1 \\ - k_2 s_1 & & +(k_2 + k_3) s_2 & & - k_3 s_3 & = Q_2 \\ & & - k_3 s_2 & & + k_3 s_3 & = Q_3 \end{aligned} \tag{6a}$$

Consequently we find

$$s_1 = \frac{Q_1 + Q_2 + Q_3}{k_1}, \quad s_2 = s_1 + \frac{Q_2 + Q_3}{k_2}, \quad s_3 = s_2 + \frac{Q_3}{k_3} \tag{6b}$$

The particular solutions, therefore, have simple physical meanings. They are equal to the static deflections of the structure under the constant forces Q_1 , Q_2 , and Q_3 .

The complete solutions of the differential equations are as follows:

$$\begin{aligned} x_1 &= A_I \cos p_I t + B_I \sin p_I t + \dots + B_{III} \sin p_{III} t + s_1 \\ x_2 &= a_I A_I \cos p_I t + a_I B_I \sin p_I t + \dots + a_{III} B_{III} \sin p_{III} t + s_2 \\ x_3 &= b_I A_I \cos p_I t + b_I B_I \sin p_I t + \dots + b_{III} B_{III} \sin p_{III} t + s_3 \end{aligned} \tag{7}$$

where the p 's are the natural frequencies and the a 's and b 's are the natural mode amplitude ratios, to be determined later.

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For the special case of a start from rest the B constants are zero and the A constants are determined by solving equations (8a).

$$\begin{aligned}
 A_I + A_{II} + A_{III} &= -s_1 \\
 a_I A_I + a_{II} A_{II} + a_{III} A_{III} &= -s_2 \\
 b_I A_I + b_{II} A_{II} + b_{III} A_{III} &= -s_3
 \end{aligned} \tag{8a}$$

Making use of determinants, we find the following

$$A_I = - \frac{\begin{vmatrix} s_1 & 1 & 1 \\ s_2 & a_{II} & a_{III} \\ s_3 & b_{II} & b_{III} \end{vmatrix}}{D}, \quad A_{II} = - \frac{\begin{vmatrix} 1 & s_1 & 1 \\ a_I & s_2 & a_{III} \\ b_I & s_3 & b_{III} \end{vmatrix}}{D}, \quad A_{III} = - \frac{\begin{vmatrix} 1 & 1 & s_1 \\ a_I & a_{II} & s_2 \\ b_I & b_{II} & s_3 \end{vmatrix}}{D} \tag{8b}$$

where

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a_I & a_{II} & a_{III} \\ b_I & b_{II} & b_{III} \end{vmatrix}. \tag{8c}$$

The determinants in the numerators in equations (8b) must, in general, be re-evaluated for each step in time. However, in most cases of practical interest the quantity that varies with time may be factored out, leaving the determinants as constants. For example, in case (a), Fig. 8, if the time-variation in external force is the same at all levels of the building, the ratios F_2/F_1 and F_3/F_1 are constant. If the force-time function F_1 is a step of constant height F_c the Q 's may then be written as follows:

$$Q_1 = F_c, \quad Q_2 = F_c (F_2/F_1), \quad Q_3 = F_c (F_3/F_1). \tag{9a}$$

Adopting the symbols,

$$f_1 = \frac{1 + (F_2/F_1) + (F_3/F_1)}{k_1}, \quad f_2 = f_1 + \frac{(F_2/F_1) + (F_3/F_1)}{k_2}, \quad f_3 = f_2 + \frac{(F_3/F_1)}{k_3} \tag{9b}$$

and making use of equations (6b) and (9a), we find,

$$s_1 = F_c f_1, \quad s_2 = F_c f_2, \quad s_3 = F_c f_3 \tag{9c}$$

where f_1 , f_2 and f_3 are constants for a given vertical distribution of external force. The expressions for the A 's are then of the following type, shown for example for A_I :

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$$A_I = -F_c \frac{\begin{vmatrix} f_1 & 1 & 1 \\ f_2 & a_{II} & a_{III} \\ f_3 & b_{II} & b_{III} \end{vmatrix}}{D} \quad (9d)$$

For ground-displacement excitation, case (b), Fig. 8, we find the following:

$$Q_1 = k_1 X_g, \quad Q_2 = Q_3 = 0; \quad s_1 = s_2 = s_3 = X_g; \quad (10a)$$

$$A_I = -X_g \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & a_{II} & a_{III} \\ 1 & b_{II} & b_{III} \end{vmatrix}}{D}; \text{ etc.} \quad (10b)$$

If the excitation is known in terms of ground acceleration, case (c):

$$Q_1 = m_1 \ddot{X}_g, \quad Q_2 = m_2 \ddot{X}_g, \quad Q_3 = m_3 \ddot{X}_g. \quad (11a)$$

Adopting the symbols,

$$\Delta_1 = \frac{w_1 + w_2 + w_3}{k_1}, \quad \Delta_2 = \Delta_1 + \frac{w_2 + w_3}{k_2}, \quad \Delta_3 = \Delta_2 + \frac{w_3}{k_3} \quad (11b)$$

and using equations (6b) and (11a), one obtains the following:

$$s_1 = G\Delta_1, \quad s_2 = G\Delta_2, \quad s_3 = G\Delta_3, \quad (11c)$$

where $G = \ddot{X}_g/g$ (the ground acceleration in g's) and where the Δ 's are the deflections of the structure when it is loaded by transverse static loads equal to the weights of the stories. Consequently,

$$A_I = -G \frac{\begin{vmatrix} \Delta_1 & 1 & 1 \\ \Delta_2 & a_{II} & a_{III} \\ \Delta_3 & b_{II} & b_{III} \end{vmatrix}}{D}; \text{ etc.} \quad (11d)$$

