# PLANE VIBRATIONS OF SATURATED SOIL IN STRUCTURAL FOUNDATION

by

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## SYNOPSIS

Presented is an algorithm of the finite difference solution to the problems of one-dimensional vibrations of a saturated soil layer. The problems of investigating dynamic vibrations and stresses are formulated for two cases: a) a layered saturated subsoil underlying a foundation for equipment with a vibratory load; b) a soil layer subjected to a seismic impact directed downward from the rigid foundation. Calculation examples are given and the results are discussed.

Numerous problems of practical importance involving the experimental or analytical dynamic stability evaluations of the subsoils and earth structures are solved with the aid of the analysis of the dynamic loading conditions and the strains in these soils. For the saturated soils the solutions to such problems are performed using a system of differential-integral equations discussed, for instance, in [1].

Analytical solution to these problems even for a simple case of plane stationary motion prove too cumbersome. In this connection it is considered efficient to apply the finite difference method which has recently gained wide popularityf for the solution of plane dynamic problems of the elasticity theory. Presented is an algorithm of the finite difference solution into problems of one-dimensional vibrations of a saturated soil layer on the elastic imprevious foundation. The problem is formulated for the following two cases:

a) dynamic stress-strain investigations in a layered saturated soil of a spread foundation subjected to vibrations with a predetermined frequency (50 Hz) and a unique amplitude of vibrational velocity;

b) investigations of stresses and strains developing in a saturated soil layer under a prescribed motion of an improvious rigid foundation. The solution of the above problems is performed with the use of the equations describing the motion of the solid and liquid components of the saturated soil.

These equations take the following form:
for the pulse equations.

$$m\rho_\kappa\frac{\partial V_\kappa}{\partial t} = -m\frac{\partial P}{\partial x} + \frac{\partial \sigma}{\partial x} + nT, \quad n\rho_\theta\frac{\partial V_\theta}{\partial t} = -n\frac{\partial P}{\partial x} - nT; \qquad (1)$$

for the equations of state:

$$\frac{\partial \sigma}{\partial t} = nK_cK^*\left(\frac{1}{K_\theta}\frac{\partial V_\kappa}{\partial x} - \frac{1}{K_\kappa}\frac{\partial V_\theta}{\partial x}\right), \quad \frac{\partial P}{\partial t} = -K^*\left(m\frac{\partial V_\kappa}{\partial x} + n\frac{\partial V_\theta}{\partial x}\right), \qquad (2)$$

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in which $m, n$ - volumetric content of the solid and liquid components in the soil; $\rho_K, \rho_B$ - the densities of the above components; $V_K$, $V_B$ - displacement velocities of the components; $K_K, K_B$ - compressive volumetric moduli; $\sigma$ - effective stresses in the skeleton; $P$ - neutral pressure; $K_c$ - skeleton uniaxial compression modulus; $T$ - volumetric interaction force of the components.

The volumetric force $T$ is associated with the values $P$ and $V_K$ by the relation;

$$T = \frac{\lambda_0}{\nu}\frac{n\rho_B g}{K_F}\int_0^t \left(-\frac{1}{\rho_B}\frac{\partial P}{\partial x} - \frac{\partial V_K}{\partial t}\right)\sum_{K=0}^{\infty} e^{-\frac{\mu_K^2 n g}{\nu K_F}(t-\xi)}\, d\xi \,, \quad (3)$$

in which

$K_F$ - soil permeability coefficient;

$\nu, \mu_K, \lambda_0$ - pore shape characteristics (for a fissured pore [1]: $\nu = 3$; $\lambda_0 = 2$ and $\mu_K = \pi(K+0.5)$.

