A FAST FOURIER TRANSFORM APPROACH TO EARTHQUAKE SOIL-STRUCTURE INTERACTION PROBLEMS

by

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SYNOPSIS

Fourier formulations of earthquake soil-structure interaction problems and solution procedures using FFT algorithms are presented. Various time- and frequency-dependent input-output relationships which can facilitate Fourier analysis of vibration problems of linear structures are introduced. FFT operations and computational errors that might occur when dealing with deterministic and probabilistic structural dynamics are discussed.

INTRODUCTION

The Fourier transform method has long been used for characterizing linear systems and for identifying frequency contents of a continuous waveform. However, not until the necessary numerical procedures and efficient algorithms for machine computations were developed has the Fourier transform method blossomed into a widely applied approach for the digital analysis of linear-system dynamics. In this paper, a new dimension is added to applications in soil-structure interaction problems where the structure-foundation systems are treated as linear discrete (lumped mass) systems with frequency-dependent soil coefficients, and the earthquake excitation and responses are sampled with discrete time intervals. In what follows, Fourier formulations and transform relationships for linear systems including some useful inequalities for various time- and frequency-dependent input or output variables are presented. Soil-structure interaction problems are then cast into the framework of Fourier formulations and the solution procedure via Fast Fourier Transform (FFT) is introduced. Numerical results for some single and multistory buildings are also presented and discussed. Finally, the FFT operations for probabilistic aspects of earthquake analysis are indicated.

FOURIER TRANSFORM APPROACH IN LINEAR SYSTEM ANALYSIS

Let $f(t)$ be a continuous signal with finite energy and $F(\omega)$ be its Fourier Transform (FT), the Fourier transform pair indicated by $f(t) \leftrightarrow F(\omega)$ can be written in the form

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)dt \qquad (1)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)d\omega$$

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for $-\infty < \omega < \infty$, $-\infty < t < \infty$, and $i = \sqrt{-1}$. Without loss of generality, the variable t is used to indicate time and ω the frequency. When the waveform (earthquake accelerogram) is sampled, or the system (structure) is to be analyzed digitally, the finite, discrete Fourier Transform (DFT) as given below must be used

$$F(j) = \frac{1}{N}\sum_{k=0}^{N-1} f(k)W^{-jk}, \qquad j = 0,1,\ldots,N-1$$

$$f(k) = \sum_{j=0}^{N-1} F(j)W^{jk}, \qquad k = 0,1,\ldots,N-1 \tag{2}$$

where $W = \exp(2\pi i/N)$. In the above, index k measures the time and index j measures the frequency. If T = total finite time duration of the waveform to be analyzed and N = total data points at an equal time interval Δt, then $\Delta t = T/(N-1)$, the real time $t = k\Delta t$, the cutoff or Nyquist frequency $= \omega_{max} = 2\pi/(2\Delta t) = \pi/\Delta t$, the incremental frequency $= \Delta\omega = 2\pi/T$, and the real frequency $\omega = j\Delta\omega = 2\pi j/T$. In practice f(t) often is a real function with finite duration e.g., $t \in [a,b]$, and $T = b - a$ for a finite a and b. For this case sometimes it is useful to define a time-varying Fourier transform or simply the running spectrum of f(t) as

$$F(t,\omega) = \int_{-\infty}^{t} f(\tau)\exp(-i\omega\tau)d\tau \tag{3}$$

and the instantaneous power spectrum as

$$\rho_f(t,\omega) = \frac{\partial}{\partial t}\,|F(t,\omega)|^2 = 2f(t)\mathrm{Re}[\exp(i\omega t)F(t,\omega)] \;. \tag{4}$$

Note that $F(t,\omega)$ is Hermitian and the relation $f(t) = \exp(i\omega t)\partial F(t,\omega)/\partial t$ holds. Discrete forms for $F(t,\omega)$ and $\rho_f(t,\omega)$ can be obtained analogously as for $F(\omega)$ in Eq. 2 with the upper limit of summation being an increasing integer variable not larger than N-1.

Fourier transform or time-varying Fourier transform are extremely useful in analyzing the dynamic behavior of a linear system; most pertinent information about the system dynamics in either the time or frequency domain can be obtained through convenient FT formulations. A special application of such formulations in soil-structure interaction problems and the solution procedures via FFT techniques will be presented next.

