

SOME APPLICATIONS OF PROBABILITY THEORY
IN ASEISMIC DESIGN

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Introduction

The most important type of destructive ground motion originates as a series of slips in extensive fault zones. The slips occur in a manner similar to a chain reaction. Each slip gives birth to a seismic wave that traverses a number of geologic formations before it affects an engineering work. Nearly all of these formations are markedly irregular, especially in seismic regions, and each seismic wave must undergo countless reflections, refractions, and irregular dispersion in its travel. It therefore reaches the structure, no longer as a single wave, but as a series of elemental pulses whose distribution in time is highly sensitive to the exact location of the slip and to other variables. Consequently one should expect detailed characteristics of different earthquakes of equal intensity to be quite different, even when their foci and places of registration coincide.

The only conditions under which marked local tendencies in detailed characteristics, or anything similar to predominating ground periods, may be operative are on thick layers of soft soil and in the proximity of massive structures.

The nature of both stages -- birth and travel of earth waves -- suggests randomness as the salient common characteristic of destructive motions. The mere inspection of accelerograms confirms this and, in a more quantitative way, so does the remarkable similarity between maximum structural responses to true earthquakes and to random series of pulses, as pointed out by Housner (Ref. 7). That the mechanism of earthquakes must be essentially as described, and that it must continue to be so as long as the earth's crust does not radically change its nature, has been conclusively proved (Refs. 8,9).

Randomness of destructive seisms implies the necessity of designing on a probabilistic basis if rational quantitative results are desired. This would hold even if one could predict the intensities and dates of occurrence of future earthquakes; the unpredictability of these variables reinforces the need for the probabilistic approach. The present paper dwells on the problem of the distribution of structural responses to earthquakes of specified intensity. Some aspects related to the probability distribution of intensities and the combination of both distributions have been treated elsewhere (Refs. 14,15,19).

It seems at first sight that a purely statistical treatment should lead to conclusions suited for aseismic design. But the amount of infor-

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mation obtainable in any reasonable period of observation is small compared with the reciprocals of permissible chances of structural failure; extrapolation of empirical distribution curves so far beyond the range of observation cannot yield useful results. Hence, statistical analyses serve only to fix empirical parameters. The type of probability distribution must be calculated on analytical bases.

Acknowledgments. Preparation of this paper was sponsored by the Institutes of Geophysics and of Engineering, Univ. of Mexico. The writer is particularly indebted to Dr. M. Moshinsky (Inst. of Phys.) and to Professor R. Valdés (Inst. of Math.) for their valuable suggestions in the analytical aspects of this work, and to X. Salazar and G. Sosa (Inst. of Eng.) for their excellent cooperation.

The Simple Structure

A basic fictitious earthquake (Fig. 1a) will be defined as a series of numerous closely spaced, instantaneous velocity pulses whose signs and magnitudes are random, and which are characterized by a constant intensity per unit time.

By velocity pulses are meant changes in ground velocity. The condition that the pulses be numerous is to be taken in the sense that

$$E\left(\sum_{i=1}^n u_i^2\right) \gg \max_i E(u_i^2), \quad (1)$$

where E denotes expectation, n = number of pulses in the motion, and u_i = magnitude of the jumps in ground velocity for the i th pulse. "Closely spaced" in the above definition signifies that the expected lapses between successive pulses be small in comparison with the natural period of a structure to be investigated.

Intensity per unit will be defined as the quantity

$$2\sqrt{\chi} = \sqrt{\frac{E(\sum u_i^2)}{\Delta t}} \quad (2)$$

where the time interval Δt is taken small in comparison with the natural period of the structure in question, but large as compared with the expected lapse between successive pulses. The summation sign extends over all pulses that occur in Δt .

From the viewpoint of the ground motion itself, it is important that the ground be at rest after the motion ends. The condition, however, is irrelevant in what concerns structural responses, provided the earthquake is long compared with the natural period of the system analyzed. Indeed, all pulses may be assumed positive (Ref. 7) if they are random in time.

Given the universe of basic earthquakes of a certain intensity, it is first of all required to find the probability distribution of responses of any simple structure. Refinements tending to make the analysis more realistic, such as consideration of variable $\chi(t)$, will be introduced later. The more refined analysis will attempt to give a design response associated with each permissible probability of failure.

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"Simple structure" refers to an undamped elastic system with a single degree of freedom (Fig. 1b). Let

$x_g(t)$ = ground displacement at time t ;
 $x(t)$ = displacement of mass M , relative to the ground, at time t ;
 K = spring constant or stiffness = force at M required to produce a unit x ;
 $\omega = \sqrt{K/M}$;

and let dots signify derivation with respect to time. Then Duhamel's integral gives the displacement of the mass relative to the ground as

$$x(t) = -\frac{1}{\omega} \int_0^t \ddot{x}_g(\tau) \sin \omega(t-\tau) d\tau; \quad (3)$$

(Ref. 14) whence,

$$\dot{x}(t) = -\int_0^t \ddot{x}_g(\tau) \cos \omega(t-\tau) d\tau, \quad (4)$$

$$\ddot{x}(t) - \ddot{x}_g(t) = \omega \int_0^t \ddot{x}_g(\tau) \sin \omega(t-\tau) d\tau, \quad (5)$$

with initial conditions $x(0) = 0$, $\dot{x}(0) = 0$.