For an elastic halfplane (elastic foundation) the pulse equations and the equations of state appear as:

$$\rho_1\frac{\partial V_1}{\partial t} = \frac{\partial \sigma_1}{\partial x} \,, \quad \frac{\partial \sigma_1}{\partial t} = (\lambda+2\mu)\frac{\partial V_1}{\partial x} \,. \quad (4)$$

The problems are solved for the zero initial conditions

$$P(x,0) = \sigma(x,0) = \sigma_1(x,0) = V_K(x,0) = V_B(x,0) = V_1(x,0) = 0 \quad (5)$$

The boundary conditions:

For problem A: on a free plane prescribed are the displacement velocities $V_K$ and $V_B$ for $t \geqslant 0$ correlated with the initial conditions. At the interface of the saturated layer and the elastic half space (at $x=\ell$) the boundary conditions are

$$\sigma(\ell,t) - P(\ell,t) = \sigma_1(\ell,t);\; V_B(\ell,t) = V_K(\ell,t) = V_1(\ell,t) \,. \quad (6)$$

As a condition of the wave transmission along a certain line inside the elastic half space elastic energe continuity equation is applied. In a plane case this equation has the following shape:

$$\frac{\partial V_1}{\partial t} = -a\frac{\partial V_1}{\partial x} \,, \quad (7)$$

where $a = \sqrt{(\lambda+2\mu)/\rho_1}$.

Problem B: the displacement velocities $V_K$ and $V_B$ are prescribed at the base of a saturated layer. At a free surface of a layer stresses are taken equal to zero.

The finite difference approximation of a system of differential equations is based on the usage of a mesh, displaced along a coordinate $x$ A step with respect to coordinate is assumed constant and equal to $\Delta x$ The nodes of a mesh are alternately related to kinematic values (displacement velocities) and stress values. The value $\Delta x$ is chosen small comparing to a minimum wavelength characteristic to a vibration process under investigation. A time step is taken from the stability condition of the difference scheme adopted for a set of wave equations

$$\Delta t \leqslant \Delta x / a_M \qquad (8)$$

where $a_M$ is the greatest propagation velocity of the vibration process.

The finite difference equations of the saturated soil motion take the form:

$$\frac{\overset{t+\Delta t}{\sigma_i} - \overset{t}{\sigma_i}}{\Delta t} = nK_c K^*\left(\frac{1}{K_B}\frac{\overset{t}{V}_{Ki+1} - \overset{t}{V}_{Ki}}{\Delta x} - \frac{1}{K_K}\frac{\overset{t}{V}_{Bi+1} - \overset{t}{V}_{Bi}}{\Delta x}\right); \qquad (9)$$

$$\frac{\overset{t+\Delta t}{P_i} - \overset{t}{P_i}}{\Delta t} = -K^*\left(m\frac{\overset{t}{V}_{Ki+1} - \overset{t}{V}_{Ki}}{\Delta x} + n\frac{\overset{t}{V}_{Bi+1} - \overset{t}{V}_{Bi}}{\Delta x}\right); \qquad (10)$$

$$m\rho_K \frac{\overset{t+\Delta t}{V}_{Ki} - \overset{t}{V}_{Ki}}{\Delta t} = -m\frac{\overset{t+\Delta t}{P_i} - \overset{t+\Delta t}{P}_{i-1}}{\Delta x} + \frac{\overset{t+\Delta t}{\sigma_i} - \overset{t+\Delta t}{\sigma}_{i-1}}{\Delta x} + n\overset{t+\Delta t}{T}; \qquad (11)$$

$$n\rho_B \frac{\overset{t+\Delta t}{V}_{Bi} - \overset{t}{V}_{Bi}}{\Delta t} = -n\frac{\overset{t+\Delta t}{P_i} - \overset{t+\Delta t}{P}_{i-1}}{\Delta x} - n\overset{t+\Delta t}{T}. \qquad (12)$$

