SEISMIC EFFECTS IN FLEXIBLE LIQUID STORAGE TANKS

by

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SYNOPSIS

A simple procedure is presented for evaluating the dynamic forces induced by the lateral component of an earthquake motion in liquid-filled cylindrical tanks of circular cross section, giving due consideration to the effects of tank flexibility. Only the so-called impulsive effects are investigated. The procedure is based on the assumptions that the tank-fluid system behaves as a single-degree-of-freedom system and that the fluid is incompressible. The effective masses for the system, the magnitudes and distributions of the hydrodynamic forces, and the base shear and moment induced by these forces are evaluated for several different assumed modes of vibration, and the results are summarized in a form convenient for design applications. It is concluded that the seismic effects in flexible tanks may be substantially greater than those induced in similarly excited rigid tanks.

INTRODUCTION

Probably the most widely used method for evaluating the seismic effects in cylindrical liquid storage tanks is the one proposed by Housner (2, 3). This method is limited to the analysis of rigid tanks. The objective herein is to describe a method which is not so limited and allows the effects of tank flexibility to be taken into account easily.

Only the so-called impulsive forces are investigated in this study. These are the forces induced on the assumption that there are no gravity surface waves. The convective forces, which are associated with the sloshing of the liquid, must be determined separately and combined appropriately with those considered herein. Because the convective effects are characterized by oscillations of much longer periods than those characterizing the impulsive effects, they cannot be influenced significantly by the flexibility of the tank. Hence, they may be evaluated by the procedure applicable to rigid tanks, as described in Refs. 2, 3, or 4.

The method presented herein is an extension of that used by Chopra (5, 6, 7) to study the effects of fluid-structure interaction in reservoir-dam systems.

SYSTEM AND ASSUMPTIONS

The tank-fluid system considered is shown in Fig. 1. It is a circular cylindrical structure of arbitrary wall thickness having radius R, height H_s, and containing fluid to a height H. The mass per unit of height of the structure without the fluid is denoted by $\mu(y)$, the total mass of the shell is denoted by m_s, and the total mass of the roof, including the appropriate live load, is denoted by m_r. A point on the middle surface of the shell is defined by the coordinates y and θ, and the plane $\theta = 0$ is taken parallel to the direction of ground motion.

Fundamental to the analysis presented is the assumption that the system

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responds as a single-degree-of-freedom system. Specifically, it is assumed that the cross section of the tank does not change shape during deformation, and that the deflection configuration of the tank at any time is of a prescribed form. It is further assumed that the liquid is incompressible and inviscid, and that the ratio of the height of the liquid to the radius of the tank, H/R, is less than about 1.2. The latter assumption makes it possible to evaluate the hydrodynamic forces on the tank from existing expressions applicable to a straight wall storing a reservoir of semi-infinite extent (5, 6, 7). This restriction may be relaxed, however, with a relatively small increase in the complexity of the resulting expressions.

HYDRODYNAMIC PRESSURES AND FORCES

Pressure on a Straight Wall. Consider first a straight vertical wall storing a reservoir of semi-infinite extent, and assume that the wall experiences a horizontal acceleration

$$a(y,t) = \psi(y)\,a(t) \tag{1}$$

where $a(t)$ = the acceleration of the wall at the reservoir level, and $\psi(y)$ = a dimensionless function defining the assumed variation of $a(y,t)$ along the wall height. It follows that at $y=H$, $\psi(y)=\psi(H)=1$.

The hydrodynamic pressure, in excess of the hydrostatic, which is induced by this motion is given (5) by the equation

$$p(y,t) = \frac{4}{\pi}\gamma H \frac{a(t)}{g}\sum_{n=1}^{\infty} d_n \cos\left[(2n-1)\frac{\pi}{2}\frac{y}{H}\right] \tag{2}$$

in which γ = the unit weight of the liquid, g = the gravitational acceleration, n = an integer, and d_n = dimensionless coefficients given by

$$d_n = \frac{1}{H(2n-1)}\int_0^H \psi(y)\cos\left[(2n-1)\frac{\pi}{2}\frac{y}{H}\right]dy \tag{3}$$

For $\psi(y)=1$ and $\psi(y)=y/H$, d_n are given, respectively, by

$$d_n = \frac{2}{\pi}\frac{(-1)^{n-1}}{(2n-1)^2} \quad \cdots\cdots\cdots (4) \quad \text{and} \quad d_n = \frac{2}{\pi}\frac{1}{(2n-1)^2}\left[(-1)^{n-1} - \frac{2}{\pi(2n-1)}\right] \cdots (5)$$

The magnitudes and distributions of the pressures corresponding to a rigid body motion, $\psi(y)=1$, and to the three additional modes of vibration indicated in Fig. 2 are shown in Fig. 3. The significance of the dashed lines in the latter figure is explained later.

