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ON THE RESPONSE SPECTRUM ANALYSIS AND THE SPECTRAL MOMENTS

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SUMMARY

We have investigated the Complete Quadratic Combination (CQC) method, and concluded that the CQC represents the true power of the White Noise (WN) in vector space. We show that the Square Root of Sum of Square (SRSS) and the Algebraic Sum (AS) methods are the particular cases of quadratic form, and that the SRSS and AS have some defects from a viewpoint of the power. We present the well classified formulae of the responses, and the five coupling conditions which the SRSS response makes unreasonable. The 0th spectral moment of the Finite Interval (FI) which tends to that of the WN is derived from the Mikusiński operator.

INTRODUCTION

Response spectrum analysis has been used in aseismatic design of long span bridges in Japan. The SRSS or so-called RMS method is widely used now, but unreasonable responses are often found in structures which have closely distributed eigenvalues. The confusion is increased because the unreasonable responses do not always arise in similar structure. In the 1980's, A.D. Kiureghian and E.L. Wilson (Refs. 1,2) introduced the CQC based on spectral moments. Wilson pointed out that the SRSS should be replaced by the CQC using spatial building model. Since the theoretical defects of the SRSS are not clear, the SRSS is still being used. Kiureghian calculated the spectral moments of the WN and the Filtered White Noise (FWN) in vector space, employing the Fourier transformation, residue theorem and statistics. In the classical control theory (Ref. 3), the Laplace-Fourier transformation is used and frequency response is easily obtained from the Paley-Wiener theorem (Ref. 4). But convergence trouble often arises as the infinite integral interval $(-\infty, +\infty)$. In the 1950's, the Polish mathematician, J. Mikusiński (Ref. 5), introduced a functional operator as

$$a \cdot b = \{\int_0^t a(t-T)b(T)dT\} = \{\int_0^t b(t-T)a(T)dT\}, \quad l \cdot a = \{\int_0^t a(T)dT\}, \quad s = \delta / l \qquad (1)$$

where l and s are the integral and differential operators, and δ is delta function. This convolution is defined over the finite interval $(0, t)$ as the Duhamel integral. The operator is an algebraic field of quotients and is easily treated in vector space; moreover it is a distribution which can avoid the convergence trouble. Urbanik applies the operator to generalized stochastic process (Ref. 6). Wolf uses the convolution in the soil-structure interaction problem (Ref. 7). We will apply s to the response spectrum analysis in the frequency (ω) domain.

ALGEBRAIC APPROACH OF THE RESPONSE SPECTRUM ANALYSIS

In the linear dynamic FEM, the 2nd order ordinary differential equation and the transformed equation by eigenvalue analysis are written, respectively, as

$$M\ddot{U}(t) + C\dot{U}(t) + KU(t) = M'\alpha(t), \quad \text{in the global space} \tag{2}$$
$$E\ddot{Y}(t) + 2D\Omega\dot{Y}(t) + \Omega^2 Y(t) = P\alpha(t), \quad \text{in the modal space.} \tag{3}$$

The notations are summarized as

M : mass matrix ,	E : unit matrix = diag(1,..,1)
C : damping matrix ,	D : damping factors = diag$(h_1,..,h_M)$
K : stiffness matrix ,	Ω : circular frequencies = diag$(\omega_1,..,\omega_M)$
M' : mass vector for (t),	Ω^2 : eigenvalues = diag$(\omega_1^2,..,\omega_M^2)$
α : scalar acceleration ,	P : participation vector
$U(t)$: displacement vector,	$Y(t)$: displacement vector
$\dot{U}(t)$: velocity vector ,	$\dot{Y}(t)$: velocity vector
$\ddot{U}(t)$: acceleration vector,	$\ddot{Y}(t)$: acceleration vector

where diag(..) is a diagonal matrix and the suffix M is the maximum modal degree. Modal matrix Φ is composed of column eigenvector ϕ_m or row vector ϕ_k^T. Φ and the vector transformations are written as

$$\Phi = (\phi_1,..,\phi_M) = \begin{bmatrix} \phi_1^T \\ \vdots \\ \phi_K^T \end{bmatrix},$$