Now let

$$r(t) = \sqrt{(\omega x)^2 + \dot{x}^2} \quad (6)$$

that is, the response r is defined so that $M r^2 / 2$ be the total energy in the system at time t . The response defined in Eq. 6 constitutes an upper bound both for ωx and for \dot{x} . Whenever ωx reaches a maximum it coincides with r . The same is not true of \dot{x} . If the natural period of the system, $T = 2\pi/\omega$, is short in comparison with the earthquake duration, say s , which is the usual case, neglect of the contribution of pulses occurring during a time interval of length $T/2$ gives the approximate relation (Ref. 7),

$$\max_t r(t) = \max_t |\omega x(t)| = \max_t |\dot{x}(t)|. \quad (7)$$

Failure of the system occurs when a certain response, say the shear in its columns, bending moments, or the displacement relative to the ground, exceeds a critical value, at least once while the ground motion lasts or shortly thereafter. The maximum of any such response is proportional to $\max_t |\omega x(t)|$. Therefore the quantity $\mathcal{R} = \max_t r(t)$ is a measure of the structural stability of the system, sufficiently accurate in most cases. First, then, the probability distribution of \mathcal{R} will be studied.

Substitution of Eqs. 3 and 4 in 6 and a simple trigonometric transformation (Ref. 7) yield

$$r(t) = \left\{ \left[\int_0^t \ddot{x}_g(\tau) \sin \omega \tau d\tau \right]^2 + \left[\int_0^t \ddot{x}_g(\tau) \cos \omega \tau d\tau \right]^2 \right\}^{1/2}. \quad (8)$$

This form suggests considering r as the position vector of a particle that starts at the origin and moves in a two-dimensional path. The projections of r on a pair of orthogonal axes equal the bracketed terms in Eq. 8.

When the ground velocity is a random step function, as defined for

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the basic earthquake, the integrals in Eq. 8 become series of terms $u_i \sin p_i \tau_i$ and $u_i \cos p_i \tau_i$, τ_i being the instant at which the i th pulse takes place. The travel of the particle becomes a two-dimensional random walk (Wiener process). As the time intervals between successive pulses tend to zero the random walk approaches a diffusion process (Ref. 5, pp. 294-295), which satisfies the heat equation,

$$\frac{\partial v}{\partial t} = \frac{1}{\kappa} \nabla^2 (v) \quad (9)$$

Here, in terms of the random walk, $v dA$ is the probability that the particle will be found in the element of area dA at a given instant. In terms of heat flow, v is the temperature in a plane, homogeneous, isotropic, heat conducting surface with diffusivity κ . In terms of structural responses, v is the probability of distribution of \mathcal{R} if proper scale factors and boundary conditions are chosen for the heat analogy.

The symbol ∇^2 is the Laplace operator, which in cylindrical coordinates with radial symmetry becomes

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)$$

(The equivalence of v in the random walk and heat flow analogies may be shown heuristically by defining temperature as the number of particles per unit area when the particles are numerous and travel at random without interfering with each other. For a sufficiently large total quantity of particles, their number in a given element of area is proportional to the probability that a single particle be found in that element.)

It is of interest to calculate the distribution of $v(s)$ when s = earthquake duration. In the random walk analogy, the probability that $v(s)$ does not exceed a certain response a is equal to the probability that a particle leaving from the center of a disk of radius a be found within that disk after a time s when its path meets no frontiers, that is, independently of whether it has, at any time smaller than s , left the disk and returned or whether it has never left the disk.

In the heat flow analogy, this is the same as the total amount of heat in a disk of radius a , a time s after a unit instantaneous heat source has operated at the center. The differential equation and boundary conditions to be satisfied are

$$\left. \begin{aligned} \frac{\partial v}{\partial t} &= \kappa \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right); \\ v(r, 0) &= \frac{1}{2\pi r} \delta(r); v(r, t) \rightarrow 0 \text{ as } r \rightarrow \infty, \end{aligned} \right\} \quad (10)$$

where $\delta(r)$ is Dirac's delta, a function equal to zero for all values of r except at $r=0$ where it is infinite; the integral of $v(r, 0)$ over any area that includes the origin is equal to unity.

The solution of Eq. 10 is

$$v(r, t) = \frac{e^{-\frac{r^2}{4\kappa t}}}{4\pi\kappa t} \quad (11)$$

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(Ref. 2, p. 218). The total heat in the disk is then, at time s ,

$$\begin{aligned} V &= 2\pi \int_0^a r \nu(r, s) dr \\ &= 1 - e^{-\alpha^2}, \end{aligned} \tag{12}$$

with $\alpha_t^2 = a^2/4\lambda t$, $\alpha = \alpha_s$. This is the probability that $r(s) < a$. Hence,

$$Pr \{ r(s) > a \} = e^{-\alpha^2}. \tag{13}$$

The same result can be established directly by using the reciprocity principle: from symmetry every path of a particle from the origin to an arbitrary point is equally likely as its reverse. Consequently the total heat at time s in an area A , due to an instantaneous point source in the origin at time zero, is equal to the temperature at the origin at time s due to the initial condition: Unit temperature at time zero ($\nu(0) = 1$) in the region A and zero elsewhere. The theorem is valid for all fixed boundary conditions when λ does not depend on t .

Application of this theorem requires computation of the temperature at the center of the disk when initially the disk is at unit temperature and the rest of the plane at zero temperature. This immediately yields Eq. 12 (Ref. 2, p. 220). Subtracting all temperatures from unity, so that the initial temperature inside the disk is zero and outside it is $\nu = 1$, then the temperature in the center at time s gives Eq. 13 directly.