Rewriting equations (7) after substituting for s_1 , s_2 and s_3 the

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expressions given by equations (8a), and setting the B's equal to zero, we find:

$$\begin{aligned}
 x_1 &= -A_I(1 - \cos p_I t) - A_{II}(1 - \cos p_{II} t) - A_{III}(1 - \cos p_{III} t) \\
 x_2 &= -a_I A_I(1 - \cos p_I t) - a_{II} A_{II}(1 - \cos p_{II} t) - a_{III} A_{III}(1 - \cos p_{III} t) \quad (12a) \\
 x_3 &= -b_I A_I(1 - \cos p_I t) - b_{II} A_{II}(1 - \cos p_{II} t) - b_{III} A_{III}(1 - \cos p_{III} t)
 \end{aligned}$$

A comparison shows that equations (12a) are composed of terms, $-A_I(1 - \cos p_I t)$, etc., of the same type as found in equation (3a) for the single-degree-of-freedom system. Furthermore, if a second step occurs in the forcing functions for the three-degree-of-freedom system at time $t = t^2$, Fig. 9b, one may derive solutions for the step containing terms of the type found in equation (3b); and so on for the n^{th} step. Moreover, each of the terms relates to vibration in only one of the natural modes and may be considered to be a principal or normal coordinate. Equations (12a), and the equations derived for succeeding steps, may therefore be represented in the following general form:

$$\begin{aligned}
 x_1 &= \theta_I + \theta_{II} + \theta_{III} \\
 x_2 &= a_I \theta_I + a_{II} \theta_{II} + a_{III} \theta_{III} \\
 x_3 &= b_I \theta_I + b_{II} \theta_{II} + b_{III} \theta_{III}
 \end{aligned} \quad (12b)$$

where θ_I , θ_{II} and θ_{III} are normal coordinates that are to be evaluated independently of each other. Their time variations may be determined in any manner that is valid for equations (3a), (3b), etc., for the single-degree-of-freedom case (23).

We will evaluate the time variation of θ_I , θ_{II} and θ_{III} by the phase-plane graphical method, and will then determine x_1 , x_2 and x_3 according to the relationships shown in equations (12b). However, it is first necessary to determine the frequencies and amplitude ratios of the natural modes.

In order to determine the natural frequencies and amplitude ratios, we return to equations (5a), set the right-hand sides equal to zero, assume solutions of the type, $x_1 = d_1 \cos pt$, and arrive at the following equations:

$$\begin{aligned}
 -m_1 d_1 p^2 + (k_1 + k_2) d_1 - k_2 d_2 &= 0 \\
 -m_2 d_2 p^2 - k_2 d_1 + (k_2 + k_3) d_2 - k_3 d_3 &= 0 \\
 -m_3 d_3 p^2 - k_3 d_2 + k_3 d_3 &= 0
 \end{aligned} \quad (13a)$$

If we proceed in the conventional manner, eliminating the amplitudes, d_1 , d_2 and d_3 , we will arrive at a cubic equation in p^2 , the roots of which

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give the natural frequencies p_I, p_{II} and p_{III} . In the course of the derivation, general expressions for the amplitude ratios, $a = d_2/d_1, b = d_3/d_1$, will be found and after the natural frequencies have been determined, the specific values of the natural mode amplitude ratios, a_I, b_I, \dots, b_{III} , may be calculated. This is a laborious process if there are more than a few degrees of freedom. Hence, in the following we look for a numerical method of solution.

The well known Holzer tabulation provides a direct calculation, by trial, of the natural frequencies and amplitude ratios. The method is shown by rewriting equations (13a) as follows:

$$\begin{aligned}
 d_2 &= d_3 - \frac{p^2}{k_3} m_3 d_3 \\
 d_1 &= d_2 - \frac{p^2}{k_2} (m_3 d_3 + m_2 d_2) \\
 0 &= d_1 - \frac{p^2}{k_1} (m_3 d_3 + m_2 d_2 + m_1 d_1) \quad ,
 \end{aligned}
 \tag{13b}$$

or, in general,

$$d_{i-1} = d_i - \frac{p^2}{k_i} (m_n d_n + \dots + m_i d_i) .$$

Equations (13b) may be solved by the tabular form below. Entering the known

1	2	3	4	5	6	7	8	9
Story i	m_i	$m_i p^2$	d_i	$m_i p^2 d_i$	$\sum_n m p^2 d$	k_i	$\frac{1}{k_i} \left(\sum_n m p^2 d \right)$	d_i/d_1
n (top)			1.00					
.								
i								
.								
1								
0 (ground)	-	-	zero	-	-	-	-	-

constants in columns 2 and 7, make a first trial estimate of one of the natural frequencies p , assign any arbitrary value to d_n , and fill out the table according to the pattern indicated by the headings and by equations (13b). If the assumed frequency value coincides with one of the natural frequencies the calculated amplitude at ground level (column 4, line 0) will be zero. When this condition has been met, column 9 giving the amplitude ratios may be completed. The Holzer calculations may be considerably shortened if good first guesses can be made of the natural frequencies. Methods for doing this have been given below.