Effects on a Cylindrical Tank. Within the range of H/R values considered, the hydrodynamic pressure induced along the generator $\theta=0$ of a cylindrical tank which is accelerated in accordance with Eq. 1 may, for all practical purposes, be considered to be the same as that given by Eq. 2. The pressure at an arbitrary point (y,θ) is then

$$p(y,\theta,t) = p(y,t)\cos\theta \tag{6}$$

and the corresponding force per unit of tank height is

$$P(y,t) = \int_0^{2\pi} [\,p(y,t)\cos\theta\,][\,R\,d\theta\,]\cos\theta = \pi R\,p(y,t) \tag{7}$$

The hydrodynamic pressure and force per unit of height for a rigid tank may also be determined from information presented in Refs. 1 to 3 but, for reasons of consistency with steps taken later, they will be evaluated from the expressions given herein by letting $\psi(y) = 1$.

EQUATION OF MOTION

Let $u(t)$ = the ground displacement at any time t, and $w(t)$ = the displacement relative to ground of a section of the tank at the liquid level. The absolute acceleration of the tank at a distance y from the base may then be expressed as

$$a(y,t) = \ddot{u}(t) + \psi(y)\,\ddot{w}(t) \tag{8}$$

where a dot superscript denotes differentiation with respect to time, and the dimensionless function $\psi(y)$ defines the assumed mode of deformation for the tank. Only functions for which $\psi(0) = 0$ are admissible in this case.

External Forces. The external forces associated with the uniform component of the acceleration, $\ddot{u}(t)$, are shown in Fig. 4a. They consist of

- the structural inertia forces; these include a distributed force of intensity $\mu(y)\,\ddot{u}(t)$ and a concentrated force at the top of magnitude $m_r\ddot{u}(t)$; and
- the hydrodynamic force, $P_u(y,t)$, which can be determined from Eqs. 7, 2, and 3 by taking $\psi(y) = 1$ and replacing $a(t)$ with $\ddot{u}(t)$.

The forces associated with the nonuniform acceleration component, $\psi(y)\,\ddot{w}(t)$, are shown in Fig. 4b. They consist of

- the inertia forces due to the mass of the structure; these include, in addition to a distributed force of magnitude $\mu(y)\,\psi(y)\,\ddot{w}(t)$ per unit of height, a concentrated force of magnitude $m_r\psi(H_s)\,\ddot{w}(t)$ at the top; and
- the hydrodynamic force $P_w(y,t)$, which can be determined from Eqs. 7, 2, and 3 using the assumed deflection function $\psi(y)$ and taking $a(t) = \ddot{w}(t)$.

Equation of Motion. With the hydrodynamic and inertia forces identified, the equation of motion for the tank-fluid system may be determined by application of the virtual work principle. By equating the work done at any time t by the external forces during a virtual displacement $\delta w(y,t) = \psi(y)\,\delta w$ to the work done by the internal forces, the following equation is obtained

$$[m^*_{w,s} + m^*_{w,\ell}]\,\ddot{w} + c^*\dot{w} + k^*w = -\,[m^*_{u,s} + m^*_{u,\ell}]\,\ddot{u}(t) \tag{9}$$

where

$$m^*_{u,s} = \int_0^{H_s} \mu(y)\,\psi(y)\,dy + m_r\,\psi(H_s) \cdots (10) \qquad m^*_{w,s} = \int_0^{H_s} \mu(y)\,\psi^2(y)\,dy + m_r\,\psi^2(H_s) \cdots (11)$$

$$m^*_{u,\ell} = \int_0^{H} S_u(y)\,\psi(y)\,dy \cdots\cdots\cdots\cdots (12) \qquad m^*_{w,\ell} = \int_0^{H} S_w(y)\,\psi(y)\,dy \cdots\cdots\cdots\cdots (13)$$

The quantities $S_u(y)$ and $S_w(y)$ in the last two equations represent, respectively, the forces $P_u(y,t)$ and $P_w(y,t)$ when $\ddot{u}(t) = \ddot{w}(t) = 1$.

