TIME DOMAIN MODELS FOR MULTI-DIMENSIONAL NONSTATIONARY STOCHASTIC PROCESSES

by

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SUMMARY

This paper proposes time domain models, AR, MA and AR-MA models, for multi-dimensional nonstationary stochastic processes which are observed in the characteristics of earthquake ground motions, and then develops the theory of identification and simulation of the system. These models can be applicable to the identification and simulation problems of many engineering systems. This paper also discusses the relationships between time domain models proposed here and frequency model for multi-dimensional nonstationary processes which is proposed by one of the authors. Finally, numerical simulations of earthquake ground motions are presented.

INTRODUCTION

Simulation methods of multi-dimensional or one-dimensional stochastic processes with nonstationary characteristics of amplitude and frequency contents have already proposed by many researchers. However, these simulation methods are normally represented by frequency domain models. On the other hand, Akaike [1], Box and Jenkins [2], Hussain and Rao [3], et.al. discussed the time domain models, namely, autoregressive (AR) model, moving average (MA) model and the mixed (AR-MA) model. These time domain models, however, have not been in suitable forms directly applicable to the stochastic processes with the nonstationarity in the amplitude and frequency domains, especially for the multi-dimensional cases.

Although both models, either in time domain or frequency domain, have their own useful advantages for estimation problems of many engineering systems, time domain models are more useful, because of their simple forms and of the forms directly applicable to such as control problems.

AUTOREGRESSIVE (AR) MODEL

An autoregressive model for multi-dimensional nonstationary stochastic processes, $x_i(t)$; $i=1,2,\ldots,m$ with zero mean is given by

$$x_i(j) = \sum_{p=1}^{i} \sum_{k=1}^{M(j)} b_{ip}(k,j)x_p(j-k) + \varepsilon_i(j) \quad ; i=1,2,\ldots,m \tag{1}$$

where, j is an index of discrete time t, that is, $t=j\Delta t$; $j=1,2,\ldots,N$ and Δt = equal time interval of the given time series. $M(j)$ is a positive integer. $\varepsilon_i(j)$ is an error function and its expectation $E[\varepsilon_i(j)]$ equals zero from Eq. 1, because of $E[x_i(j)] = 0$.

The coefficients $b_{ip}(k,j)$ are chosen in such a way that the mean

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square error $\sum_{i=1}^{m} E[\varepsilon_i^2(j)]$ should be minimum under the condition that the original data $x_i(t)$ satisfy the model given by Eq. 1. Therefore, in order to identify $b_{ip}(k,j)$ at time j, the index j is fixed first and then letting the partial derivative of $\sum_{i=1}^{m} E[\varepsilon_i^2(j)]$ with respect to $b_{np}(r,j)$ be equal to zero, we get

$$E[x_n(j)x_q(j-r)] = \sum_{p=1}^{n} \sum_{k=1}^{M(j)} b_{np}(k,j)E[x_p(j-k)x_q(j-r)] \tag{2}$$

The crosscorrelation function in Eq. 2 is a function of two variables. Therefore, using the data chosen from the neighborhood of time $j\Delta t$, and assuming that the chosen data consist of a stationary gaussian process, the crosscorrelation function can be assumed as follows.

$$E[x_n(j)x_q(j-r)] = \frac{1}{2N'} \sum_{s=j-N'}^{j+N'} x_n(s)x_q(s-r) \tag{3}$$

and

$$E[x_p(j-k)x_q(j-r)] = \frac{1}{2N'} \sum_{s=j-N'}^{j+N'} x_p(s-k)x_q(s-r) \tag{4}$$

where, N' is a positive value and N'<N. From Eqs. 2,3 and 4, we get n-th order simultaneous matrix equation as follows.

$$\begin{bmatrix} B_{n1}(j) \\ B_{n2}(j) \\ \vdots \\ B_{nn}(j) \end{bmatrix} = \begin{bmatrix} X_{11}(j) & X_{21}(j) & \cdots & X_{n1}(j) \\ X_{12}(j) & X_{22}(j) & \cdots & X_{n2}(j) \\ \vdots & \vdots & & \vdots \\ X_{1n}(j) & X_{2n}(j) & \cdots & X_{nn}(j) \end{bmatrix}^{-1} \begin{bmatrix} F_{n1}(j) \\ F_{n2}(j) \\ \vdots \\ F_{nn}(j) \end{bmatrix} \tag{5}$$