$$U(t) = \Phi Y(t), \quad \dot{U}(t) = \Phi\dot{Y}(t), \quad \ddot{U}(t) = \Phi\ddot{Y}(t), \quad \Phi^T M' = P \tag{4}$$

where Φ^T is a transpose and the suffix K is the maximum degree of freedom. We introduce a mapping between the time (t) and s domain. We define the displacement, velocity, and acceleration vectors in the global and modal space as $V(s)$, $\dot{V}(s)$, $\ddot{V}(s)$ and $Z(s)$, $\dot{Z}(s)$, $\ddot{Z}(s)$. The relations are written by $V(s)$ and $Z(s)$ as

$$\dot{V}(s) = sV(s), \quad \ddot{V}(s) = s^2V(s), \tag{5}$$
$$\dot{Z}(s) = sZ(s), \quad \ddot{Z}(s) = s^2Z(s) \tag{6}$$

where s and s^2 are the scalar operators. From Eq. (6), the orthonormalized Eq.(3) is mapped on the s domain as Eq. (7). The Z(s) is analytically solved as Eq. (8).

$$(s^2E + 2sD\Omega + \Omega^2)Z(s) = P\beta(s) \tag{7}$$

$$Z(s) = (s^2E + 2sD\Omega + \Omega^2)^{-1}P\beta(s) = H(s)^D P\beta(s), \quad \text{or}$$

$$Z(s) = \begin{bmatrix} \dfrac{1}{s^2+2h_1\omega_1 s+\omega_1^2} & & 0 \\ & \ddots & \\ 0 & & \dfrac{1}{s^2+2h_M\omega_M s+\omega_M^2} \end{bmatrix} P\beta(s) = P^D H(s)\beta(s) \tag{8}$$

where $\beta(s)$ is a mapping of scalar $\alpha(t)$ and the suffix ^{D} is diagonalization of P i.e. P^D = diag $(P_1,..,P_M)$. $H(s)^D P = P^D H(s)$ is algebraically assured. Setting all the initial conditions at 0, the global space responses are finally given as

$$V(s) = \Phi Z(s) = \Phi P^D H(s)\beta(s), \tag{9}$$
$$\dot{V}(s) = s\,\Phi Z(s) = s\,\Phi P^D H(s)\beta(s), \tag{10}$$
$$\ddot{V}(s) = s^2\Phi Z(s) = s^2\Phi P^D H(s)\beta(s). \tag{11}$$

Fig. 1 shows a block diagram of the Eqs. (8), (9), (10) and (11). If $s = j\omega$ $(j = \sqrt{-1})$, we derive the Fourier transformation. The responses are, algebraically, 1st order vectors expressed by linear combination of matrices.

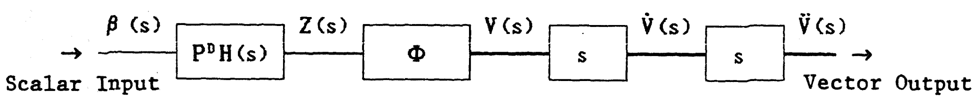


Fig.1 Block Diagram of Structural Responses

Symmetric Gram Matrix and Power   Operational calculus is continued and the results are mapped on the ω domain by $s = j\omega$. From Eq.(9), the displacement function $V_k(s)$ of the kth freedom, its conjugate function $V_k(-s)$, and the spectral density function Sf in the s domain are written, respectively, as

$$V_k(s) = \phi_k^T P^D H(s)\beta(s) = H(s)^T P^D \phi_k \beta(s), \tag{12}$$

$$V_k(-s) = \phi_k^T P^D H(-s)\beta(-s) = H(-s)^T P^D \phi_k \beta(-s), \tag{13}$$

$$Sf = V_k(s)\cdot V_k(-s) = \phi_k^T P^D (H(s)H(-s)^T) P^D \phi_k \cdot \beta(s)\beta(-s) \tag{14}$$