Evaluating integral (3) for a time instant $t$ can be presented as $t=N\Delta t$ where $N$ is an appropriate value of the intervals $\Delta t$. For the next instant of time $t+\Delta t$ the expression for a volumetric force can be presented as a sum

$$T_{\Sigma_{N+1}} = \sum_{r=1}^{N+1} A\left\{-\frac{1}{\rho_B}\frac{\overset{r\Delta t}{P_i} - \overset{r\Delta t}{P}_{i-1} + \overset{(r-1)\Delta t}{P_i} - \overset{(r-1)\Delta t}{P}_{i-1}}{2\Delta x} - \overset{r\Delta t}{V}_{Ki} - \overset{(r-1)\Delta t}{V}_{Ki}\right\}\times$$
$$\times\sum_{K=0}^{\infty}\frac{1}{(-B(K))}e^{B(K)(N+1-r)\Delta t}\left(1-e^{B(K)\Delta t}\right), \qquad (13)$$

where

$$A = \frac{\lambda_0}{\nu}\frac{n\rho_B g}{K_F}; \quad B(K) = -\frac{\mu_K^2 n g}{\nu K_F}.$$

In (13) the property of linear approximation of the values $P$ and $V_K$ is used. The value of $V_K$ at a time instant $(N+1)\Delta t$ is unknown. To obtain the relation defining this value in the total (13) entering into (11), a summand containing $V_K^{(N+1)\Delta t}$ is separated. This summand is transferred into the left side of (11) and is grouped with a similar term there. In (12) the value of an integral is determined directly from its representation (13).

The finite difference approximation of the equations for an elastic half space is formulated in a similar manner.

At the interface of a saturated soil and an elastic half space the boundary conditions (6) must be observed. A step $\Delta x$ over a coordinate $x$ is chosen such that the node with the values $V_B, V_K, V_1$ lies on this line. By summing up the pulse equations for a solid and liquid component of the saturated soil the following expression is obtained:

$$\rho_0\frac{\partial V_0}{\partial t} = \frac{\partial(\sigma - p)}{\partial x}, \qquad (14)$$

where $\rho_o = n\rho_b + m\rho_K$

By introducing the boundary value $(\sigma-p)^b = \sigma_1^b$ and utilizing a stress continuity condition, the finite difference relation for evaluating the displacements at the interface line may be obtained:

$$V_K^{t+\Delta t} = V_b^{t+\Delta t} = V_1^{t+\Delta t} = V_o^{t+\Delta t} = V_o^t + \frac{2\Delta t}{\rho_o \Delta x + \rho_1 \Delta x_1}\left[\sigma_{1,i1+1}^{t+\Delta t} - (\sigma - P)_{i1}^{t+\Delta t}\right]. \quad (15)$$

The condition of transmission of an elastic wave (7) along a line limiting the calculation region $(x = h)$ in the finite differences assumes the following shape

$$V_{1,i2}^{t+\Delta t} = V_{1,i2}^{t} - a\frac{\Delta t}{\Delta x_1}\left(V_{1,i2}^{t} - V_{1,i2-1}^{t}\right). \quad (16)$$

For a case when $\Delta t/\Delta x_1 = 1/a$ the condition of transmission reduces to a simple form of the $V_{1,i2}^{t+\Delta t} = V_{1,i2-1}^{t}$.

The numerical solution into problems A and B dealing with a vibration of a layer is carried out with the following initial values. Thickness of a soil layer is taken equal to 10 metres. For the saturated soil $\rho_K$ = 2.67 T/m$^3$; $\rho_b$ = 1 T/m$^3$; $m$ = 0.6; $K_K$ = 0.5·10$^5$ MPa; $K_b$ = 0.1·10$^4$ MPa; $K_c$ = 0.2·10$^3$ MPa; $K_F$ = 1·10$^3$; 1·10$^2$; 1·10$^1$; 1.0; 10.0 m/sec. For the elastic half space: $\rho_1$ = 2.0 T/M$^3$; $\lambda$ = 0.656·10$^4$ MPa; $\mu$ = 0.72·10$^3$ MPa; the calculation is performed up to the time instant $t$ = $\Delta t$·100.

The law of motion of the foundation base and the saturated layer footing $V(0,t) = V_K(0,t) = V_b(0,t) = F\cdot \sin \omega t$ where $F$ = 1m/sec; for problem A $\omega$ = 314·1593 rad/sec, for problem B $\omega$ = 62. 83186rad/sec. Steps with respect to coordinate are taken equal to $\Delta x = \Delta x_1$ = 1 m, steps with respect to time $\Delta t$ = 0.3·10$^{-3}$ sec.