SOIL-STRUCTURE INTERACTION AND FFT PROCEDURE

Soil-structure interaction refers to possible local distortions of the free-field earthquake motion because of the presence of a structure that is founded on the soil surface. It is well known that the vibrational responses of a building can be strongly influenced by interaction; nonetheless, most early building-vibration studies ignored it. In the first attempts to consider interaction in problems of this sort, lumped parameter representations were used to model the soil. A serious limitation to this approach exists because the numerical integration techniques used to solve the coupled time-dependent equations require that the coefficients which characterize the dynamic behavior of the soil be frequency independent. This is generally not a valid assumption as proven in Bycroft's classic paper[1] dealing with interaction forces between a circular rigid plate and elastic half space that supports it.

Parmalee et al[2] demonstrated that these frequency-dependent interaction forces defined by wave-propagation solutions could be appropriately used as foundation-force constraints in lumped parameter building-vibration studies, and a time-domain method of solution in terms of finite Fourier series was proposed. Actually, however, such building-vibration problems are better formulated in the frequency domain because of the frequency-dependent nature of the soil-structure interaction forces. Furthermore, the advent of the Fast Fourier Transform (FFT) algorithm by Cooley and Tukey[3] has made digital computer solutions using Fourier transform techniques a common engineering practice. It is therefore clear that earthquake soil-structure interaction problem is especially well suited to an FFT approach.[4,5]

The Fourier transform of the equation of motion of a building-foundation system can be expressed symbolically by the familiar matrix equation

$$[M]\{\ddot{U}(\omega)\} + [C(\omega)]\{\dot{U}(\omega)\} + [K(\omega)]\{U(\omega)\} = \{F(\omega)\} \tag{5}$$

where $[M]$, $[C(\omega)]$, and $[K(\omega)]$ are the mass, damping and stiffness matrices respectively, and $\{U(\omega)\}$, $\{\dot{U}(\omega)\}$, $\{\ddot{U}(\omega)\}$ and $\{F(\omega)\}$ are the Fourier transforms of the response vectors $\{u(t)\}$, $\{\dot{u}(t)\}$, $\{\ddot{u}(t)\}$, and the loading vector $\{f(t)\}$ respectively. Frequency-domain solutions have an inherent advantage over time-domain solutions, in that the quotient of the FT of the response variable to the FT of the input variable defines the transfer function or coefficient of the system, i.e., $H_j(\omega) = U_j(\omega)/F_j(\omega)$, which simplifies the form of Eq. 5 to

$$[Z(\omega)]\{H(\omega)\} = \{1\} \tag{6}$$

with $[Z(\omega)]$ to be determined from building and soil coefficients.

The effect of the soil on the structure can be approximated by using set of influence functions derived from solutions of a harmonically excited rigid-plate foundation model supported on an elastic half-space representing the soil medium. Such solutions were provided by Bycroft[1] who considered a circular plate model under stress distributions corresponding to static loading conditions, and by Oien[6] who considered a strip-plate model bonded to the soil half-space. The elements of $[Z(\omega)]$ are expressed in terms of these influence functions with the foundation geometry parameters (radius r for circular plates and half-width b and length d for rectangular plates) and soil parameters (shear wave velocity v_s, soil density ρ, and

Poisson's ratio ν) as explicit variables. Exact expressions of $[Z(\omega)]$ for single- and multi-degree-of-freedom shear building models can be found in Liu and Fagel.[4,5]

Having defined $[Z(\omega)]$ in Eq. 6, the transfer function vector $\{H(\omega)\}$, whose output parameters include base rocking and displacement, and building story displacements, is solved numerically by a standard matrix inversion routine. Note that Eq. 6 can be solved independent of the earthquake loading vector. These transfer functions are essentially the steady-state responses of the interaction system to harmonic excitations. The time-history of the corresponding response parameters $u_j(t)$, $j = 1,2,\ldots,n$ can be subsequently calculated for any input $f_j(t)$ by obtaining the inverse transform of the quantity $U_j(\omega) = H_j(\omega)F_j(\omega)$. It is at this Fourier inversion stage, and the stage when $F_j(\omega)$ is calculated that FFT technique is used for its high computation efficiency.