where k depends on ϕ_k alone and the other terms are independent of k. $\beta(s)\beta(-s)$ is indeterministic and the other terms are deterministic. Sf is, algebraically, a 2nd order "real" scalar function in the ω domain because of $V_k(j\omega)\cdot V_k(-j\omega)$, and is expressed by quadratic form. $G(s) \equiv (H(s)H(-s)^T)$ is an asymmetric Gram matrix; notice that the imaginary parts remain in the ω domain. This fact seems to be a contradiction. We find that the residue theorem is needed on account of the asymmetric Gram matrix. Sf is rewritten as

$$Sf = \phi_k^T P^D (\{H(s)H(-s)^T + H(-s)H(s)^T\}/2) P^D \phi_k \cdot \beta(s)\beta(-s), \quad \text{or}$$

$$Sf = \phi_k^T P^D G(s) P^D \phi_k \cdot \beta(s)\beta(-s). \tag{15}$$

$G(s) \equiv (\{H(s)H(-s)^T + H(-s)H(s)^T\}/2)$ is a symmetric Gram matrix and all the imaginary parts vanish in the ω domain. $G(\omega)$ is referred to as

$$G(\omega) = \left[\frac{\omega^4 - (\omega_l^2 + \omega_m^2 - 4h_l h_m \omega_l \omega_m)\omega^2 + \omega_l^2 \omega_m^2}{\{(-\omega^2 + \omega_l^2)^2 + 4h_l^2 \omega_l^2 \omega^2\}\{(-\omega^2 + \omega_m^2)^2 + 4h_m^2 \omega_m^2 \omega^2\}} \right] \tag{16}$$

where the suffixes l and m are the integers of $(1,..,M)$. We will integrate Sf in the s domain using $ds = jd\omega$, and the power function Pf is written as

$$Pf = \frac{1}{j}\int_{j\omega_n}^{j\omega_{n+1}} Sf\, ds = \phi_k^T P^D \left(\frac{1}{j}\int_{j\omega_n}^{j\omega_{n+1}} G(s)\cdot \beta(s)\beta(-s)\, ds\right) P^D \phi_k = \phi_k^T P^D \lambda_0 P^D \phi_k \tag{17}$$

where (ω_n, ω_{n+1}) is a Finite Interval (FI). λ_0 is the 0th spectral moment of the FI, and depends on $G(s)$ and the seismic spectrum $\beta(s)\beta(-s)$.

Spectral Moments of WN, FWN and FI   Kiureghian presented the spectral moments of the WN and FWN. The WN is physically curious, for instance it includes even the frequencies as 0.000001Hz and 1MHz. Seismic mechanism of the FWN is approximated by a 2nd order ordinary differential equation. As the descending characteristics of the FWN are not sharp, we may not use the FWN to simulate the sharp peaks of $\beta(s)\beta(-s)$. The spectral moments of the WN and FWN are derived from the infinate interval Fourier transformation. In Eq.(17), if the limits of $\omega_n \to +0$ and $\omega_{n+1} \to +\infty$ and $\beta(s)\beta(-s) = \text{const.} = g_n$, then λ_0 of the FI tends to that of the WN. Eq.(17) is expressed in the s domain, but we think that this equation is a kind of the finite interval Fourier transformation or the infinate interval Fourier transformation with a window function. The FI has the sharpest descending characteristics as $-\infty$ db/oct. (Fig.2). We will calculate λ_0 of the FI in a concrete form. The spectral density functions of the velocity and acceleration are $-s^2 Sf$ and $s^4 Sf$, but the calculation of their powers are abandoned in the face of the difficulty of analytical integration. Now, we point out that the direct spectrum analysis of arbitrary $\beta(s)\beta(-s)$ (Fig.3) may be possible, if λ_2 and λ_4 of the FI are obtained analytically.