Results of the effective stress calculations for problems A and B respectively are shown in Figs 1, 2 (solid line). The dashed lines in the same Figures indicate the total stresses plotted to a reduced scale $\sigma_*^t = (\beta-1)/\beta \times \sigma^t$ where $\beta = 1 + nK_c(1/K_b - 1/K_K)$.

The calculation results for the displacement velocities $V_K$ (solid line) and $V_b$ (dashed line) of A and B respectively are given in Figs 3, 4.

In Fig. 1, 3 the curves designated by 1 correspond to the evaluations for $K_F$ = 10 m/sec. The curves designated by 2 correspond to the calculations for the value of $K_F$ = 0.1 m/sec, and the curves 3 correspond to $K_F$ = 1·10$^{-3}$ m/sec. In Figs 2, 4 I stand for the curves corresponding to the calculation results for $K_F$ = 10 m/sec, 2 stands for the data corresponding to 1 m/sec, 3 - $K_F$ = 0.1 m/sec, 4 - $K_F$ = 0.01 m/sec. The scale of reduction $(\beta-1)/\beta$ when constructing the total stresses is adopted equal to a coefficient for $\sigma^t$ in a relation for recalculating the total stresses into the effective stresses:

$$\sigma = (\beta-1)/\beta \times \sigma^t - (nK_c/\beta)\tilde{\varepsilon} \quad (17)$$

which was established earlier [2] and where $\tilde{\varepsilon} = \varepsilon_{в} - \varepsilon_{к}$ is the difference of the total strains of water and the solid soil component.

As determined earlier, the permissibility of the quasi-uniform saturated soil model depends on the condition whether the value $B = \omega K_F / ng$ can be neglected. For the selected values of $K_F$ and the remaining values of the parameter $B$ accepted for the solution of the problem the magnitudes B are: $B_1^A = 800$ ($K_F = 10$ m/sec); $B_2^A = 8$ (0.1 m/sec); $B_3^A = 0.08$ ($1 \cdot 10^{-3}$ m/sec); $B_1^B = 160$ (10 m/sec); $B_2^B = 16$ (1 m/sec); $B_3^B = 1.6$ (0.1 m/sec); $B_4^B = 0.16$ (0.01 m/sec). The calculation results reveal that for $B_3^A = 0.08$, the difference (except for the boundary zones of a saturated layer) can adequately be defined from (17), assuming that $\tilde{\varepsilon} = 0$.

With the large values of B the difference in the velocities $V_в, V_к$ becomes notable. The curves $V$ and $\sigma$ assume the additional minimum and maximum, which is due to the propagation of the waves of the second order for large $B$ with $\lambda \simeq \frac{2\pi}{\omega} \sqrt{nK_c / \rho_в \beta} \simeq 5.7$ м as the wave length (Problem A). In such a situation effective stresses, the displacement and strain velocities in the soil solid component must be evaluated by the saturated soil theory taking into account the mutual penetration of the solid and liquid components. In this case, as is seen from Fig. 1, the conditions may arise for which the total stresses with a certain error in the vicinity of the Payer boundary can be defined on the basis of the quasi-uniform soil model. However, the recalculation of these stresses by the condition of the quasi-uniform model into the effective stresses may involve inadmissible error. Besides, as it comes out from Fig. 2, the utilization of quasi-uniform model for the boundary conditions of Problem B may result in a large error when defining the total stresses.