From a design viewpoint it is of interest to compute the distribution of R . This is attained by changing the boundary conditions. In the random walk analogy there is now an absorbing barrier at the boundary of the disk, so that a particle that leaves the disk does not re-enter. The probability that a particle be found inside the disk of radius a a time s after leaving the origin, with an absorbing barrier at $r = a$, equals the probability that the particle has not once left the disk. This is the probability that $r(t) < a$ for all t , or $R < a$. In the heat flow analogy the problem amounts to solving Eq. 10 with zero temperature at the circumference of the disk at all times, and calculating the total heat in the disk at time s .

Again subtracting all temperatures and probabilities from unity and using the reciprocity principle, the probability that a structure fail when subjected to a basic earthquake of duration s , when the resistance of the structure is a (in the sense that it resists without failure all $r < a$), equals the temperature at the center of a disk whose interior is initially at zero temperature and its edge is kept at unit temperature. The solution of Eq. 10 with the new boundary conditions is (Ref. 2, p. 175),

$$\begin{aligned} Pr (R_{ob} \geq a) &= \nu(0, s) \\ &= 1 - 2 \sum_{n=1}^{\infty} \frac{e^{-4\lambda_n^2/\alpha^2}}{\lambda_n J(\lambda_n)} \end{aligned} \tag{14}$$

where J_n is a Bessel function and the λ_n are the positive roots of $J_n(\lambda) = 0$. The subscripts of R refer to zero damping and basic earth-

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quakes.

For design purposes, only small probabilities of failure, corresponding to large α , are of interest. In that range the series in Eq. 14 converges slowly and it is preferable to use the asymptotic approximation

$$\begin{aligned} P_r(R_{ob} \geq \alpha) &\approx \frac{\alpha}{\sqrt{\pi}} e^{-2\alpha^2} K_{1/4}(2\alpha^2) \\ &\approx 2 e^{-\alpha^2} \left(1 - \frac{3}{16\alpha^2}\right) \\ &\approx 2 e^{-\alpha^2} \end{aligned} \tag{15}$$

(Ref. 19, pp. 274,375) where $K_{1/4}$ denotes a modified Bessel function of the second kind. The last form is twice the second member of Eq. 13. Consequently, for small probabilities of failure the distribution of the maximum response is asymptotically proportional to the distribution of the response at the end of the motion. The equivalence is not surprising for the effect of the boundary condition at $r=a$ when $\alpha \gg 1$ is merely to double the probability that particles leave the disk.

The expectation of α may be computed from Eqs. 14 and 15. By numerical integration one obtains 1.175. Therefore,

$$E(R_{ob}) = 1.175 \sqrt{4\pi s}. \tag{16}$$

The intensity of the basic earthquake will be defined arbitrarily as

$$I = 1.175 \sqrt{\int_0^s 4\pi dt} = 1.175 \sqrt{E\left(\sum_{i=1}^n u_i^2\right)}. \tag{17}$$

Its units are cm/sec. For basic earthquakes, π is constant and $I = E(R)$.

The factor of safety depends only on the probability of failure:

$$G = \frac{R_d}{E(R)} \tag{18}$$

where the numerator is the response associated with some given permissible probability of failure.

From Eqs. 16-18,

$$\frac{R_{obd}}{IG} = 1, \tag{19}$$

independently of the natural period T . Now, if $T \ll \epsilon$, the design responses that correspond to displacement X and velocity \dot{X} both relative to the ground, and to the acceleration \ddot{X} , all associated with the same probability of failure as R_d , are given by

$$\frac{2\pi}{T} X_{obd} = \dot{X}_{obd} = \frac{T}{2\pi} \ddot{X}_{obd} = R_{obd} = IG \tag{20}$$

(see Eq. 7). Full lines in Fig. 2 depict design responses to basic earthquakes as determined from Eq. 20, in terms of IG .

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Some Refinements

To bring idealized motions into a better agreement with true earthquakes it is necessary to permit a variable $\kappa(t)$. In Eq. 10 with either the boundary conditions specified therein or $w(a, t) = 0$, the transformation

$$\tau = \frac{1}{\kappa_0} \int_0^t \kappa(\theta) d\theta, \quad (21)$$

where κ_0 is arbitrary, gives the diffusion equation with constant κ . The change in time scale merely changes α into $\alpha I / \sqrt{4\kappa_0 s}$, with I defined as in Eq. 17. Hence, the distribution of responses to a motion of variable intensity per unit time is the same as that of responses to basic earthquakes. This is true of undamped systems only.

Next consider a structure subject to a basic earthquake when τ is not much smaller than s ; then $R - \rho X$ may not be negligible. If ρX and R occur at times t_1 and t_2 respectively, one can write

$$\rho X = |\rho x(t_1)| = r(t_1) \pm r(t_2) = R. \quad (22)$$

The maxima of $|\rho x|$ coincide with r and occur at intervals approximately equal to $T/2$. The probability that $R - \rho X$ exceed an arbitrary value ξ is therefore approximately the same as the probability that a particle, leaving from the circumference $r = \rho X$, be found inside the circle of same radius a time $T/2$ later, having crossed the circumference $r = \rho X + \xi$ at least once. This equals the probability that the particle be found inside the same circle with an absorbing barrier at $r = \rho X + \xi$. Also, it is the temperature $w(\rho X, T/2)$ when $w(r, 0) = \frac{2(r - \rho X)}{2\pi \rho X}$ subject to $w(r, t) = 0$ as $r \rightarrow \infty$, minus the temperature $w(\rho X, T/2)$ with the same initial condition and $w(\rho X + \xi, t) = 0$.