The right hand member of Eq.9 and the first term on the left hand member represent, respectively, the work done by the forces in Figs. 4a and 4b, whereas the second and third terms represent the work done by the damping forces and internal forces. The sum $m^*_{u,s} + m^*_{u,\ell} = m^*_u$ represents the effective mass of the system for the rigid body component of motion, and $m^*_{u,s}$ and $m^*_{u,\ell}$ represent the contributions of the structural mass and liquid mass, respectively. In an analogous manner, $m^*_{w,s} + m^*_{w,\ell} = m^*_w$ represents the effective mass of the system for the motion specified by $\psi(y)$, and $m^*_{w,s}$ and $m^*_{w,\ell}$ represent the contributions of the structural mass and liquid mass. The quantities c^* and k^*, which stand for the effective damping coefficient and the effective stiffness of the system, may also be expressed in terms of $\psi(y)$, but this is not generally necessary.

Dividing Eq. 9 by m^*_w leads to

$$\ddot{w} + 2\zeta\omega\dot{w} + \omega^2 w = - C\,\ddot{u}(t) \tag{14}$$

in which ω = the circular natural frequency of the system corresponding to the assumed vibration mode, ζ = the damping factor, and

$$C = \frac{m^*_u}{m^*_w} = \frac{m^*_{u,s} + m^*_{u,\ell}}{m^*_{w,s} + m^*_{w,\ell}} \tag{15}$$

is the participation factor. The evaluation of the natural frequency, ω, is considered in a later section.

Effective Masses. The values of m^*_u and m^*_w corresponding to the four deflection functions referred to previously are listed in Table 1 for two different values of H_s/H. The deflection function between $y = H$ and $y = H_s$ is assumed to be a straight line tangent to the deflection curve at $y = H$, and the wall thickness of the tank is assumed to be constant.

In Table 1, note that the value of $m^*_{w,\ell}$ corresponding to a given $\psi(y)$ is significantly smaller than the associated value of $m^*_{u,\ell}$. Note further that, whereas both of these values are sensitive to variations in $\psi(y)$, the participation factor, C, is relatively insensitive to such variations.

SOLUTION OF EQUATION

The solution of Eq. 14 can be obtained by analogy to that governing the motion of a simple mass-spring-dashpot oscillator.

Let D and A be, respectively, the maximum deformation and pseudo-acceleration of a simple oscillator which has the same natural frequency and damping factor and is subjected to the same excitation as the actual tank-fluid system. The maximum value of w(t) and the associated pseudo-acceleration are then given by

$$w_o = CD = CA/\omega^2 \quad \cdots\cdots\cdots \quad (16) \quad \text{and} \quad \omega^2 w_o = CA \quad \cdots\cdots\cdots \quad (17)$$

With w_o determined, the maximum relative displacement of a general section of the tank is $\psi(y)\,w_o$, and the associated pseudo-acceleration is $\psi(y)\,CA$

The values of U and A for the oscillator may be determined from the response spectrum corresponding to the particular input motion considered. A representative spectrum, plotted in the familiar tripartite form, is shown in solid lines in Fig. 5, where $V = \omega D = A/\omega$.

Equivalent Static Forces. The maximum forces induced in the structure may be determined either by differentiation of the deflection function, or by first determing the equivalent static forces corresponding to the computed maximum deflection and by evaluating the desired effect from these equivalent static forces. The latter approach is the more reliable of the two and will be used exclusively herein. Two different schemes of implementing this approach are described.

Scheme 1. This is the conventional approach in which the equivalent static earthquake forces are taken proportional to the inertia forces and the hydrodynamic forces induced by the deformation of the structure, $w(y,t)$. The distributions of these forces are shown in Fig. 4b, and their magnitudes are determined by replacing $\ddot{w}(t)$ with $\omega^2 w_o = CA$.

It must be emphasized that only these equivalent static forces must be used to determine the maximum effects induced by the earthquake. The effect of the exciting dynamic forces have already been incorporated in evaluating the equivalent static forces and should not be reconsidered.