The results obtained may be summarized in short as:

- a method is presented for a numerical solution to one-dimensional dynamic problems for saturated soils with an elastic skeleton and the law of unstationary permeability effects of the components described by an integral equation (3);
- shown is the necessity of the regard made for the relative motion of the saturated soil components for the values of $B = \frac{\omega K_F}{ng}$ compared to unity or substantially exceeding unity;
- indicated is the importance of utilizing a quasi-uniform model neglecting the relative component motion for a case

## REFERENCES

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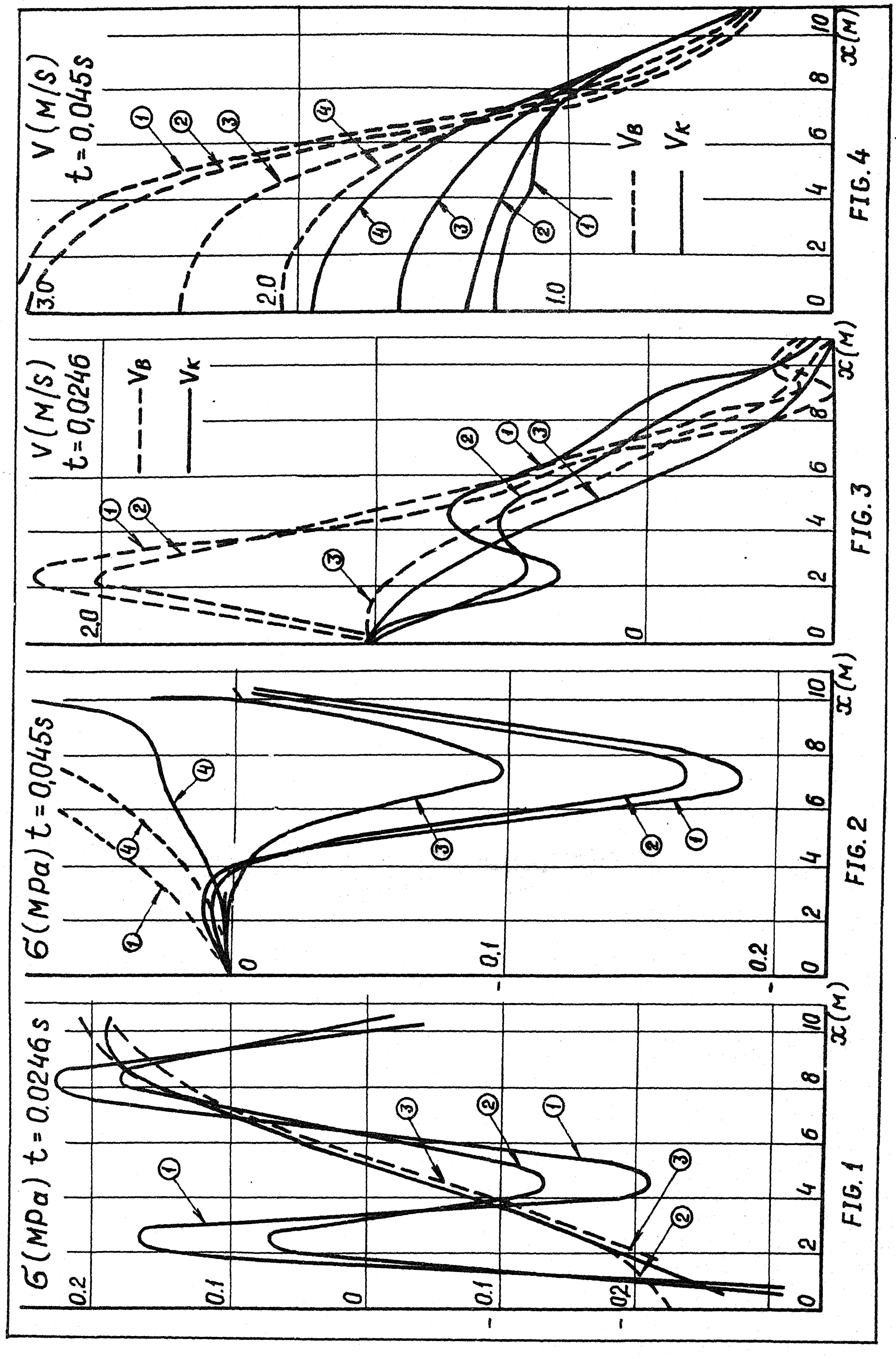

FIG.1

FIG.2

FIG.3

FIG.4