where,

$$X_{pq}(j) = \sum_{s=j-N'}^{j+N'} \begin{bmatrix} x_p(s-1)x_q(s-1) & x_p(s-2)x_q(s-1) & \cdots & x_p(s-M)x_q(s-1) \\ x_p(s-1)x_q(s-2) & x_p(s-2)x_q(s-2) & \cdots & x_p(s-M)x_q(s-2) \\ \vdots & \vdots & & \vdots \\ x_p(s-1)x_q(s-M) & x_p(s-2)x_q(s-M) & \cdots & x_p(s-M)x_q(s-M) \end{bmatrix} \tag{6}$$

$$B_{np}(j) = \begin{bmatrix} b_{np}(1,j) \\ b_{np}(2,j) \\ \vdots \\ b_{np}(M,j) \end{bmatrix} \quad \cdots (7) \qquad F_{nq}(j) = \sum_{s=j-N'}^{j+N'} \begin{bmatrix} x_n(s)x_q(s-1) \\ x_n(s)x_q(s-2) \\ \vdots \\ x_n(s)x_q(s-M) \end{bmatrix} \quad \cdots (8)$$

where, s=1,2,...,N n=1,2,...,m r=1,2,...,M(j) q=1,2,...,n j=1,2,...,N, and M = M(j) in Eqs. 5 to 8.

If this AR model is applied to such as an earthquake accelerogram, $B_{np}(j)$ can be assumed to be nearly equal to $B_{np}(u)$ when j and u are close, since the nonstationarity of the earthquake accelerogram changes gradually in the vicinity of time j=1,2,...,N [4]. Therefore, $B_{np}(j)$ are identified from Eq. 5 at the specified time with a constant interval ΔT which is bigger than Δt, that is, $\Delta T=a\Delta t$; a = integer and a > 1. And then, $B_{np}(u)$ between $j\Delta t - (\Delta T/2)$ and $j\Delta t + (\Delta T/2)$ are considered that they have the same constant value of $B_{np}(j)$, or are determined by interpolation. Using the above procedure, the computational time can be greatly reduced. In the calculation of Eq. 5, the values of $x_i(j)$ at negative time are needed when time j stays in early stages. In this case, proper dummy values of $x_i(j)$, e.g. 0.0, can be used for the calculation.

If Eq. 5 is satisfied, we can see that $\varepsilon_i(j)$ and $x_q(j-r)$ are mutually independent for $r=1,2,\ldots,M(j)$. But, if $x_i(j)$ are applied to the engineering fields, for example ; earthquake motions, it is natural that the autocorrelation decreases rapidly when the time lag increases. Therefore, it can be assumed that $\varepsilon_i(j)$ and $x_q(j-r)$ are mutually independent for $r=1,2,\ldots,M(j),M(j)+1,\ldots,\infty$. Since $\varepsilon_q(j-v)$ is given by the linear summation of $x_q(j-v),x_p(j-v-1),\ldots,x_p(j-v-M(j))$; $v\geq 1$, as shown in Eq. 1, and also satisfies the above relation of independence, it is seen that $\varepsilon_i(j)$ and $\varepsilon_q(j-v)$; $v\geq 1$, are mutually independent. Therefore, we get

$$E[\varepsilon_i(j)\varepsilon_q(j-v)] = 0 \quad ; v\geq 1, \quad \text{any i and q} \tag{9}$$

In order to generate the error function $\varepsilon_i(j)$, the following cross-correlation matrix can be used.

$$\boldsymbol{\sigma}^2(j) = \begin{bmatrix} \sigma_{11}^2(j) & \sigma_{12}^2(j) & \cdots & \sigma_{1m}^2(j) \\ \sigma_{21}^2(j) & \sigma_{22}^2(j) & \cdots & \sigma_{2m}^2(j) \\ \vdots & \vdots & & \vdots \\ \sigma_{m1}^2(j) & \sigma_{m2}^2(j) & \cdots & \sigma_{mm}^2(j) \end{bmatrix} \tag{10}$$

where,

$$\begin{aligned} \sigma_{iq}^2(j) &\cong E[\varepsilon_i(j)\varepsilon_q(j)] = \frac{1}{2N'}\sum_{s=j-N'}^{j+N'}\varepsilon_i(s)\varepsilon_q(s) \\ &= \frac{1}{2N'}\sum_{s=j-N'}^{j+N'}\left\{x_i(s) - \sum_{p=1}^{i}\sum_{k=1}^{M(j)} b_{ip}(k,s)x_p(s-k)\right\} \\ &\quad \cdot\left\{x_q(s) - \sum_{p=1}^{q}\sum_{k=1}^{M(j)} b_{qp}(k,s)x_p(s-k)\right\} \end{aligned} \tag{11}$$