Scheme 2. For an extremely stiff system for which $A = \ddot{u}_o$, it might be expected that the equivalent static forces would be equal to the inertia and hydrodynamic forces associated with a rigid body motion of the tank. It might also be expected that Eq. 16 would yield the exact static deflection due to these forces. Because of the approximate nature of the analysis, however, this is not generally the case. Instead, the results obtained represent the best, in the least squares sense, one-term approximation of the desired effect for the assumed configuration.

A more conservative solution may be obtained by expressing the equivalent static earthquake forces as the sum of the following components: (a) the maximum forces induced on a rigid tank; and (b) a set of forces proportional to those computed by scheme 1, with their magnitudes determined by replacing CA with $C(A - \ddot{u}_o)$. The first set of forces is shown in Fig. 4a, where $\ddot{u}(t)$ must be interpreted as the maximum ground acceleration, $\ddot{u}_o$. These effects will be referred to as the static or rigid tank effects and will be identified with the subscript st. Any desired effect may then be determined by superposing on the static value of the effect a dynamic increment corresponding to an effective acceleration $(A - \ddot{u}_o)$. For example, the value of w_o determined by this scheme is

$$w_o = (w_{st})_o + C(A - \ddot{u}_o)/\omega^2 \tag{18}$$

Considering that A may be greater than $\ddot{u}_o$, it follows from Eq. 18 that the seismic effects in a flexible tank may be greater than those induced in a rigid tank.

Choice of Schemes. It is recommended that Scheme 2 be used when the natural frequency of the system is greater than the

frequency corresponding to point f in Fig. 5;
Scheme 1 be used within the frequency range defined by points c and e; the latter point corresponds to a frequency value which is at one-third the distance between points c and f; and that
For all other frequencies greater than that corresponding to point b, the response be determined by averaging the results obtained by schemes 1 and 2.

For systems having natural frequencies smaller than the value corresponding to point b, the contributions of the higher natural modes are generally important, and the single-degree-of-freedom solution presented herein may not be sufficiently accurate. However, it is unlikely that tanks of practical proportions will have such a low natural frequency.

MAXIMUM BASE SHEAR AND MOMENT

The maximum base shear, Q_o, and the maximum base moment, M_o, can be determined from the equivalent static forces by integration.

Computation by Scheme 1. When evaluated by this approach

$$Q_o = \left[\int_0^{H_s} \mu(y)\,\psi(y)\,dy \right] CA + m_r\,\psi(H_s)\,CA + Q_{o,\ell} \tag{19}$$

and

$$M_o = \left[\int_0^{H_s} \mu(y)\,\psi(y)\,y\,dy \right] CA + m_r\,\psi(H_s)\,H_s\,CA + M_{o,\ell} \tag{20}$$

where $Q_{o,\ell}$ and $M_{o,\ell}$ represent the contributions of the liquid, and are given by

$$Q_{o,\ell} = \int_0^H P(y,t)\,dy \quad \cdots\cdots\cdots \quad (21) \quad \text{and} \quad M_{o,\ell} = \int_0^H P(y,t)\,y\,dy \quad \cdots\cdots\cdots \quad (22)$$

The force $P(y,t)$ in the last two equations is determined from Eqs. 7, 2 and 3 using the function $\psi(y)$ under consideration and taking $a(t) = \ddot{w}(t) = CA$. By making use of Eq. 10, Eq. 19 can also be written as

$$Q_o = m^*_{u,s}\,CA + Q_{o,\ell} \tag{23}$$

Values of $Q_{o,\ell}$, $M_{o,\ell}$ and $\Delta M_{o,\ell}$. Eqs. 21 and 22 have been evaluated for a rigid body motion, $\psi(y) = 1$, and for the three additional modes of vibration shown in Fig. 2. The results are summarized in Table 2, where W_ℓ represents the total weight of the liquid in the tank, and $\ddot{w}_o$ represents the maximum value of $\ddot{w}(t)$.

It is now possible to comment on the significance of the diagrams shown in dashed lines in Fig. 3. They represent pressure distributions for which the values of $Q_{o,\ell}$ and $M_{o,\ell}$ are in good agreement with those listed in Table 2. The constant intensity portions of these diagrams extend to $y = 0.85H$.