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The approximate method of Lord Rayleigh for determining the fundamental natural frequency is well known. It consists in equating the maximum kinetic energy to the maximum potential energy for free vibration in the fundamental natural mode, and solving for the frequency. If the maximum dynamic displacements are assumed to be equal to the static deflections of the system when loaded by its own weight, Fig. 10, the Rayleigh expression for p_I takes the form of equation (14). In the case of the multi-story building, or other

$$p_I \approx \sqrt{g \frac{\sum_1^n w_i \Delta_i}{\sum_1^n w_i \Delta_i^2}}, \text{ radians/second.} \quad (14)$$

equivalent system with only one end tied to ground, the Δ 's have already been given by equations (11b). The Rayleigh method results in a value for p_I which is higher than the true value. Frequently the error is less than one per cent.

The natural frequencies of the higher modes may be estimated by use of Figs. 11 and 12, which show the square of the dimensionless natural frequency, $p^2 m/k$, against the total number of mass concentrations or stories, n , in the structure. In both cases, the masses and spring constants are constant throughout the structure, except that in Fig. 12, $k_1 \rightarrow 0$, that is, the "structure" is not tied to ground at either end. The natural frequencies of "shear" systems, having approximately constant mass and elastic properties except in the first story, often lie between the frequencies shown for these two limiting cases. Curves have been drawn through the discrete points to aid in interpolation between the limiting cases and to suggest an extrapolation to greater values of n . Only integer values of the abscissae have meaning. In cases of non-uniform mass and elastic properties, one may calculate the average value of m/k for the building and arrive at a preliminary rough estimate of the natural frequencies.

The foregoing development has been shown for buildings having only a few stories. It may be extended to apply to shear buildings having any finite number of stories.

UNSYMMETRICAL STRUCTURES. Fig. 13 shows a single-story structure for which the center of rigidity C.R. (center of static twist) does not coincide with the center of gravity C.G. The coordinate system and the principal elastic axes aa and bb have been shown in Fig. 14. More general cases in which the spring directions are not all at right angles to each other and in which, consequently, the directions of the principal elastic axes are not obvious from the geometry, have been discussed elsewhere (14).

The analysis is applicable to buildings having a rigid roof structure supported by a "single-story" frame, including some types of aircraft hangars, grand-stand roofs, industrial buildings, and public halls.

The coordinates of plane motion are in general coupled together and

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the system lacks dynamic symmetry. The differential equations of motion are:

$$\begin{aligned} m\ddot{x} + k_x x - k_x r_y \theta &= q_x \\ m\ddot{y} + k_y y - k_y r_x \theta &= q_y \\ m\rho^2 \ddot{\theta} + k_\theta \theta - k_x r_y x + k_y r_x y &= q_\theta \end{aligned} \quad (15a)$$

where

$$\begin{aligned} k_x &= k_1 + k_2 = \sum_x k_i, \quad k_y = k_3 + k_4 = \sum_y k_i \\ k_\theta &= k_1 l_1^2 + \dots + k_4 l_4^2 = \sum k_i l_i^2 \\ r_x &= (k_3 l_3 - k_4 l_4) / (k_3 + k_4) = (\sum_y k_i l_i) / k_y \\ r_y &= (k_2 l_2 - k_1 l_1) / (k_1 + k_2) = (\sum_x k_i l_i) / k_x \end{aligned} \quad (15b)$$

For excitation forces q_x and q_y and moment q_θ we find:

$$\begin{aligned} \text{for case (a): } q_x &= F_x, \quad q_y = F_y, \quad q_\theta = M \\ \text{(b): } q_x &= k_x x_g, \quad q_y = k_y y_g, \quad q_\theta = -k_x r_y x_g + k_y r_x y_g \\ \text{(c): } q_x &= m\ddot{x}_g, \quad q_y = m\ddot{y}_g, \quad q_\theta = 0 \end{aligned} \quad (15c)$$

Note that rotational ground motions have been omitted. They may be included in other applications for which they have significance.

Proceeding as in the derivation for the multi-story symmetrical structure, we represent the q functions by steps of constant height Q , and determine the particular solutions as follows:

$$s_x = \frac{Q_x}{k_x} + r_y s_\theta, \quad s_y = \frac{Q_y}{k_y} - r_x s_\theta, \quad s_\theta = \frac{Q_\theta + Q_x r_y - Q_y r_x}{k_\theta - k_x r_y^2 - k_y r_x^2} \quad (16)$$

The denominator of s_θ is the torsional stiffness of the structure calculated with respect to the center of static twist. We find, as before, that the particular solutions are equal to the static deflections of the structure under the constant forces Q_x, \dots . For case (b), for example, the particular solutions are: $s_x = X_g, s_y = Y_g, s_\theta = 0$.

The complete solutions are given by equations (7) provided x, y and θ for x_1, x_2 and $x_3; s_x, s_y$ and s_θ for s_1, s_2 and

ANALYSIS OF STRUCTURAL RESPONSE

tively. For a start from rest the B constants are zero and the A constants may be determined from equations (8b) by appropriate changes in the symbols.