The FFT is simply an efficient method for computing the DFT as in Eq. 2. Therefore the FFT can be used in place of continuous FT only to the extent that DFT could, but with a substantial reduction in machine operations and therefore computer time. Generally the reduction results from the removal of redundant computations in DFT operations. It can be achieved by using the Cooley-Tukey recursive FFT formulations in terms of W and binary representation of the index variables k and j when $N = 2^m$ for an integer m. Savings are also achieved in the FFT algorithm by repetitive elimination of operations that are actually nothing more than a multiplication by +1 and by recognizing that $W^{\ell} = W^{\ell+N/2}$ for $\ell = 0,1,2,\ldots$. The total reduction of operations of the FFT increases very rapidly with N.

When the time series f(k) such as a digitized earthquake accelerogram being Fourier analyzed is real, the real part of F(j) is symmetric about the folding frequency $\omega_f = \omega_{max}/2$ and the imaginary part is antisymmetric. Therefore, in this case, the computation and storage requirements of complex FFT algorithms can be cut in half. Subroutines which eliminate redundant operations for real series were developed by Bergland[7] according to FFT algorithms. These subroutines evaluate real sequence A(k) and complex sequence X(j) from each other:

$$X(j) = \sum_{k=0}^{N-1} A(k)W^{jk}, \quad j = 0,1,\ldots,N/2 \tag{7-1}$$

$$A(k) = \sum_{j=0}^{N/2} X(j)W^{-jk}, \quad k = 0,1,\ldots,N-1. \tag{7-2}$$

For a specified ground acceleration, $\ddot{u}_g(t)$, which constitutes the set A(k), its FT is given by using Eq. 7-1 as

$$F(j) = \frac{1}{N} X(j)^c, \tag{8}$$

where the superscript c indicates the complex conjugate. Let the transfer functions solved from Eq. 6 be $\{H(j)\}$; the corresponding response time-history are obtained as $A(k)^c$ by Eq. 7-2 with the corresponding vector of $X(j)$ sequences given by

$$\{X(j)\} = F(j)\{H(j)\} \; . \tag{9}$$

It should be pointed out that all FFT algorithms approximate continuous wave forms as discrete data with a finite sample rate and assume that the waveforms repeat periodically. Thus, transient signals such as earthquakes could cause errors such as aliasing, leakage, and cyclic convolution effects[7] in Fourier analysis using an FFT approach. The aliasing indicates that if the sampling rate is too low (or ΔT is too large), high-frequency components of a time function can impersonate low frequencies. This error can be removed by selecting a sufficiently high sampling rate. The leakage is caused by finite record of data which is equivalently formed by multiplying an infinitely long record of data by a rectangular data window which, in turn, is equivalent to performing a convolution in the frequency domain. The approach to reduce leakage would be to use the time function being analyzed with a data window which has low sidelobes in the frequency domain. The cyclic convolution error is caused by treating a finite record of data as being periodic. This error, which might likely be induced during the frequency-domain multiplication step indicated in the right-hand-side operation of Eq. 9, can be side-stepped by allowing the original data to be followed by N extra zeroes, i.e., using $\hat{A}(k)$ in place of $A(k)$ in Eq. 7-1.

$$\begin{aligned} \hat{A}(k) &= A(k); \quad && 0 \le k < N \\ \hat{A}(k) &= 0; && N \le k < 2N \; . \end{aligned} \tag{10}$$

For illustration, a 20-story shear type building with realistic structural and soil parameters is analyzed with the presented FFT approach. Acceleration transfer functions throughout the building are given in Fig. 1. For the soft foundation case (v_s = 200 fps) the fundamental frequency is lower, the ground-floor transfer function deviates from unity by a considerable amount, and the amplitude of each resonant peak is considerably smaller than that in the case of a stiff foundation (v_s = 4000 fps). Note that Fig. 1a resembles typical lightly damped resonance curves with peaks occurring at natural frequencies of lower modes. When this building is considered to be subjected to a synthesized accelerogram (Figs. 2a and 2b), its acceleration-response time histories at the roof differ considerably for soft and stiff foundations as shown in Figs. 2c and 2d. Figure 3 shows the maximum story displacements, shears, and acceleration responses of this building, and indicates significant response variations which occur because of different soil conditions. In general the results indicate that soil-structure interaction effects are very important in structural response if the founding soil is very soft and if there is a close match between the building's fundamental frequency and the dominant frequency of the earthquake accelerogram.[5]