Now, let us introduce the following matrix equation

$$\boldsymbol{\mathcal{E}}(j) = \begin{bmatrix} \varepsilon_1(j) \\ \varepsilon_2(j) \\ \vdots \\ \varepsilon_m(j) \end{bmatrix} = \begin{bmatrix} C_{11}(j) & & & 0 \\ C_{21}(j) & C_{22}(j) & & \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1}(j) & C_{m2}(j) & \cdots & C_{mm}(j) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix} = C\boldsymbol{\xi} \tag{12}$$

where, the matrix C is a lower triangular linear transformation matrix, and ξ_i ; $i=1,2,\ldots,m$ are mutually independent random variables with zero mean and its variance is unity. From Eq. 12, we get

$$E[\boldsymbol{\mathcal{E}}(j)\boldsymbol{\mathcal{E}}^T(j)] = C(j)E[\boldsymbol{\xi}\,\boldsymbol{\xi}^T]C^T(j) = C(j)C^T(j) \tag{13}$$

Therefore, from Eqs. 10,11 and 13, we get the following equation.

$$\boldsymbol{\sigma}^2(j) = C(j)C^T(j) \tag{14}$$

From Eqs. 10,11 and 14, $\varepsilon_i(j)$ can be generated.

Nonstationary cross spectrum is defined as follows [5].

$$S_{ij}(\omega,t) = E[\frac{1}{2\pi}X_i(\omega,t)X_j^*(\omega,t)] \quad ; -\infty<\omega<\infty \quad -\infty<t<\infty \tag{15}$$

where,

$$X_i(\omega,t) = \int_{-\infty}^{\infty} W(t-u)x_i(u)e^{-i\omega u}du \quad ; i^2 = -1 \tag{16}$$

Although details on the data window W(t) can be seen in Fef. 4, its characteristic is considered as the weighting function. Therefore, appling the weighted Fourier transformation to the both sides of Eq. 1, we have

$$X_i(\omega,j) = \sum_{r=-\infty}^{\infty} W(j-r)x_i(r)e^{-i\omega r\Delta t}\Delta t$$
$$= \sum_{p=1}^{i}\sum_{k=1}^{M(j)}\sum_{r=-\infty}^{\infty} W(j-r)b_{ip}(k,r)x_p(r-k)e^{-i\omega r\Delta t}\Delta t$$
$$+ \sum_{r=-\infty}^{\infty} W(j-r)\varepsilon_i(r)e^{-i\omega r\Delta t}\Delta t \tag{17}$$

Let assume that $b_{ip}(k,r)$ changes gradually with the change of time r. And since $\varepsilon_i(r)$ is a nonstationary white noise, letting

$$E_i(\omega,j) \equiv \sum_{r=-\infty}^{\infty} W(j-r)\varepsilon_i(r)e^{-i\omega r\Delta t}\Delta t \tag{18}$$

we get the following equation in matrix form.

$$X(\omega,j) = B(\omega,j)X(\omega,j) + E(\omega,j) \tag{19}$$

where,

$$X(\omega,j) = [X_1(\omega,j) \quad X_2(\omega,j) \quad \ldots. \quad X_m(\omega,j)]^T \tag{20}$$

$$X_p(\omega,j) \cong \sum_{s=-\infty}^{\infty} W(j-s-k)x_p(s)e^{-i\omega s\Delta t}\Delta t \tag{21}$$

$$E(\omega,j) = [E_1(\omega,j) \quad E_2(\omega,j) \quad \ldots. \quad E_m(\omega,j)]^T \tag{22}$$

$$B(\omega,j) = \begin{bmatrix} \sum_{k=1}^{M(j)} b_{11}(k,j)e^{-i\omega k\Delta t} & & & 0 \\ \sum_{k=1}^{M(j)} b_{21}(k,j)e^{-i\omega k\Delta t} & \sum_{k=1}^{M(j)} b_{22}(k,j)e^{-i\omega k\Delta t} & & \\ \vdots & \vdots & \ddots & \\ \sum_{k=1}^{M(j)} b_{m1}(k,j)e^{-i\omega k\Delta t} & \cdots\cdots\cdots & & \sum_{k=1}^{M(j)} b_{mm}(k,j)e^{-i\omega k\Delta t} \end{bmatrix} \tag{23}$$