The natural mode frequencies and amplitude ratios are found in the general manner shown for the previous case, except that the approximate methods given there are not directly applicable to this case. The natural frequencies are the three positive roots of equation (17a),

$$p^6 - p^4 \left(\frac{k_x + k_y + k_\theta / \rho^2}{m} \right) + p^2 \left(\frac{k_x k_y + (k_x + k_y) k_\theta / \rho^2}{m^2} \right) - \frac{k_x k_y k_\theta}{m^3 \rho^2} = 0 \quad (17a)$$

and the amplitude ratios are given by two of the following:

$$\frac{d_x}{d_\theta} = \frac{r_y}{1 - mp^2/k_x}, \quad \frac{d_y}{d_\theta} = \frac{-r_x}{1 - mp^2/k_y}, \quad \frac{d_y}{d_x} = \frac{-r_x(1 - mp^2/k_x)}{r_y(1 - mp^2/k_y)} \quad (17b)$$

When the system is vibrating in a natural mode the instantaneous center of plane motion is stationary and its location is given by d_x/d_θ and d_y/d_θ . The direction of the normal to the static equilibrium position of a line passing through the center of gravity and through the stationary center is given by d_y/d_x . This is the direction of maximum effect of the excitation, or critical direction, for the natural mode (19).

In rewriting equations (7) through (12) to apply to this case the amplitude ratios a and b should be expressed by $a = d_y/d_x$ and $b = d_\theta/d_x$.

If either r_x or r_y equals zero the structure has one-fold "symmetry" and only two of equations (15a) remain coupled. Several examples have been shown in Fig. 15.

If both r_x and r_y are zero the structure has two-fold "symmetry", and equations (15a) are independent of each other, with the resulting simplification of the dynamic analysis and the elimination of torsion or twisting of the structure caused by translational ground motions. However, there are many cases in which it is not feasible to obtain dynamic symmetry of design.

Another application of equations (15a) has been shown in Fig. 16. The structure is a rigid block supported by an elastic foundation that allows "rocking" as well as translation of the block (27). In most applications of this type, r_x is zero and consequently y is not coupled with x and θ . In a more complete statement of the problem a gravitational moment term (negative) should be added to the left side of the θ equation.

In more refined analyses of the multi-story shear building elastic rocking of the entire structure on the foundation has been coupled with the shear displacements (22), (34). Only the shear displacements have been included in equations (5a).

There are many multi-story structures having great lack of symmetry, in fact, most are unsymmetrical to some extent. The condition for de-coupling of rotation from translation requires that the centers of gravity and centers of statical twist for all stories lie on a vertical line. If this condition is not met, then in general the $3n$ coordinates of the shear building of n stories are all coupled. The simplest case for the n -storied building is that in which the principal elastic axes for all stories lie on two common vertical planes and in which the centers of gravity all lie on the intersection line of the two planes. This case results in de-coupling of the three sets of coordinates (complete dynamic symmetry). We then have three independent sets of n differential equations, one set in x_1, \dots, x_n , one in y_1, \dots, y_n , and one in $\theta_1, \dots, \theta_n$. This is the assumption that was made in writing equations (5a).

It is impractical to solve the $3n$ coupled equations by anything but automatic computer methods if $n > 2$. However, it seems worthwhile to show the differential equations for the two-story structure since they not only have direct application but also set the pattern for the case of n stories. The equations (18a) have been written for the semi-general case in which the principal elastic axes are parallel but do not lie in two common vertical planes. An example has been shown in Fig. 17. The coordinate system and the principal axes of the elastic system appear in Fig. 18. The origin of coordinates has been taken at the center of gravity of the first story.

$$\begin{aligned}
 m_1 \ddot{x}_1 + (k_{1x} + k_{2x})x_1 - k_{2x}x_2 - (k_{1x}r_{1y} + k_{2x}r_{2y})\theta_1 + k_{2x}r_{2y}\theta_2 &= q_{1x} \\
 m_1 \ddot{y}_1 + (k_{1y} + k_{2y})y_1 - k_{2y}y_2 + (k_{1y}r_{1x} + k_{2y}r_{2x})\theta_1 - k_{2y}r_{2x}\theta_2 &= q_{1y} \\
 m_1 \rho_1^2 \ddot{\theta}_1 + (k_{1\theta} + k_{2\theta})\theta_1 - k_{2\theta}\theta_2 - (k_{1x}r_{1y} + k_{2x}r_{2y})x_1 + k_{2x}r_{2y}x_2 \\
 + (k_{1y}r_{1x} + k_{2y}r_{2x})y_1 - k_{2y}r_{2x}y_2 &= q_{1\theta} \quad (18a)
 \end{aligned}$$

$$\begin{aligned}
 m_2 (\ddot{x}_2 - c_{2y}\ddot{\theta}_2) + k_{2x}x_2 - k_{2x}x_1 - k_{2x}r_{2y}\theta_2 + k_{2x}r_{2y}\theta_1 &= q_{2x} \\
 m_2 (\ddot{y}_2 + c_{2x}\ddot{\theta}_2) + k_{2y}y_2 - k_{2y}y_1 + k_{2y}r_{2x}\theta_2 - k_{2y}r_{2x}\theta_1 &= q_{2y} \\
 m_2 (\rho_2^2 \ddot{\theta}_2 - c_{2y}\ddot{x}_2 + c_{2x}\ddot{y}_2) + k_{2\theta}\theta_2 - k_{2\theta}\theta_1 - k_{2x}r_{2y}x_2 + k_{2x}r_{2y}x_1 \\
 + k_{2y}r_{2x}y_2 - k_{2y}r_{2x}y_1 &= q_{2\theta}
 \end{aligned}$$