FURTHER APPLICATIONS OF FFT METHOD

The effectiveness of the Fourier method in solving vibration and mechanics problems depends not only on the availability of FFT algorithms, but also on the existence of simple, tractable input-output relationships for linear structures. These relationships include some useful inequalities in terms of various time- and frequency-dependent input or output parameters, Denoting the FT pairs of the input, output, and transfer function of the system as $f(t) \leftrightarrow F(\omega)$, $u(t) \leftrightarrow U(\omega)$, and $h(t) \leftrightarrow H(\omega)$ respectively, some of the more important relations between these quantities are given below without proof. Note that most of these relations can be proved by using Schwartz inequality and Parseval's theorem.

The energy function of input waveform.

$$E_f(t) = \int_{-\infty}^{t} |f(\tau)|^2 d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(t,\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{t}\int_{-\infty}^{\infty} \rho_f(\tau,\omega)\, d\omega\, d\tau \ . \tag{11}$$

From above the total energy of input signal $f(t)$, $t \in [0,T]$ is

$$E = \int_{0}^{T} |f(\tau)|^2 d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \ . \tag{12}$$

The energy function of a simple linear causal system with natural frequency ω_o and damping λ.

$$E_s(t) = \frac{1}{2}\, |V(t,p)|^2 \exp(2\lambda\omega_o t) \ , \tag{13}$$

where $p = (1-\lambda^2)^{1/2}\omega_o$ and $V(t,\omega)$ is the time-varying Fourier transform of $v(t) = f(t)\exp(\lambda\omega_o t)$.

Upper bounds for Fourier spectrum and related quantities.

$$|F(\omega)| \leq (ET)^{1/2} \tag{14-1}$$

$$|F(\omega+\delta) - F(\omega)| \leq \left[2ET\left(1 - \frac{\sin \delta T}{\delta T}\right)\right]^{1/2} , \qquad \delta \geq 0 \tag{14-2}$$

$$\left|\frac{d^nF(\omega)}{d\omega^n}\right| \leq T^n \left(\frac{ET}{2n+1}\right)^{1/2} \tag{14-3}$$

Bounds for output u(t) and related quantities.

$$|u(t)| \leq \frac{|\Omega(\omega_c)| + |\Omega(-\omega_c)|}{2\pi t} \tag{15-1}$$

where $\Omega(\omega) = F(\omega)H(\omega)$ and ω_c is the cutoff frequency of $F(\omega)$ or $H(\omega)$ whichever is smaller.

$$|u(t)|^2 \leq EN \ , \quad N = \int_{-\infty}^{\infty} |h(\tau)|^2 d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega \qquad (15\text{-}2)$$

$$\langle |u(t)|^2 \rangle \leq P|H(\omega)|^2_{max} \ , \quad P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)d\omega \qquad (15\text{-}3)$$

where the symbol $\langle\ \rangle$ indicates ensemble average, $S_f(\omega)$ is the power spectral density of a stationary random process $f(t)$.

Bounds for response spectra $S_d(\omega_o,\lambda) = \sup_t |u(t)|$ and $S_v(\omega_o,\lambda) = \sup_t |\dot{u}(t)|$.

$$\frac{1}{p}|F(p)| \leq S_d(\omega_o,\lambda) \leq \sup_t \left[\frac{1}{p}|V(t,p)|\exp(-\lambda\omega_o t)\right] \leq \sup_t \frac{1}{p}|V(t,p)| \qquad (16\text{-}1)$$

$$|F(p)| \leq S_v(\omega_o,\lambda) \leq \sup_t \left[\frac{1}{(1-\lambda^2)^{1/2}}|V(t,p)|\exp(-\lambda\omega_o t)\right] \leq \sup_t |V(t,p)|. \qquad (16\text{-}2)$$

With the help of the above relations, a wide range of structural response analyses can be cast into the framework of Fourier formulation. The FFT method has great application potential in many other important earthquake engineering problems, particularly in dealing with the probabilistic aspects when ground excitations are treated as random processes. The handicap which had prevented the nondeterministic approach of earthquake analysis from being widely accepted was due to lack of efficient algorithms to handle all the computations of both input and response ensemble statistics. The FFT approach, if used with caution and correctly, can perform operations such as convolution, digital filtering, correlation and spectrum analysis with great computational savings. Some of these FFT-aided operations are discussed briefly below.