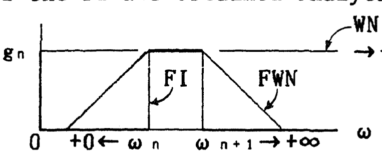


Fig.2 Seismic Spectrum

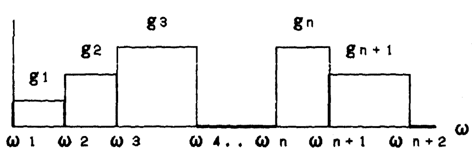


Fig.3 Arbitrary Seismic Spectrum

Oth Spectral Moment of FI As the analytical integration is difficult, we consider the following existent conditions. ① : Sf is a real function in the ω domain. ② : Pf is integrable over (ω_n, ω_{n+1}) as ①. ③ : Each element of matrix λ_0 is integrable as Eq.(16) in case of $\beta(s)\beta(-s) = g_n$. After the weary calculation, we obtain λ_0 and a primitive function F as

$$\lambda_0 = (\lambda_{0lm}) = (F(\omega))_{\omega_n}^{\omega_{n+1}} = (F(\omega_{n+1})) - (F(\omega_n)), \tag{18}$$

$$F(\omega) = g_n \cdot \Big[\frac{h_l\omega_l + h_m\omega_m}{Klm}\{\mathrm{mod}(\arctan(Al)+\pi, \pi) + \mathrm{mod}(\arctan(Am)+\pi, \pi)\} - \frac{2h_l\omega_l(h_l\omega_l+h_m\omega_m)-\omega_l^2+\omega_m^2}{4\omega_l(1-h_l^2)^{1/2}Klm}\ln\Big(\frac{\omega^2-2\omega_l(1-h_l^2)^{1/2}\omega+\omega_l^2}{\omega^2+2\omega_l(1-h_l^2)^{1/2}\omega+\omega_l^2}\Big) - \frac{2h_m\omega_m(h_l\omega_l+h_m\omega_m)+\omega_l^2-\omega_m^2}{4\omega_m(1-h_m^2)^{1/2}Klm}\ln\Big(\frac{\omega^2-2\omega_m(1-h_m^2)^{1/2}\omega+\omega_m^2}{\omega^2+2\omega_m(1-h_m^2)^{1/2}\omega+\omega_m^2}\Big)\Big], \tag{19}$$

$$Klm = (\omega_l^2-\omega_m^2)^2 + 4\omega_l\omega_m(h_m\omega_l+h_l\omega_m)(h_l\omega_l+h_m\omega_m), \tag{20}$$

$$Al = 2h_l\omega_l\omega/(-\omega^2+\omega_l^2), \qquad Am = 2h_m\omega_m\omega/(-\omega^2+\omega_m^2)$$

where F is a real symmetric matrix and mod(,) is a remainder. Both the sharp peaks of $\beta(s)\beta(-s)$ and G(s) are considered by the narrow band of (ω_n, ω_{n+1}). In the limits of $\omega_n \to +0$ and $\omega_{n+1} \to +\infty$, λ_0 of the WN, which is the same that of Kiureghian, is derived as Eq.(21). Then Eq.(18) is a generalization of the WN.

$$\lambda_0 = (\lambda_{0lm}) = (2\pi g_n(h_l\omega_l+h_m\omega_m)/Klm). \tag{21}$$

Well Classified Formula and States of CQC, SRSS and AS We can show the CQC response in an algebraically well classified formula. The modal cross-correlation ρ is derived from Eq.(21) by normalization as Eq.(22); Eq.(23) is a concrete form of Eq.(22). The square root of the power $\sqrt{Pf}$ in Eq.(24) is a "RMS-value".

$$\rho = (\rho_{lm}) = (\lambda_{0lm}/(\sqrt{\lambda_{0ll}}\sqrt{\lambda_{0mm}})), \text{ or} \tag{22}$$

$$(\rho_{lm}) = (8(h_lh_m)^{1/2}(h_l+\gamma h_m)\gamma^{3/2}/\{(1-\gamma^2)^2+4\gamma(h_l+h_m\gamma)(h_l\gamma+h_m)\}) \tag{23}$$

$$\sqrt{Pf} = (\phi_k^TP^D\lambda_0P^D\phi_k)^{1/2} = (\phi_k^TP^DQ^D\rho Q^DP^D\phi_k)^{1/2} \tag{24}$$