where

$$\begin{aligned}
 k_{1x} &= k_1 + k_2 = \sum_{1x} k_i ; \dots ; k_{2y} = k_7 + k_8 = \sum_{2y} k_i \\
 \dots k_{2\theta} &= k_5 l_5^2 + k_6 l_6^2 + k_7 l_7^2 + k_8 l_8^2 = \sum_{2\theta} k_i l_i^2 \\
 r_{1x} &= (k_3 l_3 - k_4 l_4) / (k_3 + k_4) = \sum_{1y} k_i l_i / k_{1y} ; \dots \\
 r_{2y} &= (k_6 l_6 - k_5 l_5) / (k_5 + k_6) = \sum_{2x} k_i l_i / k_{2x}
 \end{aligned} \quad (18b)$$

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The excitation forces and moments, omitting rotational ground motions, are as follows:

$$\begin{aligned}
 \text{case (a): } & q_{1x} = F_{1x}, \dots, q_{2y} = F_{2y} \\
 & q_{1\theta} = M_1, \quad q_{2\theta} = M_2 - F_{2x} c_{2y} + F_{2y} c_{2x} \\
 \text{case (b): } & q_{1x} = k_{1x} x_g, \quad q_{1y} = k_{1y} y_g, \quad q_{1\theta} = -k_{1x} r_{1y} x_g + k_{1y} r_{1x} y_g \\
 & q_{2x} = q_{2y} = q_{2\theta} = 0 \qquad (18c) \\
 \text{case (c): } & q_{1x} = m_1 \ddot{x}_g, \quad q_{1y} = m_1 \ddot{y}_g, \quad q_{1\theta} = 0 \\
 & q_{2x} = m_2 \ddot{x}_g, \quad q_{2y} = m_2 \ddot{y}_g, \quad q_{2\theta} = m_2 (-c_{2y} \ddot{x}_g + c_{2x} \ddot{y}_g)
 \end{aligned}$$

The remaining procedure is in general the same as has already been shown.

It should be noted that in many special cases, equations (18a) can be greatly simplified. If, for example, the structure has one-fold dynamic symmetry with respect to the x-z plane, then $r_{1y} = r_{2y} = c_{2y} = 0$, and the equations fall into two independent groups, one group of two equations in x_1 and x_2 , and the other of four in y_1, y_2, θ_1 and θ_2 .

SUMMARY. The solution for the transient response of the symmetrical multi-story building may consist of the following steps: Rayleigh approximation for the fundamental natural frequency p_{I} ; preliminary estimation of the higher natural frequencies $p_{II} \dots$; Holzer-table calculations for refinement of the natural frequency values and determination of the natural mode amplitude ratios $a_I, a_{II}, a_{III}, \dots, b_I, \dots$; evaluation of the determinants giving the normal mode coefficients A_I, A_{II}, \dots ; step-wise graphical construction of the phase-trajectory for each normal mode (coordinates $\theta_I/p_I, \theta_I; \dots$); superposition of the normal mode responses to give the resultant responses in x_1, x_2, \dots .

The extensions to unsymmetrical structures and to cases involving Coulomb damping have also been shown.

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FIGURE CAPTIONS

Fig. 1. Multi-story shear building (symmetrical).

Fig. 2. Single-story shear building (symmetrical)

- a) Excitation force, F .
- b) Ground displacement excitation, x_g .
- c) Ground acceleration excitation, \ddot{x}_g .

Fig. 3. Step-wise representation of general excitation function.

Fig. 4. Graphical construction for the n^{th} step in the phase-plane.

Fig. 5. Example of construction of phase trajectory and displacement-time curve.

Fig. 6. Effect of Coulomb (constant) damping on free vibration of oscillator.

Fig. 7. Transient forced response of Coulomb-damped oscillator.
(Line marked, $C = 0$, shows undamped response.)

Fig. 8. Three-story shear building (symmetrical)

- a) Excitation forces, $F_1, \dots; x_1$.
- b) Ground displacement excitation, $x_g; x_1$.
- c) Ground acceleration excitation, $\ddot{x}_g; x_1$.

Fig. 9. Step-wise representation of general excitation functions for multi-story building.

Fig. 10. Assumed fundamental-mode deflection shape for use in Rayleigh approximation.

Fig. 11. Natural frequencies for building of n stories; k and m constant for all stories.

Fig. 12. Natural frequencies for "building" of n stories; k and m constant for all stories except that $k_1 \rightarrow 0$.

Fig. 13. Unsymmetrical single-story building.

Fig. 14. Coordinate system and principal elastic axes for unsymmetrical single-story building.

Fig. 15. Examples of unsymmetrical plans

- a), b) One-fold symmetry
- c) No symmetry.

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Fig. 16. Elevation view of a rigid block-type structure on an elastic foundation; an example of one-fold symmetry in the vertical plane.

Fig. 17. Unsymmetrical two-story building; for restrictions on coordinates see Fig. 8.

Fig. 18. Coordinate system and principal elastic axes for unsymmetrical two-story building.