Digital Filtering, Convolution and Power Spectrum. The problem of determining the output or the impulse response of a linear system can be represented as a convolution in the time domain

$$u(k) = \frac{1}{N}\sum_{j=0}^{N-1} f(j)h(k-j) \ . \qquad (17)$$

The above is also the form of the cross-correlation function of $f(k)$ and $h(k)$, and when $h(k) = f(k)$ it is the autocorrelation function. Equivalently Eq. 17 can be written as $U(j) = f(j)H(j)$ therefore

$$u(k) = \sum_{j=0}^{N=1} [F(j)H(j)]W^{jk} \qquad (18)$$

which can be solved by FFT using Eq. 7-2 with $X(j) = F(j)H(j)$ and $u(k) = A(k)^c$. Note that in performing convolution and correlation with

FFT, cyclical convolution error as discussed earlier is likely to occur but can be avoided as suggested by Eq. 10. Let $r(t)$ be the correlation function of a time series $f(t)$; the power spectrum density $S(\omega)$ is the FT of $r(t)$. To determine $S(\omega)$ through the estimate of $r(t)$ from a finite sample of $f(j)$, special care has to be paid to avoid aliasing and to smooth the sample spectral density by utilizing a proper spectral window.[8]

<u>Digital Synthesis of Accelerograms Under Various Soil Conditions</u>. To simulate earthquake accelerograms under various soil conditions taking interaction effects into account, FFT again can be used effectively. Assume $\phi(t)$ to be a deterministic shaping function, $n(t)$ a normal stationary process with spectral density $S_n(\omega)$, and the earthquake process at the bedrock to be represented by a multiplicative process in the form

$$f(t) = \phi(t)n(t) \ . \tag{19}$$

Further assume that the transfer function pair $h_b(t) \leftrightarrow H_b(\omega)$ for the acceleration at the interface of the structure's base and soil half-space as given in Eq. 6 is also known. The modified ground acceleration and its FT are then given by

$$f_b(t) = h_b(t)*f(t) \tag{20-1}$$

$$F_b(\omega) = H_b(\omega)F(\omega) = \frac{1}{2\pi} H_b(\omega)[\Phi(\omega)*N(\omega)] \tag{20-2}$$

in which * indicates convolution. Therefore $f_b(t)$ can be digitally synthesized using Eq. 7-2 with $X(j) = H_b(j)$ and $f_b(k) = A(k)^c$.

<u>Time-Varying FT and Response Spectra</u>. As shown in Eq. 16-2 the velocity response spectrum is bounded by the Fourier spectra of input accelerogram $f(t) = \ddot{u}_g(t)$ from below and by the maximum value of the time-varying Fourier spectrum of the modified waveform $v(t) = \ddot{u}_g(t)\exp(\lambda\omega_o t)$ from above. Therefore the time-varying FT with FFT algorithm can be utilized to estimate response spectra if large volumes of data processing are involved. Let the time-varying FT of $v(t)$ be expressed in terms of its real part and imaginary part

$$V(j,k) = \begin{Bmatrix} R(j,k) \\ I(j,k) \end{Bmatrix} = \begin{Bmatrix} \sum_{m=0}^{k=1} v(m)\cos(jm\Delta\omega\Delta t) \\ \sum_{m=0}^{k=1} v(m)\sin(jm\Delta\omega\Delta t) \end{Bmatrix} , \tag{21-1}$$

and define the following matrices[9]

$$B = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{21-2}$$

$$L(j\Delta\omega) = \begin{bmatrix} \cos(j\Delta\omega\Delta t) & -\sin(j\Delta\omega\Delta t) \\ \sin(j\Delta\omega\Delta t) & \cos(j\Delta\omega\Delta t) \end{bmatrix} \tag{21-3}$$

It can be easily shown by induction that the following relationship holds:

$$V(j,k) = V(k,k-1) + L^k(j\Delta\omega)Bv(k) \quad k = 0,1,\ldots,(N-1) \tag{22}$$

with $V(j,-1)^T = (0,0)$ where T indicates transpose. Note that $L(j\Delta\omega)$ is orthogonal and $L^k(j\Delta\omega) = L(kj\Delta\omega)$. Therefore Eqs. 7 and 22 can be conveniently used to bound response spectra according to Eq. 16.