Expressions for these temperatures as functions of time and of the two radii have been published (Ref. 2, pp. 174, 220). The most interesting case corresponds to large values of ρX , for which the boundaries approach straight lines. The solution obtains, then through a simple application of explicit formulas for linear flow in a semi-infinite solid (Ref. 2, p. 40). This gives

$$\left. \begin{aligned} \Pr(R - \rho X > \xi) &= \frac{1}{2} \operatorname{erfc} \xi \sqrt{\frac{2}{\pi T}} \cdot \xi > 0, \\ \Pr(R - \rho X = 0) &= \frac{1}{2}. \end{aligned} \right\} \quad (23)$$

Since ξ is practically independent of ρX , the distribution of ρX for small chances of failure is, asymptotically,

$$\Pr(\rho X \leq a') = \int_0^\infty \Pr(R \leq a' + \xi') \frac{\partial}{\partial \xi'} \Pr(\xi \leq \xi') d\xi'. \quad (24)$$

From Eqs. 15 and 23, then

$$\Pr(\rho X \leq a') = \frac{1}{2} \left(1 - e^{-\frac{a'^2}{4\kappa s}} \right) + \sqrt{\frac{2}{\pi \kappa T}} \int_0^\infty \left[1 - e^{-\frac{(a' + \xi)^2}{4\kappa s}} \right] e^{-\frac{2\xi^2}{\kappa T}} d\xi \quad (25)$$

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which gives, after making $\text{Pr}(\rho X > a') = \text{Pr}(R > a)$,

$$\text{Pr}(\rho X > a') = \frac{1}{2} e^{-\alpha'^2} \left[1 + \frac{e^{\frac{\theta \alpha'}{1+\theta}}}{\sqrt{1+\theta}} \text{erfc} \left(\alpha' \sqrt{\frac{\theta}{1+\theta}} \right) \right] = e^{-\alpha'^2}, \quad (26)$$

where $\alpha'^2 = a'^2 / 4 \kappa s$, $\theta = T / 8s$, and $\rho X_d / R_d = \alpha' / \alpha$. The ratio $\rho X_d / R_d$ increases with the probability of failure and tends to unity.

Values of $\rho X_d / R_d$ appear in Fig. 2 for $s = 10$ sec (which seems a conservative estimate for durations of basic earthquakes equivalent to true temblors) and for permissible probabilities of failure of 10^{-2} and 10^{-4} . Even with $T = 20$ sec the error introduced by taking $\rho X_d = R_d$ amounts to less than 4% with a probability of failure of 10^{-2} . Since actual designs will rarely be based on more than one chance of failure in a hundred, it is justified to take $\rho X_d = R_d$ in all cases.

It is simple to show that $\rho X_d = \ddot{X}_d / \rho < \dot{X}_d < R_d$. Consequently Eq. 21 is always sufficiently accurate. However, the distributions of ρX , \dot{X} , and R for larger probabilities of failure than are usual in design and large T/s differ appreciably.

At this point it is convenient to introduce the concept of "spectrum". The spectrum of a ground motion for any responses Q (with $Q = R$ or X or \dot{X} or a linear function of these) is defined as the graph that represents the maximum absolute values of the responses Q of simple structures as ordinates and the corresponding natural periods of the structures as abscissas. The "average spectrum" for a response Q is the graph of $E(Q)$ as a function of T .

In particular the average R -spectrum of basic earthquakes is a horizontal line in the entire range $0 < T < \infty$. Approximately the same is true of average ρX - and \dot{X} -spectra, for, according to Eq. 23, $E(R - \rho X) = 0.170 E(R) \sqrt{T/s}$, and for $T = 4$ sec with $s = 10$ sec, $E(\rho X)$ differs from $E(R)$ by less than 11% (dot-dash line in Fig. 2).

On the other hand the weighted average of ρX_0 -spectra of real earthquakes differs appreciably from a horizontal line (Fig. 2 and Refs. 8,9). This indicates an important effect due to the shape and duration of elemental pulses and to their systematic reflections.

When all pulses have the same length and shape, so that the i th pulse u_i of a basic earthquake is replaced by the acceleration

$$\begin{aligned} \ddot{x}_i(t) &= b_i \ddot{x}_0(t-t_i), \quad t_i \leq t \leq t_i + t_0, \\ \ddot{x}_i(t) &= 0, \quad t_i > t \text{ or } t > t_i + t_0, \\ \ddot{x}(t) &= \sum_i \ddot{x}_i(t), \end{aligned}$$

$$b_i = \frac{u_i}{\int_{t_i}^{t_i+t_0} \ddot{x}_0(t) dt},$$

that is, a pulse of duration t_0 , magnitude u_i , and shape \ddot{x}_0 , the distributions of all $Q/E(Q)$ are approximately the same as for a basic earthquake provided $t_0 \ll s$. This can be seen from Eqs. 3-8, where the substitution does not change the form of the equations. However, from Eq.