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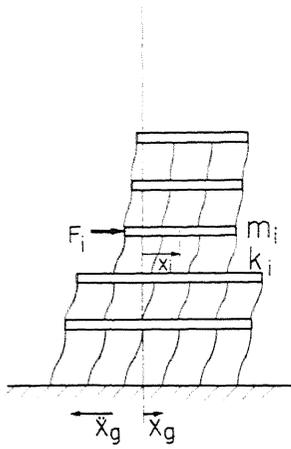


FIG.1

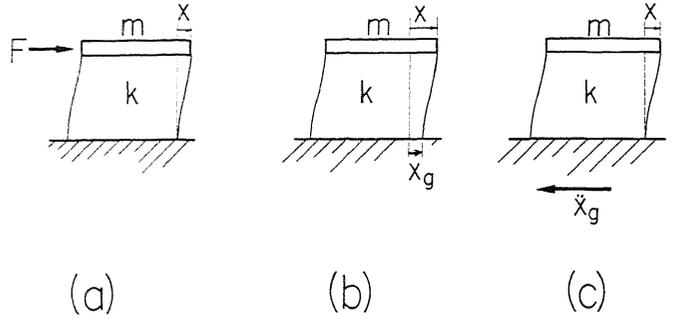


FIG.2

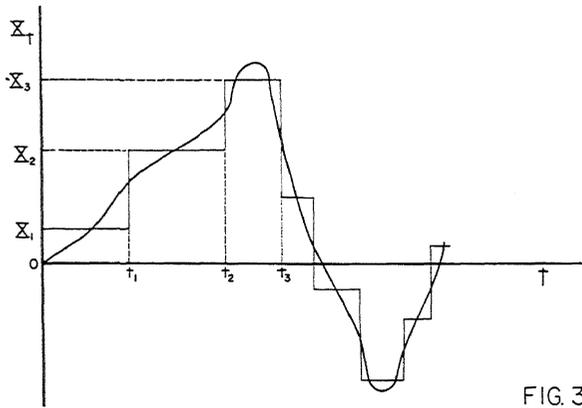


FIG.3

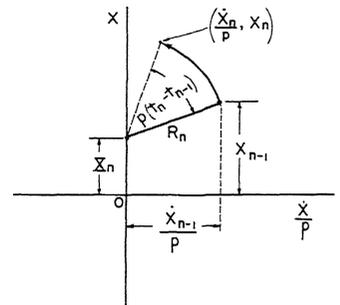


FIG.4

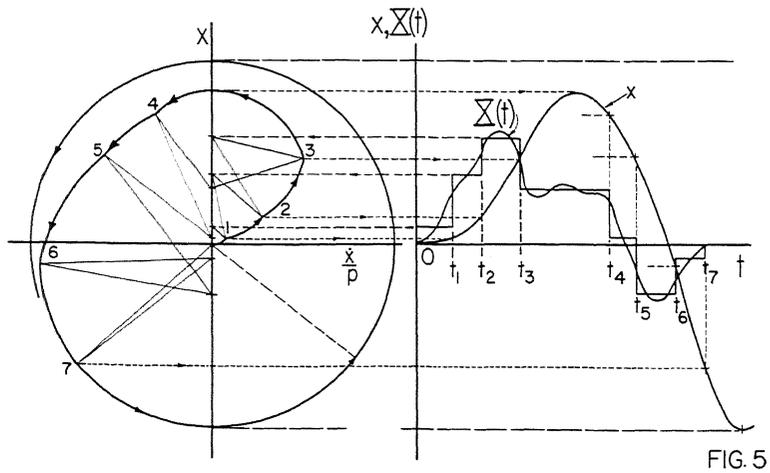


FIG.5

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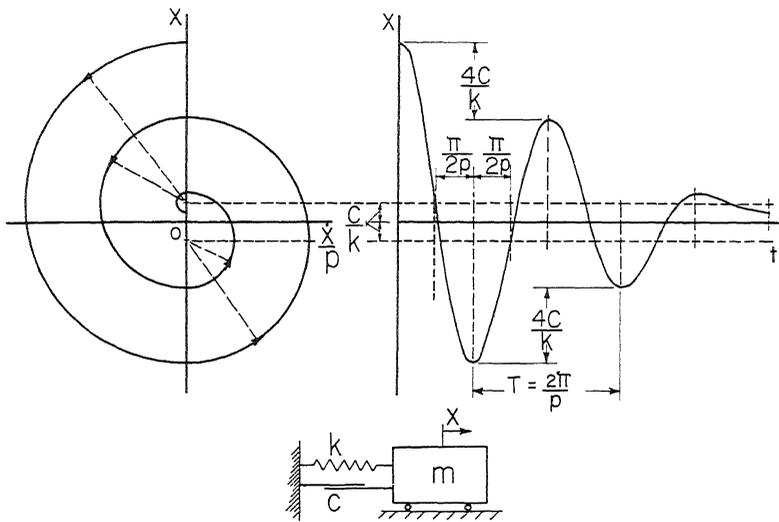