The values of $M_{o,\ell}$ defined by Eq. 22 and listed in Table 2 refer to a section immediately above the base, and do not include the effect of the hydrodynamic pressure on the base of the tank. The additional moment, $\Delta M_{o,\ell}$, contributed by the pressure on the base can be determined approximately from the following expression used by Housner in the analysis of rigid tanks

$$\Delta M_{o,\ell} = 0.884\, p_o R^3 \tag{24}$$

in which p_o = the maximum hydrodynamic pressure at the junction of the base and the wall of the tank. The values of p_o corresponding to the four functions $\psi(y)$ considered were evaluated from Eq. 2 and are listed in the following

Deflection Function, $\psi(y)$	1	$\sin \frac{\pi}{2}\frac{y}{H}$	$\frac{y}{H}$	$1-\cos\frac{\pi}{2}\frac{y}{H}$
Value of p_o, in terms of $\gamma H \ddot{w}_o/g$	0.743	0.282	0.200	0.106

For $\psi(y) = 1$, $\ddot{w}_o$ should be interpreted as $\ddot{u}_o$.

It should be noted that the moment determined from Eq. 24 may be quite conservative for tanks having values of H/R less than about 0.5. The base pressure in such cases tends to be highly concentrated near the edge, and the distribution implied in the use of this equation tends to exaggerate the moment.

Computation by Scheme 2. The static or rigid tank component of Q_o in this approach is given by

$$(Q_{st})_o = [m_s + m_r]\,\ddot{u}_o + [0.542\,\frac{H}{R}]\,m_\ell\,\ddot{u}_o \tag{25}$$

and the associated component of M_o is given by

$$(M_{st})_o = \left[\int_0^{H_s} \mu(y)\, y\, dy + m_r H_s\right]\ddot{u}_o + [0.217\,\frac{H}{R}]\, m_\ell H \ddot{u}_o \tag{26}$$

where the terms on the extreme right were obtained from Table 2. The remaining components of Q_o and M_o, representing the effects of tank flexibility, may be determined from the results obtained by Scheme 1 by replacing A with $A - \ddot{u}_o$.

Vibrational Mode and Natural Frequency. The configuration $\psi(y)$ which is appropriate in a given case depends on the relative magnitudes of the flexural and shearing deformations for the structure. These magnitudes depend, in turn, on the dimensions of the tank and on the relative weights of the roof system and of the contained liquid. The following procedure is recommended for selecting $\psi(y)$:

1. Assume a trial configuration $\psi(y)$; for convenience, it may be taken equal to one of the functions considered in Fig. 2.
2. Compute the resulting inertia and hydrodynamic forces taking, for simplicity, $\ddot{w}(t) = g$; the hydrodynamic force may be determined directly from Fig. 3 using either the exact curve or the trapezoidal approximation.
3. Compute the deflection of the tank due to the forces determined in step 2, considering the effects of both flexural and shearing deformations.
4. The desired $\psi(y)$ is the deflection determined in step 3, normalized with respect to the deflection value computed at $y = H$.

The circular natural frequency of the system, ω, may then be determined from Rayleigh's quotient, $\omega^2 = V/T_o$, where V is the integral of the product of the forces identified in step 2 and the deflection computed in step 3, and T_o, the pseudo-kinetic energy of the system, is given by

$$T_0 = m_w^* w^2 = [m_{w,s}^* + m_{w,\ell}^*] w^2 \tag{27}$$

In this equation, m_w^* is the effective mass associated with the deflection function computed in step 4, and w is the deflection at y=H computed in step 3.

SUMMARY AND CONCLUSION

The principal steps of the method may be summarized as follows:

1. By application of the procedure described in the preceding section, select a mode of vibration, $\psi(y)$, and evaluate the natural frequency of the system corresponding to this mode.
2. From the data presented in Table 1 or by integration of the relevant expressions, determine the effective masses $m_u^* = m_{u,s}^* + m_{u,\ell}^*$ and $m_w^* = m_{w,s}^* + m_{w,\ell}^*$.
3. Determine the participation factor C as the ratio m_u^*/m_w^*.
4. From the response spectrum applicable to the particular ground motion under consideration, determine the pseudo-acceleration, A, corresponding to the natural frequency determined in step 1. The pseudo-acceleration of a general section located at a distance y from the base is then $\psi(y)\,CA$.
5. Determine the equivalent static earthquake forces by Scheme 1 or 2, as appropriate. In Scheme 1, these forces are taken proportional to those due to the deformation of the tank, and their magnitudes are determined by setting $\ddot{w}(t) = CA$. In Scheme 2, these forces are expressed as the sum of those exerted on a rigid tank and those induced by the deformation of the tank when $\ddot{w}(t) = C(A - \ddot{u}_0)$. In either case, the hydrodynamic pressures are determined from Fig. 3.
6. With the equivalent external static earthquake forces determined, the internal forces are computed by standard methods. The magnitudes of the maximum base shear and maximum overturning base moment, both excluding and including the effect of the hydrodynamic pressure on the base, may be determined from the data presented in Table 2.