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17, the intensity I and hence all responses become affected by a factor

$$\beta_{nr} = \max_t \int_0^{t_0} \ddot{x}_o(\tau) \cos p(t-\tau) d\tau. \quad (27)$$

Factors β_{nr} , as functions of the natural period T , have been computed for various assumed wave shapes (Ref. 6). If the pulses vary in shape and duration throughout the motion, β_{nr} can be got from the square root of the weighted square average of terms similar to the second member in Eq. 27; the order in which the pulses occur is unimportant. In any case, whatever be the response Q in question and the permissible probability of failure, Q_d is equal to β_{nr} times the design value Q_{od} of responses to basic earthquakes of the same intensity as the true motions considered.

Given an empirical $\beta_{nr}(T)$, there are an infinity of combinations of shapes, intensities, durations, and relative frequencies of elemental pulses that would give the same factor β_{nr} . These variables are of no practical importance so long as they cannot be accurately predicted by purely analytical methods. The only significant variable is β_{nr} itself.

The empirical curve for $\beta_{nr}(T)$ shown in Fig. 2 was deduced from the weighted average of a number of earthquake spectra of widely different intensities, all registered on firm ground. Actually, the variation of β_{nr} as a function of T seems to depend somewhat on the intensity. But available records are still meager, and the systematic departures from the average are not large. Hence it is justified, at least for the present, to adopt a single curve, independent of intensity.

The scale of β_{nr} cannot be established empirically, for $\beta_{nr}I$ always appears as a product. This is not serious, however, since it is also the product that is of interest in design.

An important difference between true earthquakes and the idealized motions considered lies in the systematic wave reflections that take place in strata of rock whose physical characteristics differ from those of neighboring formations. Ground motions at the free surface of a horizontal layer of soil that rests on a semi-infinite base rock are given by

$$x_g(t) = (1+\eta) \sum_{n=0}^{\infty} (-\eta)^n x_r(t - \frac{2n+1}{4} T_g) \quad (28)$$

(Refs. 13,17,20) where η = a reflection constant of the ground formations, T_g = fundamental period of top layer, and x_r = ground displacement that would take place at the rock surface if the top layer were not present. Eq. 28 assumes elasticity in both formations and considers effects of only horizontal shear waves. Spectra have been computed for a variety of subsoil conditions (Refs. 11,13,17) and have received qualitative empirical confirmation.

In particular, an instantaneous velocity pulse is changed into a damped chain of pulses. The response r of a simple structure becomes ζr after a sufficiently long time, with

$$\zeta = (1+\eta) \left\{ \left[\sum_{n=0}^{\infty} (-\eta)^n \sin \frac{n\pi T_0}{2} \right]^2 + \left[\sum_{n=0}^{\infty} (-\eta)^n \cos \frac{n\pi T_0}{2} \right]^2 \right\}^{1/2}$$

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$$\xi = \frac{\sqrt{1 + 2\eta \cos \frac{pT_g}{2} + \eta^2}}{1 - \eta} \quad (29)$$

(The first expression results from substituting Eq. 28 in Eq. 8 and moving the time origin an interval $+T_g/4$.) Since responses to all the pulses are affected by the same factor, the ordinates of the spectrum itself must be multiplied by approximately this same factor. Eq. 29 shows that ξ is a periodic function of the natural circular frequency p , with relatively flat maxima at $T = T_g / (2n - 1)$, $n = 1, 2, 3, \dots$

Actually T_g and η depend on the wave velocity, the location of the epicenter, and other variables. Moreover, dynamic structural characteristics are never known precisely. It is therefore indicated to regard ξ as a random function depending on T_g , which in turn should be assigned a uniform probability distribution, as this is sufficiently accurate for any smooth distribution of pT_g . The distribution of R will be the combination of the one found previously for basic earthquakes and the distribution of ξ , regarded as independent variables. From Eq. 29,

$$\Pr(\xi \geq z) = \frac{1}{\pi} \cos^{-1} \frac{z^2(1-\eta)^2 - 1 + \eta^2}{2\eta} \quad (30)$$

$$\Pr\left\{\frac{R}{E(R)} \geq \theta\right\} = 1 - \int_0^{\frac{1+\eta}{1-\eta}} \Pr\left\{\frac{R_0}{E(R_0)} \leq \frac{\theta \bar{\xi}}{z}\right\} \frac{d}{dz} \Pr(\xi \leq z) dz \quad (31)$$

Here the integrand is computed from Eqs. 14, 15, 16, and 30, and

$$\bar{\xi} = \frac{2(1+\eta)}{\pi(1-\eta)} \mathcal{E}\left(\frac{2\sqrt{\eta}}{1+\eta}\right) \quad (32)$$

(\mathcal{E} represents the complete elliptic integral.) Eq. 31 can be evaluated numerically. For small probabilities of failure it becomes a Gauss exponential. Thus, with $\eta = 0.9$, Eq. 31 tends to a probability of failure of $0.29 e^{-0.57\theta^2}$.

Empirical Distribution

In Fig. 3 the theoretical distribution of $R/E(R)$ without systematic wave reflection (curve 0) was tested in comparison with several empirical distributions. Scales of this graph are such that the theoretical distributions tend asymptotically to straight lines for all η . Curve 1 shows the distribution of $pX/E(pX)$ as responses of 20 simple structures to 21 fictitious basic earthquakes ranging from 20 to 36 pulses each. These motions were constructed from tables of random digits. Pulse intensities tended to simulate those of true earthquakes (Refs. 14, 16). It was assumed that $E(pX)$ would be proportional to $\sqrt{\Sigma u_i^2}$. The ratio $E(pX)/\sqrt{\Sigma u_i^2}$ depends somewhat on the number of pulses in a motion and on their intensities when their number is small. This and the difference between $E(R)$ and $\mathcal{E}(pX)$ explain the fact that curve 1 should lie to the right of curve 0 for small probabilities of failure. Abscissas of the dash line are proportional to those of the theoretical distribution; it fits the empirical curve satisfactorily.