FIG.6

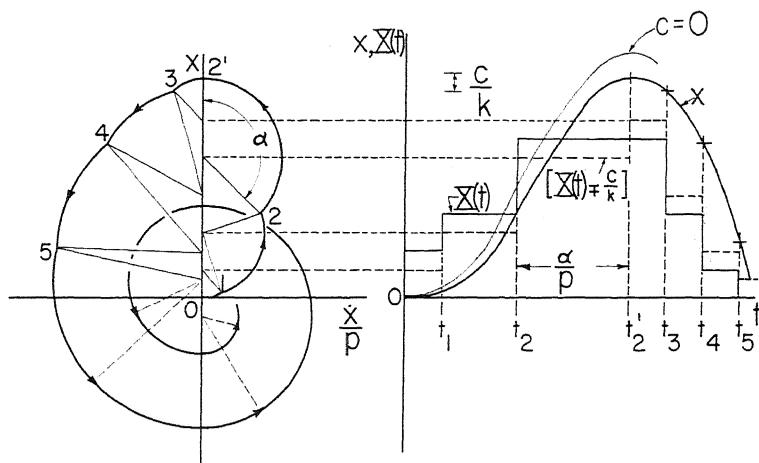


FIG.7

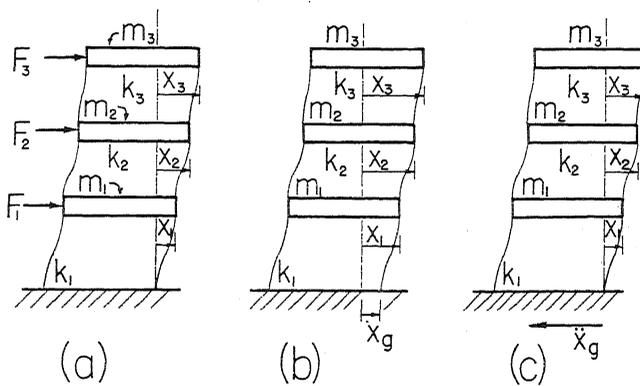


FIG.8

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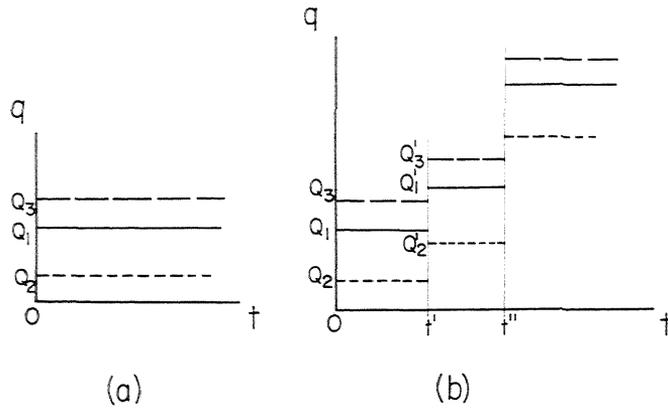