Therefore, letting $I - B(\omega,j) = H(\omega,j)$, we get $H(\omega,j)X(\omega,j) = E(\omega,j)$. From this equation, we get

$$H(\omega,j)\ X(\omega,j)X^{*T}(\omega,j)H^{*T}(\omega,j) = E(\omega,j)E^{*T}(\omega,j) \tag{24}$$

Taking the expectation of both sides of Eq. 24, and following the definition of nonstationary spectral matrix, we get

$$S_x(\omega,j) = E[\frac{1}{2\pi}X(\omega,j)X^{*T}(\omega,j)] = H^{-1}(\omega,j)S_\varepsilon(\omega,j)\{H^{*T}(\omega,j)\}^{-1} \tag{25}$$

where, $S_\varepsilon(\omega,j)$ is a nonstationary cross spectral matrix and given by

$$S_\varepsilon(\omega,j) = \frac{\Delta t}{2\pi}\sigma^2(j) = \frac{\Delta t}{2\pi}\begin{bmatrix} \sigma_{11}^2(j) & \sigma_{12}^2(j) & \cdots & \sigma_{1m}^2(j) \\ \sigma_{21}^2(j) & \sigma_{22}^2(j) & \cdots & \sigma_{2m}^2(j) \\ \vdots & \vdots & & \vdots \\ \sigma_{m1}^2(j) & \sigma_{m2}^2(j) & \cdots & \sigma_{mm}^2(j) \end{bmatrix} \tag{26}$$

$$; \ -(\pi/\Delta t) \leq \omega \leq (\pi/\Delta t)$$

After identifing $b_{ip}(k,j)$, we can immediately calculate the nonstationary cross spectrum by using Eq. 25.

MOVING AVERAGE (MA) MODEL

A multi-dimensional nonstationary MA model is given by [6]

$$x_i(j) = \sum_{p=1}^{i}\sum_{k=1}^{M(j)} h_{ip}(k,j)a_p(j-k) + \varepsilon_i(j) \quad ; \ i=1,2,\ldots,m \tag{27}$$

where, symbols j and $M(j)$ have the same meanings in AR model. And $a_p(j)$ are m mutually independent random variables (white noise) with zero mean and its variances $E[a_p^2(j)]$ are equal to σ^2.

Following the criterion of minimizing the mean square error $\sum_{i=1}^{m} E[\varepsilon_i^2(j)]$, and pursuing the same derivation in AR model, we get

$$\sum_{s=j-N'}^{j+N'} x_n(s)a_q(s-r) = \sum_{p=1}^{n}\sum_{k=1}^{M(j)} h_{np}(k,j)\sum_{s=j-N'}^{j+N'} a_p(s-k)a_q(s-r) \tag{28}$$

However, since $a_p(j)$ and $a_q(u)$ are mutually independent if $p \neq q$, we have

$$\sum_{s=j-N'}^{j+N'} a_p(s-k)a_q(s-r) = \begin{cases} 2N'\sigma^2 & ; \ p = q \\ 0 & ; \ p \neq q \end{cases} \tag{29}$$

Therefore, Eq. 28 can be written as follows.

$$H_{nq}(j) = A_{qq}^{-1}F_{nq}(j) \quad ; \ n=1,2,\ldots,m \quad q=1,2,\ldots,n \tag{30}$$

where,

$$H_{nq}(j) = \begin{bmatrix} h_{nq}(1,j) \\ h_{nq}(2,j) \\ \vdots \\ h_{nq}(M,j) \end{bmatrix} \ \ldots(31) \qquad F_{nq}(j) = \sum_{s=j-N'}^{j+N'} \begin{bmatrix} x_n(s)a_q(s-1) \\ x_n(s)a_q(s-2) \\ \vdots \\ x_n(s)a_q(s-M) \end{bmatrix} \ \ldots(32)$$

$$A_{qq} = 2N'\sigma^2 I \ \ldots (33), \qquad M = M(j)$$

Hence, $h_{ip}(k,j)$ can be identified from Eq. 30.