Curve 2 represents a similar study of responses $R/E(R)$. Curve 3

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shows the same results as curve 2 but using the average \bar{R} of each motion instead of $E(R)$. The dotted line is based on a correction factor $\sqrt{n/(n-1)}$, n = number of values of R per motion. (This factor applies strictly to the asymptotic approximation.) Agreement with curve 0 is good.

Curves 4 and 5 were obtained from 21 velocity spectra of true earthquakes*. Straight-line variation was assumed between reported successive maxima and minima of \dot{X} for these spectra. Thence the curves in Fig. 3 were constructed after division of each \dot{X} by the β_{nr} corresponding to the same natural period (from Fig. 2) and by the earthquake intensity I (from the average \dot{X}/β_{nr} of each spectrum). Curve 5 differs from curve 4 by the omission of responses in the neighborhood of $\tau = 1.10$ sec in the N88° W component of the Seattle, Wash. earthquake of 22 June 1950, which shows an abnormally pronounced peak. The peak is in all likelihood due to systematic wave reflections characteristic of the locality (see G. W. Housner et al. Bul. SSA 43, 2, 97-119) and is not at all typical of the other spectra. Therefore curve 5 may be regarded as representative of earthquakes registered on firm ground. For small probabilities of failure, responses from curves 0 and 5 differ by about 15%. The difference between $E(\dot{X})$ and $E(R)$ and local variations in β_{nr} are probably insufficient to account for it, and the most reasonable explanation lies in that $\gamma \neq 0$.

The agreement of curve with the theoretical distribution for $\gamma = 1/3$ (curve 6) is very good. It is safe, hence, to recommend design based on curve 6 for structures resting on ground not characterized by markedly systematic wave reflections.

Damping

Linear damping requires the introduction of a factor $e^{-c(t-\tau)}$ in the integrands of Eqs. 3-5 and 8. Here $c = \mu \rho$, μ = damping ratio in terms of critical damping, ρ_0 = undamped natural circular frequency = $\sqrt{K/M}$, and $\rho = \rho_0 / \sqrt{1-\mu^2}$. For responses at the end of the motion, $t = s$ is fixed, and the introduction of the exponential factor amounts to changing $\ddot{x}_g(\tau)$ into $\ddot{x}_g(\tau) e^{-c(s-\tau)}$. Since basic fictitious motions are statistically symmetric, the transformation is equivalent to changing $\ddot{x}_g(\tau)$ into $\ddot{x}_g(\tau) e^{-c\tau}$. Hence, in the case of basic motions, viscous damping merely changes the intensity I into $I' = \beta_{nr} I$ with

$$\beta_{nr} = \sqrt{\frac{1-e^{-2cs}}{2cs}} \quad (33)$$

which obtains directly from the second member in Eq. 17 after putting $\kappa e^{-2c\tau}$ for κ (see Eq. 2).

Equation 33 is valid for $r(s)$. But since the asymptotic approximation to the probability of failure (Eq. 15) is twice the probability that $r(s)$ exceed the structural resistance (Eq. 14), Eq. 33 may be used for a correction factor in design of damped systems when considering small probabilities of failure.

Examination of Eq. 27, where the factor $e^{-c\tau}$ should be introduced in the integrand to include viscous damping, shows that β_{nr} and β_{nr} are multi-

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plicative provided $c t_0 \ll 1$. Moreover, neglect of damping in the consideration of systematic wave reflections introduces errors on the safe side. Hence, damped structures may be designed to resist responses given by

$$Q_d = \beta Q_{od}, \quad \beta = \beta_w \beta_{nr}. \quad (34)$$

Figure 4 depicts values of β as a function of the damped natural period. In this graph β_w was taken from Fig. 2, and β_{nr} from Eq. 33 with the conservative estimate of 10 sec for s .

In order to calculate the complete probability distribution of damped responses to basic motions, it is convenient to multiply both members of Eq. 8 by e^{ct} . The analogous diffusion equation becomes

$$\left. \begin{aligned} \frac{\partial v}{\partial t} &= \alpha e^{2ct} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right); \\ v(r, 0) &= \frac{1}{2\pi r} \delta(r), \quad v(2e^{ct}, t) = 0. \end{aligned} \right\} \quad (35)$$

Due to the second boundary condition, the reciprocity principle does not apply. Eq. 35 may be solved numerically by introducing exterior instantaneous circular sinks (negative heat sources) with intensities such as to satisfy the second boundary condition at a finite number of instants. To this end it is convenient to use the transformation $r = \frac{e^{2ct} - 1}{2c}$ which changes the differential equation into the ordinary diffusion equation with a moving boundary. The further transformation $\rho = r e^{-ct}$ permits solution by relaxation or by separation of variables. The latter method leads to a form similar to Eq. 14, save that the Bessel functions become confluent hypergeometric functions.