Application of this procedure to the analysis of practical systems reveals that the seismic effects in flexible tanks may be significantly greater than those induced in similarly excited rigid tanks.

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TABLES

TABLE 1. EFFECTIVE MASSES m_u^* AND m_w^*

Deflection Function $\psi(y)$	$\frac{H_s}{H}$	Components of m_w^*		Components of m_u^*	
		$m_{w,s}^*$	$m_{w,\ell}^*$	$m_{u,s}^*$	$m_{u,\ell}^*$
$\sin\frac{\pi}{2}\frac{y}{H}$	1.0	$0.5m_s + m_r$	$0.178\,\frac{H}{R}\,m_\ell$	$\frac{2}{\pi}m_s + m_r$	$0.293\,\frac{H}{R}\,m_\ell$
	1.1	$0.6m_s + m_r$		$0.74\,m_s + m_r$	
$\frac{y}{H}$	1.0	$0.33\,m_s + m_r$	$0.103\,\frac{H}{R}\,m_\ell$	$0.5m_s + m_r$	$0.217\,\frac{H}{R}\,m_\ell$
	1.1	$0.44\,m_s + 1.21\,m_r$		$0.61\,m_s + 1.1\,m_r$	
$1-\cos\frac{\pi}{2}\frac{y}{H}$	1.0	$0.23\,m_s + m_r$	$0.050\,\frac{H}{R}\,m_\ell$	$0.36m_s + m_r$	$0.137\,\frac{H}{R}\,m_\ell$
	1.1	$0.35m_s + 1.34m_r$		$0.47\,m_s + 1.16m_r$	

TABLE 2. MAXIMUM HYDRODYNAMIC SHEAR AND MOMENTS AT BASE

$\psi(y)$	Shear, $Q_{o,\ell}$	Moment $M_{o,\ell}$	Moment $\Delta M_{o,\ell}$
1	$[0.542\,\frac{H}{R}]\,W_\ell\,\frac{\ddot{u}_o}{g}$	$[0.217\,\frac{H}{R}]\,W_\ell H\,\frac{\ddot{u}_o}{g}$	$\frac{0.209}{H/R}\,W_\ell\,H\,\frac{\ddot{u}_o}{g}$
$\sin\frac{\pi}{2}\frac{y}{H}$	$[0.294\,\frac{H}{R}]\,W_\ell\,\frac{\ddot{w}_o}{g}$	$[0.135\,\frac{H}{R}]\,W_\ell H\,\frac{\ddot{w}_o}{g}$	$\frac{0.079}{H/R}\,W_\ell H\,\frac{\ddot{w}_o}{g}$
$\frac{y}{H}$	$[0.218\,\frac{H}{R}]\,W_\ell\,\frac{\ddot{w}_o}{g}$	$[0.103\,\frac{H}{R}]\,W_\ell H\,\frac{\ddot{w}_o}{g}$	$\frac{0.056}{H/R}\,W_\ell H\,\frac{\ddot{w}_o}{g}$
$1-\cos\frac{\pi}{2}\frac{y}{H}$	$[0.137\,\frac{H}{R}]\,W_\ell\,\frac{\ddot{w}_o}{g}$	$[0.070\,\frac{H}{R}]\,W_\ell H\,\frac{\ddot{w}_o}{g}$	$\frac{0.030}{H/R}\,W_\ell H\,\frac{\ddot{w}_o}{g}$