The relationships between the nonstationary cross spectral matrix $S_x(\omega,j)$ and the identified $h_{ip}(k,j)$ can be derived by assuming the same approximation in the previous section as follows.

$$X(\omega,j) = H(\omega,j)A(\omega) + E(\omega,t) \tag{34}$$

where,

$$X(\omega,j) = [X_1(\omega,j) \ \ X_2(\omega,j) \ \ \ldots \ \ X_m(\omega,j)]^T \tag{35}$$

$$A(\omega) = [A_1(\omega) \quad A_2(\omega) \quad \ldots \quad A_m(\omega)]^T \tag{36}$$

$$H(\omega,j) = \sum_{k=1}^{M(j)} \begin{bmatrix} h_{11}(k,j)e^{-i\omega k\Delta t} & & 0 \\ h_{21}(k,j)e^{-i\omega k\Delta t} & h_{22}(k,j)e^{-i\omega k\Delta t} & \\ \vdots & \cdot & \cdot \\ h_{m1}(k,j)e^{-i\omega k\Delta t} & \cdots\cdots & h_{mm}(k,j)e^{-i\omega k\Delta t} \end{bmatrix} \tag{37}$$

From Eq. 34, we get

$$S(\omega,j) = \frac{1}{2\pi}E[X(\omega,j)X^{*T}(\omega,j)] = H(\omega,j)S_a(\omega)H^{*T}(\omega,j) + S_\varepsilon(\omega,j)$$
$$= \frac{\sigma^2\Delta t}{2\pi}H(\omega,j)H^{*T}(\omega,j) + S_\varepsilon(\omega,j) \quad ; \quad -(\pi/\Delta t) \leq \omega \leq (\pi/\Delta t) \tag{38}$$

NUMERICAL CALCULATIONS AND DISCUSSIONS

In this section, an appropriateness of the simulation model proposed by Eq. 1 is discusses. The earthquake acceleration records used for the numerical calculations are three components of San Fernando earthquake in Feb. 9, 1971 [7] which were observed at a basement of Millikan library, Calf. Inst. of Tech. These records which are digitized with equal time interval of $\Delta t = 0.04$ sec are illustrated in Fig. 1 and their physical spectra are shown in Figs. 2 to 4.

In order to determine the nonstationarity in time domain, the acceleration record is divided into small groups with equal time length of $2N'\Delta t$ sec by shifting the central time $j\Delta t$ with constant interval $\Delta T = 10\Delta t$ sec as mentioned in the previous section. Then the effects of $M(j)$ and N' are first examined for one-dimensional model, that is, m=1 in Eq. 1. In this case, $M(j)$ at time $j\Delta t$ is determined by appling the Final Prediction Error (FPE) method proposed by Akaike [1] under the assumption of partial stationarity of the processes for each small group. And then, $M(u)$ for $j\Delta t-(\Delta T/2) \leq u \leq j\Delta t+(\Delta T/2)$ are considered to be the same constant value of $M(j)$ at time $j\Delta t$. $B_{np}(j)$ at time $j\Delta t$ are determined by Eq. 5 and $B_{np}(u)$ for $j\Delta t \leq u \leq j\Delta t+\Delta T$ are interpolated by second order polynormials.

From the above examinations, it seems that $M(j)$ does not give any serious effects to the simulation results. Therefore, we choose that $M(j)$ is constant and equals to 4. And we also choose N' to be equal to 25. With these values, the simulations of three-dimensional waves are carried out with different random variables ξ. One of these simulated waves is shown in Fig. 5 and the physical spectra of its each component are shown in Figs. 6 to 8. As can be seen from Figs. 5 to 8, it seems that each simulated wave contains higher frequency content waves than the original waves do and the maximum acceleration of U-D component of the simulated wave becomes higher than that of the original U-D component.

CONCLUSIONS

The multi-dimensional nonstationary stochastic process models in time domain (AR and MA models) were proposed in this paper. These models can be applicable to the identification and simulation problems of many engineering systems. And the following conclusions can be made from the study.

1). The multi-dimensional AR and MA models were led for the phenomena that the nonstationarity changes gradually along time axis. Stationary model as well as one dimensional model was induced easily from the above multi-dimensional model as a special case.
2). The appropriateness of the multi-dimensional nonstationary AR model was demonstrated by the simulation of the observed three dimensional earthquake records. And despite of some defects observed in this study, it is totally considered that the AR model proposed in this paper, Eq. 1 is reasonable.
3). Though a multi-dimensional nonstationary mixed AR-MA model was not discussed in this paper, it is obvious that the mixed AR-MA model can be given by the following equation.

$$x_i(j) = \sum_{p=1}^{i} \sum_{k=1}^{M(j)} b_{ip}(k,j)x_p(j-k) + \sum_{p=1}^{i} \sum_{k=1}^{L(j)} h_{ip}(k,j)a_p(j-k) + \varepsilon_i(j)$$