FIG.9

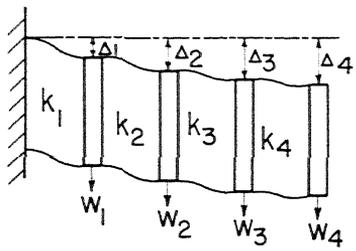


FIG.10

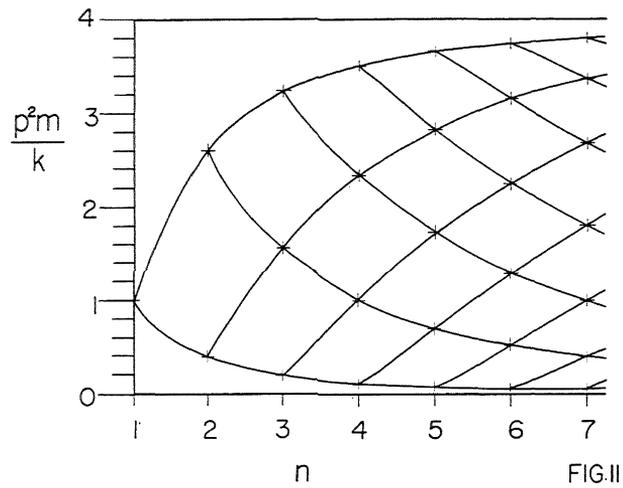


FIG.11

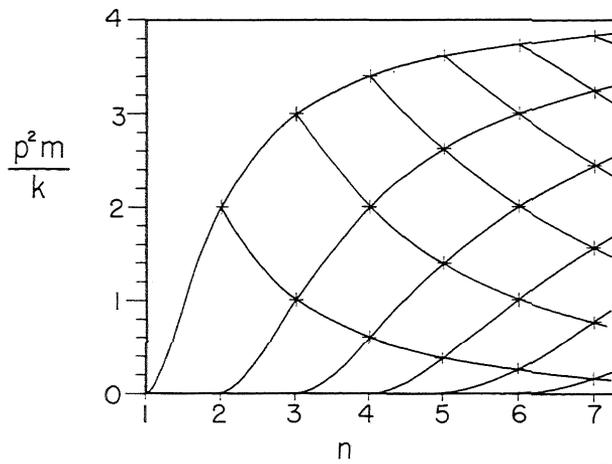


FIG.12

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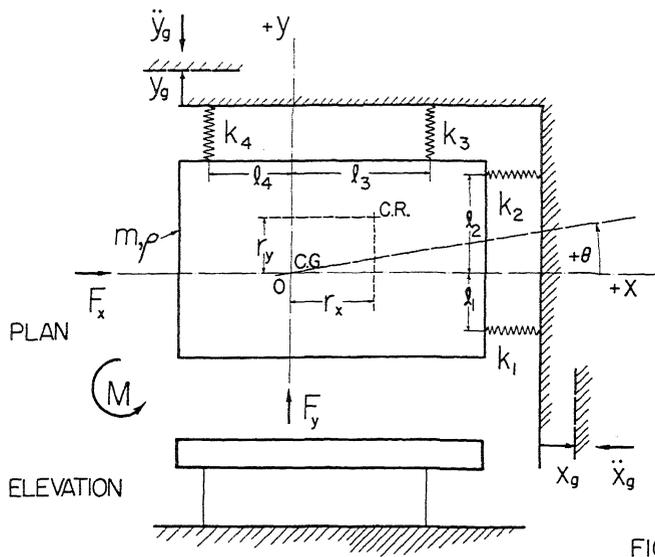


FIG.13

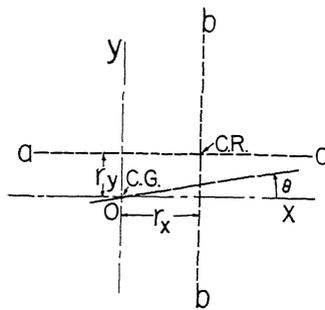


FIG.14

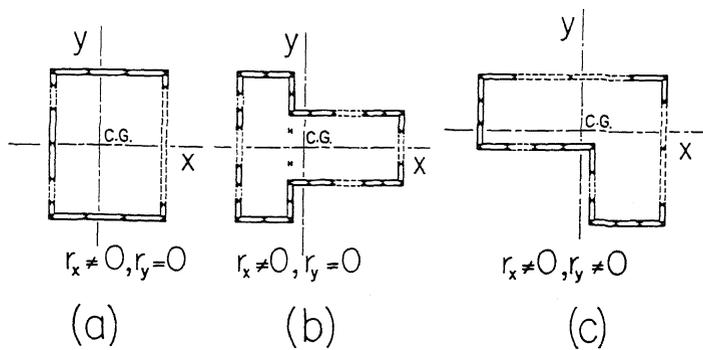


FIG.15

ANALYSIS OF STRUCTURAL RESPONSE

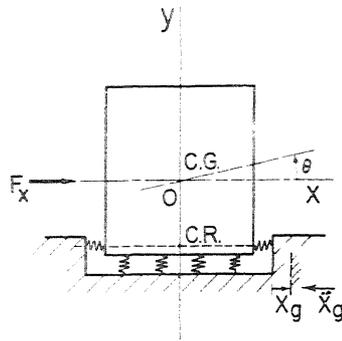


FIG. 16

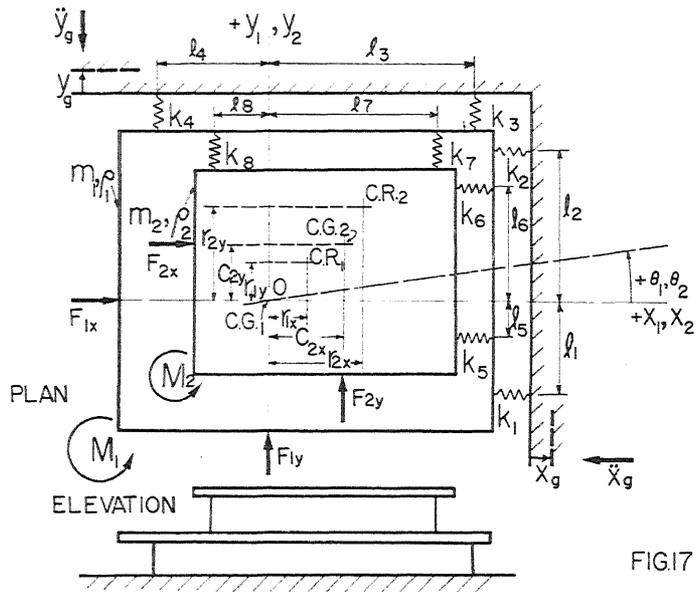


FIG. 17

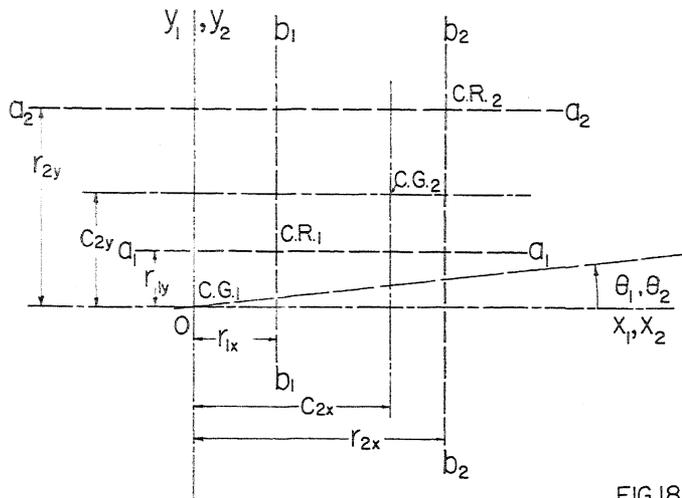


FIG. 18