Research is being carried on to evaluate the theoretical distribution, refine the treatment of systematic reflections, combine this with linear damping, and correlate with damped spectra of true earthquakes.

Structures with Several Degrees of Freedom

Responses of the general linearly damped system equal the sum of responses of its natural modes, each regarded as an independent system with a single degree of freedom. Hence, $r(t) = \sum_i r_i(t)$ and $Q < \sum_i Q_i$.

Now, $\sum_i r_i(t)$ may be regarded as the response of a simple structure to an earthquake of duration t consisting of the superposition of n motions, each with the same characteristics as the original motion and an intensity proportional to $E[r_i(t)]$. If the natural periods differ sufficiently, the r_i 's may be regarded as independent variables and it follows from the definition of α that the r_i 's are additive. Therefore

$$Q_d = \sqrt{\sum_i Q_{id}^2} \quad (36)$$

provided the responses in the natural modes may be regarded as independent variables. This condition is satisfied even for relatively close natural periods; witness the satisfactory agreement between the theoretical probability distribution and that of responses to fictitious motions (Fig. 3) when in that study the differences between natural periods of the various structures did not exceed 0.2 sec.

Clearly, Eq. 36 holds for undamped systems designed on the basis of

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probabilities of failure smaller than about 0.1 (see Fig. 3). Damping decreases the influence of systematic reflections. Hence Eq. 36 may be expected to hold with sufficient accuracy in practically all cases of interest in design.

To find the amplitude of each mode one must make use of excitation coefficients. Let y_i denote the shape of the i th mode to an arbitrary scale. Associated with y_i there is a response Q_i (stress, deflection, etc.) proportional to the design response Q_{id} . Its true magnitude may be found by multiplying all ordinates y_i and associated responses by $\phi_i \beta_i I G$, where ϕ_i is the i th coefficient of excitation:

$$\phi_i = \frac{1}{\beta_i} \frac{\int y_i dm}{\int y_i^2 dm}. \quad (37)$$

Here dm denotes differential mass, and the integrals cover the entire system. Thus, $X_{id} = \phi_i \beta_i y_i I G$ is the shape of the i th mode to design scale; the responses associated therewith are the design responses in that mode. Some applications of this method have been presented elsewhere (Refs. 13,17,18,19).

Design Criteria

Application of the design method suggested by the foregoing study leads to extremely high design stresses. For example, the S 45° E component of the Ferndale earthquake of 9 February 1941 gives $I = 5.6$ ft/sec when β_w is drawn to the scale of Fig. 2. Horizontal shears in typical buildings designed to resist this earthquake with any reasonable values of μ and G are many times greater than those specified in current building codes. A similar result has been found by other methods of dynamic analysis (Refs. 3,4,10). Yet buildings resist these ground motions.

Part of the explanation lies in the large factors of safety specified, especially for static loads. But there are more important reserves of strength. The structural frame may undergo large plastic deformations without signs of failure, and deformations of ductile systems are appreciably smaller than those of elastic structures (this is shown conclusively in Ref. 12, which errs on the safe side by neglecting systematic wave reflections). When brick walls are present, their rigidity and resistance are often many times greater than those of the structural frame.

It is clear that rational designs must be based on permissible deformations rather than stresses and should include effects of non-bearing walls on rigidity and resistance. But types of failure other than those due to excessive deformation should also be considered. One common type consists in the discomfort or panic of the occupants. A convenient measure in this respect is the maximum velocity of the structure. Since this criterion calls for comparison with reactions of persons on the ground or in extremely rigid buildings, the velocity relative to the ground constitutes a better basis for design than the absolute velocity. The relative velocity may be computed from $X_d = \sqrt{\Sigma \beta_i^2} X_{id}$. A permissible 1 ft/sec has been suggested for apartment buildings (Ref. 18).

Damping ratios may usually be assumed independent of the natural period (Ref. 1).

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Conclusions Applicable in Design

Erring (perhaps excessively) on the safe side, the following method of design is suggested.

1. Taking in consideration the rigidity of partitions, compute the shapes y_i and circular frequencies $p_i = p_{i0} / \sqrt{1 - \mu^2}$ of natural modes.
2. Find the excitation coefficients ϕ_i and the correction factors for damping and finite wave duration. Hence $x_{id} = \phi_i \beta_i y_i I G$.
3. Compute pertinent responses Q_{id} from x_{id} , such as velocities relative to the ground, displacements of each floor relative to the one below, shears in the columns of each story; and design for the responses $Q_d = \sqrt{\sum Q_{id}^2}$. It is indicated to make liberal use of limit design, to lower the factor of safety in permissible stresses, and to pay greater heed to relative displacements than to stresses, using as main criterion of design that permissible deformations not be exceeded. The general tendency should be towards the greatest rigidity and the greatest damping.

This method has three main drawbacks. It is unsuited for routine design, although its application to typical structures may serve as a guide. It applies only to structures founded on firm ground; proposed correction factors for structures resting on soft layers lack qualitative empirical confirmation. Finally, the method may be overly conservative due to its radical simplifications, particularly in the calculation of β . Still it offers a rational quantitative basis for aseismic design.