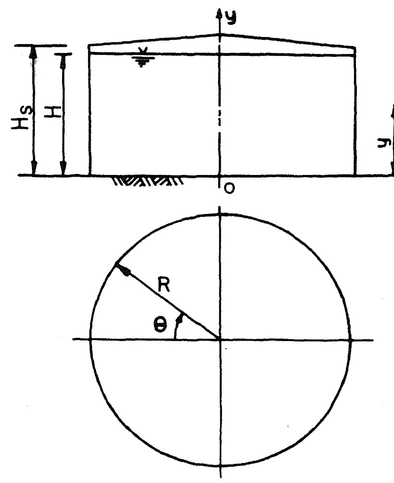

FIG.1 SYSTEM CONSIDERED

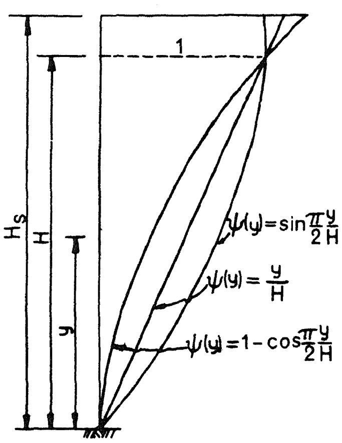

FIG.2 VIBRATION MODES CONSIDERED

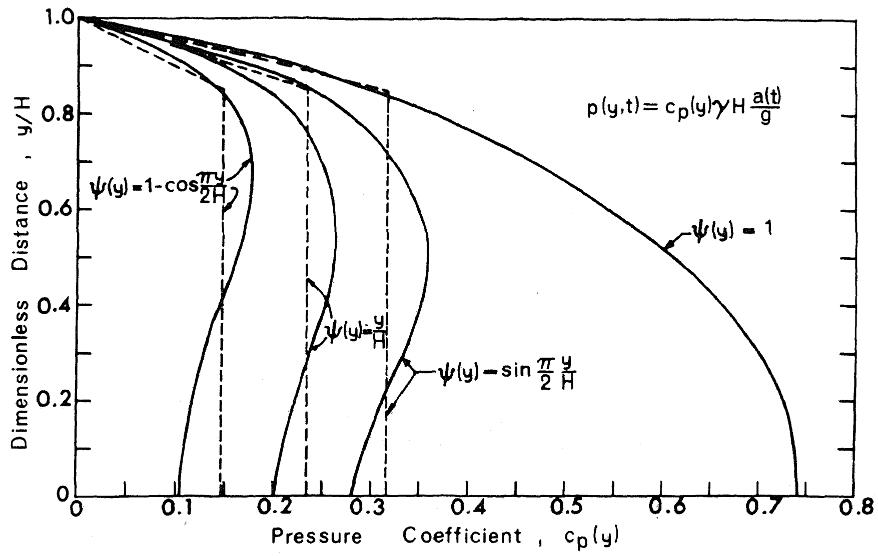

FIG.3 HYDRODYNAMIC PRESSURES FOR DIFFERENT MODES OF VIBRATION

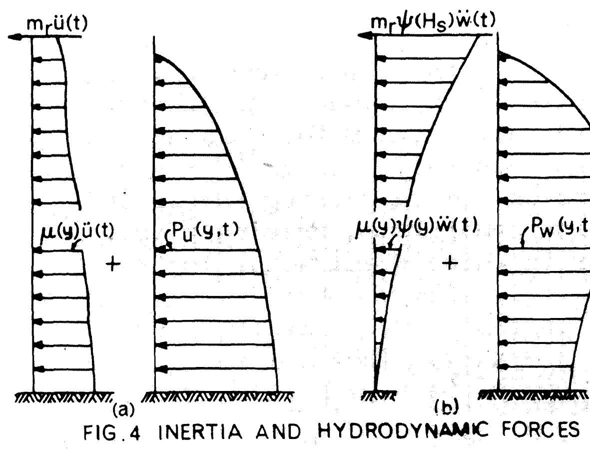

FIG.4 INERTIA AND HYDRODYNAMIC FORCES

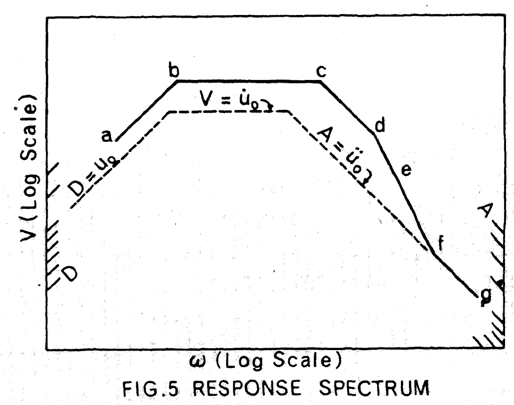

FIG.5 RESPONSE SPECTRUM