where $Q^D = \mathrm{diag}(\sqrt{\lambda_{011}},..,\sqrt{\lambda_{0MM}})$ and $\gamma = \omega_m/\omega_l$. Q^D represents the response spectrum, and the CQC response Rc of the kth freedom is well formulated as

$$Rc = (\phi_k^TP^DS^D\rho S^DP^D\phi_k)^{1/2} = (X^T\rho X)^{1/2}, \quad X = S^DP^D\phi_k \tag{25}$$

where $S^D = \mathrm{diag}(S_1,...,S_M)$ is the given input response spectrum. The CQC, SRSS and AS are summarized by quadratic form. Supposing the displacement responses of the SRSS and AS are Rs and Ra, the three responses are written, respectively, as

$$Rc = (X^T\rho X)^{1/2}, \tag{26}$$

$$Rs = (\Sigma X_m^2)^{1/2} = (X^TEX)^{1/2}, \tag{27}$$

$$|Ra| = ((\Sigma X_m)^2)^{1/2} = (X^Te\cdot e^TX)^{1/2} = (X^T(ee^T)X)^{1/2} \tag{28}$$

where $| \ |$ is an absolute value, E is a unit matrix, $e = (1,..1)$ is a column vector, and all the elements of Gram matrix (ee^T) are 1. Eq.(28) is a perfect square formula and Ra is also written in quadratic form. The off-diagonals of the SRSS or RMS are too weak, and those of the AS are too strong (Fig.4). Consequently, the three methods are stated as SRSS $\equiv$ RMS $<$ CQC $<$ AS. The CQC method is naturally derived from the algebraic operations, and represents the true RMS-value if the WN assumption is admitted. The SRSS or so-called RMS method does not represent the RMS-value despite its name. Because the derivation of the SRSS requires that $(H(s)H(-s)^T) = H(s)^{D2}$. But it is a well known fact that $(H(s)H(-s)^T) \neq H(s)^{D2}$. As for the AS method, the condition of $\rho = (ee^T)$ yields $\omega_m = \omega_l$ and $h_m = h_l$ for all l and m, which is an obvious contradiction. These are the theoretical defects of the SRSS and AS.

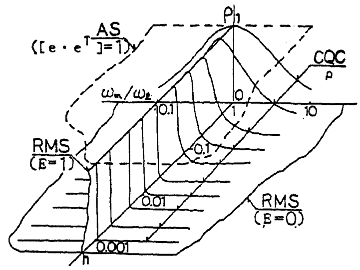


Fig.4 Modal Cross-Correlation

RESPONSES OF CQC, SRSS AND TIME HISTORY

We can show that unreasonable SRSS responses are found in long span bridges as in the spatial buildings studied by Wilson. Figs.5 and 6 show spatial cable stayed bridge models which have closely distributed eigenvalues. Model 1 is a single plane cable system and model 2 is a double one. Fig.7 shows the CQC, SRSS and Time History (TH) responses of model 1. Fig.8 shows the CQC and SRSS responses of model 2. The responses are girder moments of longitudinal acceleration. The given input response spectrum and seismic wave of the TH are taken from the Taft earthquake of 1969.

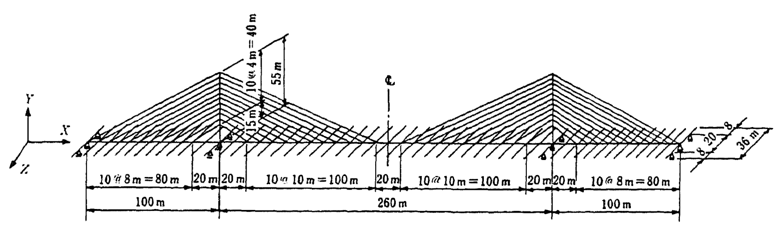


Fig.5 Model 1

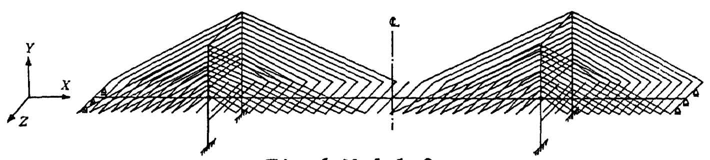
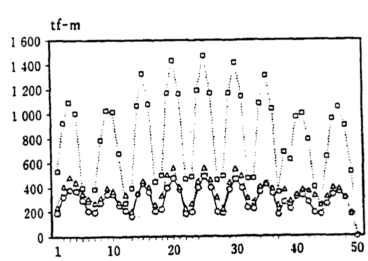