CONCLUDING REMARKS

The Fourier transform, when used in conjunction with FFT algorithms and with simple, tractable input-output relationships, can solve a wide range of vibration problems of linear structures. Fourier formulation and solution procedures for earthquake soil-structure interaction with frequency-dependent soil coefficients are presented. The FFT approach is computationally efficient particularly when large quantities of computations are required, such as in the case of nondeterministic analyses of earthquake responses. The approach is also flexible because it can be used with any linear structural or interaction model, and it is effective because it simultaneously provides all response information in both the time and frequency domains. It is also possible to extend the FFT approach for nonlinear cases if some appropriate equivalently linearized transfer functions for the soil structure system can be found.

REFERENCES

1. Bycroft, G. N., "Forced Vibrations of a Rigid Circular Plate on a Semi-Infinite Elastic Space and on an Elastic Stratum," Philosophical Trans., Royal Soc. London, Series A., No. 948, vol. 248, January 1956, pp. 327-368.
2. Parmalee, R. A., Perleman, D. S., Lee, S. L., and Keer, L., M., "Seismic Response of Structure-Foundation Systems," J. EM Div., ASCE vol. 94, No EM6, December 1968, pp. 1295-1315.
3. Cooley, J. W., and Tukey, J. W., "An Algorithm for the Machine Calculation of Complex Fourier Series," Math. Computation, vol. 19, April 1965, pp. 297-301.
4. Liu, S. C., and Fagel, L. W., "Earthquake Interaction by Fast Fourier Transform," J. EM Div., ASCE, vol. 97, No. EM4, August 1971, pp. 1223-1237.
5. Fagel, L. W., and Liu, S. C., "Earthquake Interaction for Multistory Buildings," J. EM Div., ASCE, vol. 98, No. EM4, August 1972.
6. Oien, M. A., "Steady-Motion of a Rigid Strip Bounded to an Elastic Half-Space," J. App. Mech., ASME, vol. 38, June 1971, pp. 328-334.
7. Bergland, G. D., "A Guided Tour of the Fast Fourier Transform," IEEE Spectrum, 6(7), July 1969, pp. 41-52.
8. Cooley, J. W., Lewis, P. A. W., and Welch, P. D., "The Application of the Fast Fourier Transform Algorithm to the Estimation of Spectra and Cross Spectra," J. Sound V.b., 12(3), 1970, pp. 339-352.
9. Ahmed, N., Rao, K. R., and Tjoe, S. K., "Time-Varying Fourier Transform," Electronic Letters, 7(18), September 1971, pp. 535-536.