ACKNOWLEDGMENT

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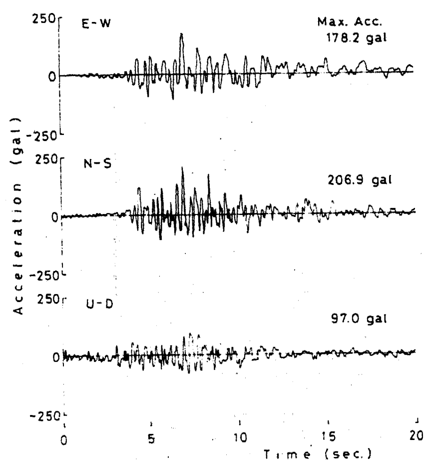

Fig. 1 Acceleration Records of Millikan Library

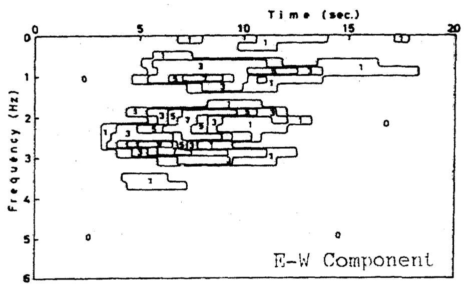

Fig. 2 Spectrum (Original Wave)

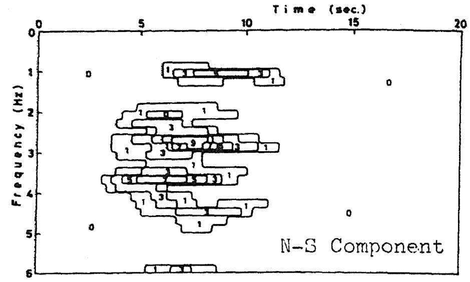

Fig. 3 Spectrum (Original Wave)

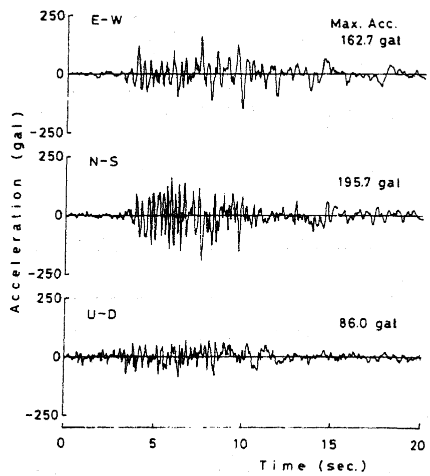

Fig. 5 Simulated Waves

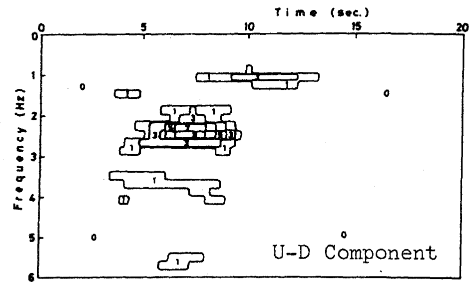

Fig. 4 Spectrum (Original Wave)

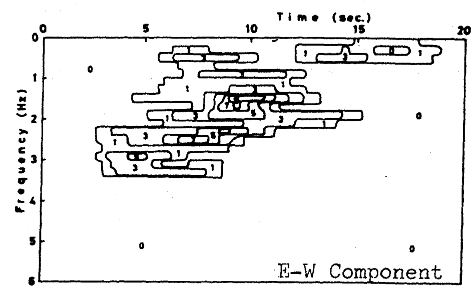

Fig. 6 Spectrum (Simulated Wave)

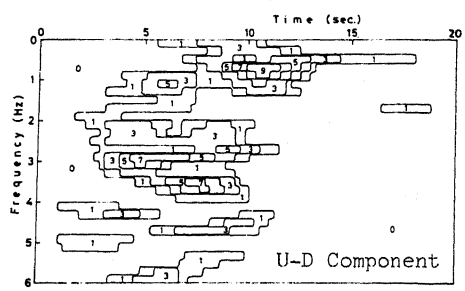

Fig. 8 Spectrum (Simulated Wave)

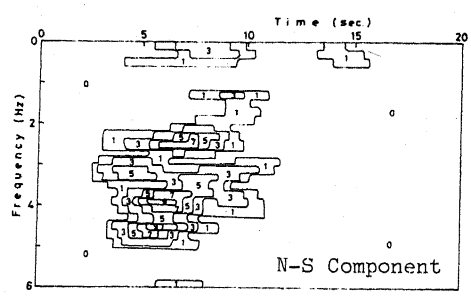

Fig. 7 Spectrum (Simulated Wave)