Fig.6 Model 2



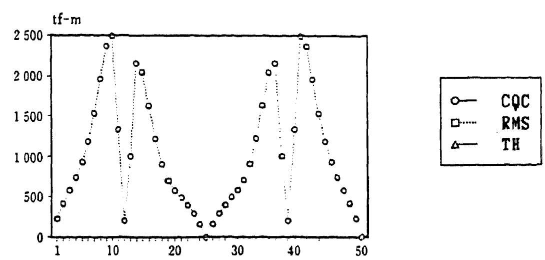


Fig.7 Moment responses of Model 1   Fig.8 Moment responses of Model 2

In model 1, the SRSS is found to give unreasonable responses, while the CQC & TH give reasonable responses. The CQC and SRSS have the same responses in model 2, despite its closely distributed eigenvalues. That is to say, unreasonable SRSS responses do not always arise in similar structures. This is a part of the uncertainty of the SRSS. From the theoretical inspection and the numerical models, we will stand for the CQC method.

UNREASONABLE RESPONSES OF SRSS

In the previous numerical models, we showed the uncertainty of the SRSS. We will clarify the reasons why the unreasonable SRSS responses are occasionally revealed. Eqs.(26) and (27) are rewritten as

$$Rc = (X^T \rho X)^{1/2}, \tag{29}$$

$$Rs = (X^T E X)^{1/2}, \quad X = S^D P^D \phi_k \tag{30}$$

where the vector X is composed of given input response spectrum S^D, participation factor P^D, and kth freedom modal vector ϕ_k. The differences between them are ρ and E. ρ is a function of ω_l, ω_m, h_l and h_m, whereas E is a unit matrix. We have come to the conclusion that the following five coupling conditions (the AND conditions) make the SRSS responses uncertain.

① : Multiple roots or closely distributed eigenvalues existed.
② : The damping factors are relatively large.
③ : The elements of ϕ_k are large.
④ : The elements of participation P^D are large.
⑤ : The elements of input response spectrum S^D are large.

From the conditions ① and ②, off-diagonals of ρ tend to 1, whereas those of E are all 0. In the quadratic forms as Eqs.(29) and (30), the elements of X also influence the values of Rc and Rs. The coupling conditions ③, ④ and ⑤ yield that the elements of X differ from 0. The confusion of the SRSS derives from the latter three conditions. The following striking examples show that the SRSS results in underestimated, equal, and overestimated responses as compared with the CQC. The conditions ① and ② hold, all the elements of ρ are 1, suppose the two dimensional case i.e. $X = (X_1, X_2)$, and that the values of X are selected from -1, 0 and 1. The three examples are as follows:

Ex. 1: underestimated case ($X = (1,1)$) → $Rc = 2$, $Rs = \sqrt{2}$, $Rc/Rs = \sqrt{2}$
Ex. 2: equal case ($X = (0,1)$) → $Rc = 1$, $Rs = 1$, $Rc/Rs = 1$
Ex. 3: overestimated case ($X = (-1,1)$) → $Rc = 0$, $Rs = \sqrt{2}$, $Rc/Rs = 0$

Ex. 1 is the dangerous case in aseismatic design. Since X is the multiplication of ϕ_k, P^D, and S^D as $X = S^D P^D \phi_k$, if one of the ϕ_k, P^D, or S^D elements are almost 0 on account of the joint of mode shape, the asymmetric mode, or the very low or high frequencies respectively, then Ex. 2 results in. Ex. 3 shows unreasonable responses when it arises as model 1. The most important defect of the SRSS is the uncertainty of its numerical results. We conclude that the SRSS should be replaced by the CQC in aseismatic design.

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