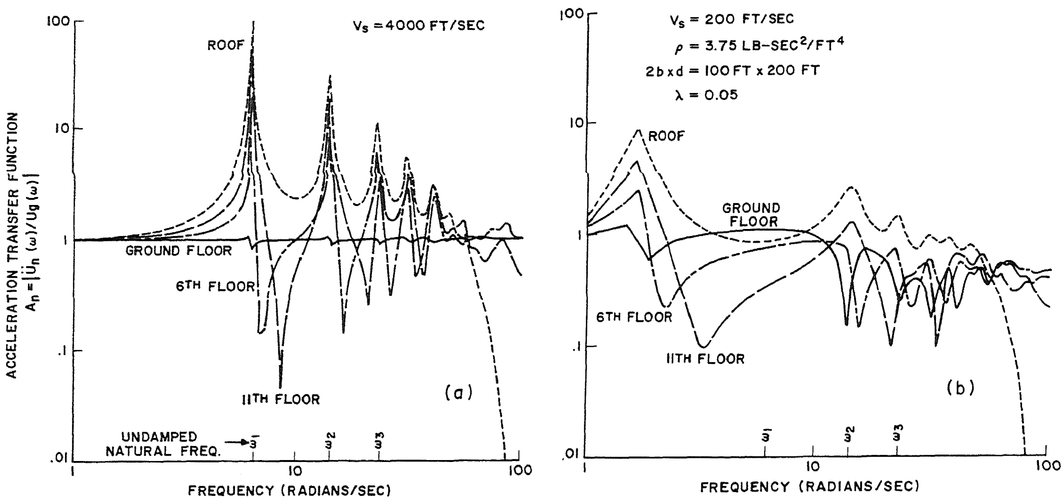

FIG. 1 – ACCELERATION TRANSFER FUNCTIONS FOR A 20-STORY BUILDING

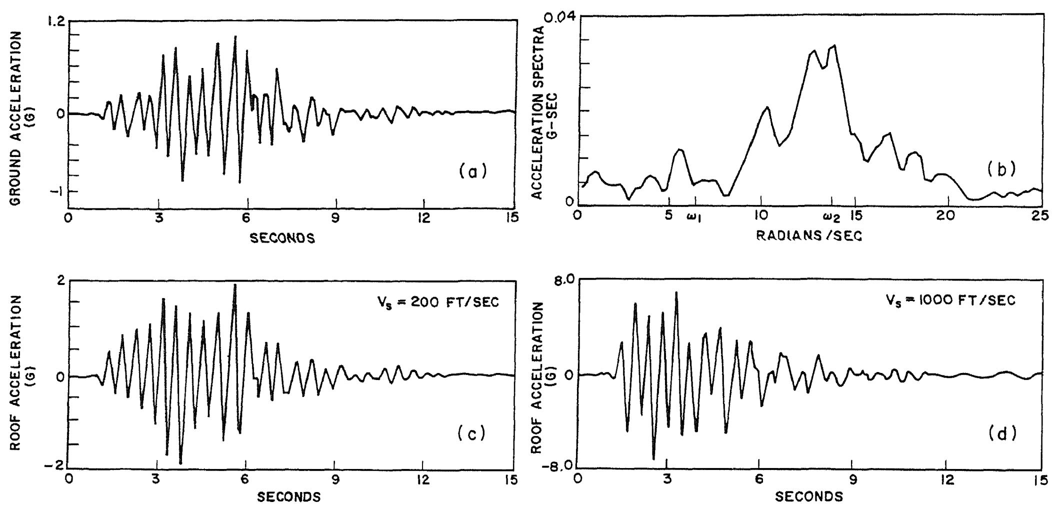

FIG. 2 – SYNTHESIZED FREE-FIELD ACCELEROGRAM (a), CORRESPONDING FOURIER SPECTRA (b), AND ACCELERATION RESPONSES (c) AND (d) OF A 20-STORY BUILDING

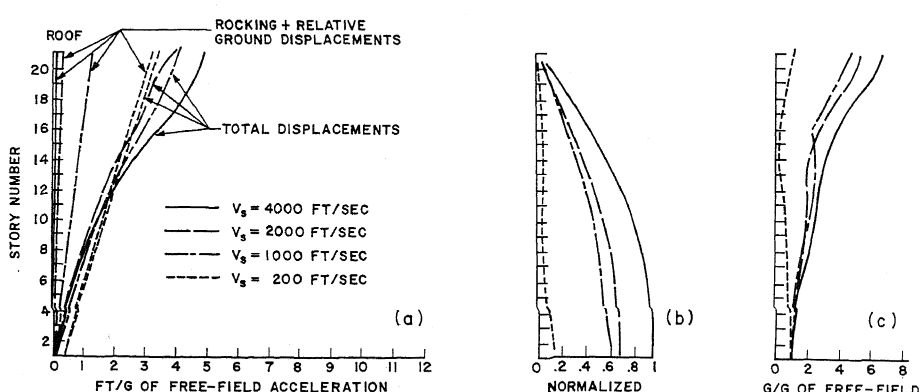

FIG. 3 – MAXIMUM RESPONSES OF A 20-STORY BUILDING ON VARIOUS SOILS