Nomenclature

a (cm/sec)	= a particular value of the response r or \mathcal{R} ;
b (subscript)	= reference to basic fictitious motions;
c (sec ⁻¹)	= μp ;
d (subscript)	= reference to design;
E (operator)	= expectation;
G (dimensionless)	= factor of safety;
I (cm/sec)	= intensity;
K (kg/cm)	= spring constant;
M, m (kg sec ² /cm)	= mass;
P_r (operator)	= probability (distribution function);
p (rad/sec)	= natural circular frequency;

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Q (any units)	= structural response;
r (cm/sec)	$= \sqrt{(\dot{\rho} x)^2 + \dot{x}^2}$;
s (sec)	= earthquake duration;
T (sec)	= natural period;
t (sec)	= time;
u (cm/sec)	= pulse magnitude;
v (deg)	= temperature;
x (cm)	= displacement relative to the ground;
x_g (cm)	= ground displacement;
y (cm)	= natural mode shape;
α_s (dimensionless)	$= \frac{a^2}{4\pi s}$, $\alpha = \alpha_s$;
β (dimensionless)	$= \beta_v \beta_w$;
β_v (dimensionless)	= correction factor due to viscous damping;
β_w (dimensionless)	= correction factor due to pulse shape;
κ (cm ² /sec ²)	= diffusivity;
μ (dimensionless)	= damping ratio in terms of critical damping;
ξ (cm/sec)	$= R - \rho X$;
ϕ (dimensionless)	= excitation coefficient.

Dots signify differentiation with respect to time. Maximum absolute values are capitalized.

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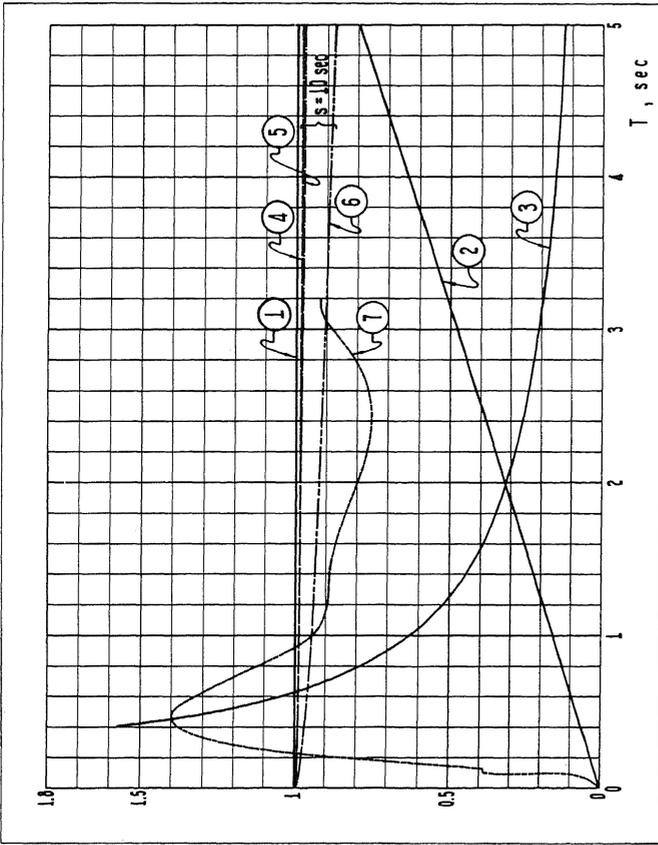


Fig. 2. Average and design spectra.

- (1) R_d/IG .
- (2) $X_d/IG, T/s \ll 1$.
- (3) $X_d/10IG, T/s \ll 1$.
- (4) $pX_d/IG, 10^{-4}$ chances of failure.
- (5) $pX_d/IG, 10^{-2}$ chances of failure.
- (6) $E(pX)/I$.
- (7) β_w (from Ref. 9, slightly modified).

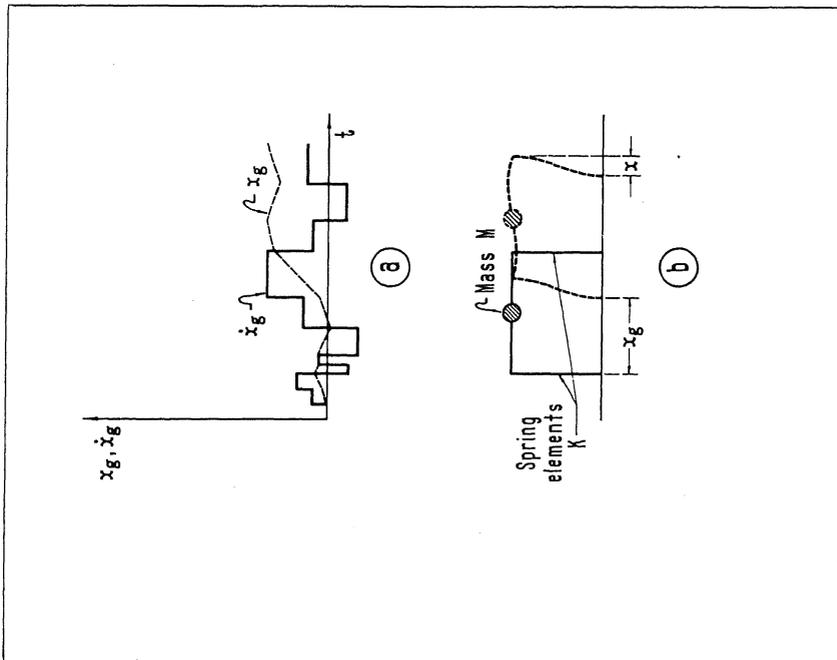


Fig. 1. Basic earthquake and simple structure.

