Added mass model for vertical circular cylinder partially immersed in water

E. Da Lozzo
UME Graduate School - IUSS Pavia, Pavia, Italy

F. Auricchio
University of Pavia, Department of Structural Mechanics, Pavia, Italy

G.M. Calvi
UME Graduate School - IUSS Pavia, Pavia, Italy

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SUMMARY:
The present paper deals with a method for studying the dynamic response of partially submerged vertical circular cantilever beam undergoing flexural oscillations. The cylinder is idealized as a one-dimensional Euler-Bernoulli beam, whereas the fluid is modeled by potential theory. The sectional added mass, equivalent to the inertial effects of hydrodynamic forces, is firstly determined assuming a cosine-type deflected shape of the cantilever. Then, a lumped mass model is set up and the Finite Element Method is applied for calculating the fundamental "wet" natural frequency. A parametric study is conducted by varying the immersion ratio of the beam and its slenderness. The results obtained are compared with the "exact" solution available for the special case of cylinder immersed up to the height. Finally, for the purpose to enable engineers in estimating the fundamental "wet" frequency, a simple formula valid for any fluid depth is drawn.

Keywords: added mass model, cantilever circular cylinder, partially immersed beam, natural frequency

1. INTRODUCTION

A class of structures posing particular difficulties for their design is the one where a strong interaction takes place between the structure and the surrounding environment. Partially submerged structures, such as reinforced concrete bridge piers, intake towers, and in general any off-shore and coastal installations, belong to this category. The strategic role played by these structures in the socio-economic field requires protecting their integrity against earthquake-related damage.
The seismic analysis of partially immersed structures needs special considerations, which do not arise for structure on land. In fact, when a structure is partially or totally immersed, it interacts with the surrounding fluid through a process referred to as fluid-structure interaction (FSI). The modelling of such a mechanism may result of paramount importance in the reliable prediction of dynamic characteristic of the coupled system. The accurate modeling of FSI demonstrates to be highly challenging because of the inherent computational difficulties in analyzing two coupled fields which mutually interact each other in time. On the other hand, simple added mass model may be adopted to circumvent this issue. In fact, it is well known that when a structure vibrates in water, it induces motions and consequently accelerations in the fluid, thus producing an extra force on the structure called hydrodynamic pressure. This extra force can be conveniently modelled as the product of an equivalent mass of fluid and the acceleration of the structure, as stated in the earlier work of Lamb [1932] and recently discussed in Han and Xu [1996]. The term *added mass* given to the body of water mobilized by the structure, should not confused with the term *virtual mass*, which refers of the total effective mass of the system that participates in the vibrations, and is defined as the sum of the structural and the added mass.
In the past, studies on water-structure interaction was conducted with a different level of complexity. Assuming potential flow, Jacobsen [1949] applied the technique of separation of variable and the eigenfunction expansion to solve the Laplace equation associated to ideal incompressible fluid surrounding a cylindrical undeformable structure subjected to ground excitation. The significance of water compressibility and presence of surface waves on the earthquake response of cantilever towers

was initially investigated by Liaw and Chopra [1973]. Authors concluded that compressibility effect is negligible for slender towers, but important for squat towers vibrating at high frequency, because of the shorter natural period of the latter. The effect of surface waves was found to be significant only at very low frequencies, typically far from the fundamental frequency of the structure, and hence negligible in most of the practical cases of interest in earthquake engineering. Therefore, there is a considerable range over which neither compressibility nor surface wave effects are significant. The exact solution for free vibration of flexural beams immersed to the height was reported in Han and Xu [1996], who solved analytically the integro-differential equation associated to structure-water coupled system when the fluid is assumed ideal and inviscid. The solution is limited to the case where both the effects of compressibility of fluid and presence of surface waves are negligible. Authors investigated also an approximate estimation of added mass coefficient useful for rapid computation of natural frequencies of submerged structures.

It is generally accepted that, the natural frequencies of flexible structures in contact with water, namely "wet" frequencies, decrease compared with the frequencies which occur in "dry" conditions (see e.g. Xing at al. [1996]). This directly results in a potentially modified seismic response, as different vibrational characteristics of the structure in water correspond to different spectral acceleration ordinate, as well as different spectral displacement ordinate.

The purpose of this paper is to develop a simple formula for calculating the added mass to be used in estimation of natural frequencies of uniform circular cantilever beam partially submerged in water. The oscillating system is firstly discretized in a number of finite elements with lumped mass, which accounts for both the structural and the added hydrodynamic mass mobilized by the interaction with fluid. Then, the Finite Element Method (FEM) is used to analyze the natural frequencies of the equivalent coupled fluid-structure system. A parametric study is conducted with the purpose to explore the dependence of fundamental frequency on the slenderness and the immersion ratio.

Finally, the fundamental "wet" natural frequency is assumed to be suitably computed using the same formula valid for calculating the fundamental "dry" frequency, but conveniently modified to account for the added mass through an added mass coefficient. A further parametric study is conducted with the purpose to investigate the dependence of such coefficient on the slenderness, the immersion ratio, the height of the cantilever and the modulus of elasticity. An approximated expression is also derived as function of the slenderness and the immersion ratio, only. It allows the quick calculation of the first natural frequency of partially immersed structure, and thus resulting very useful for designers working in the dynamics of submerged structures subjected to earthquake excitation.

2. MODEL ASSUMTIONS AND FORMULATION OF GOVERNING EQUATIONS

We consider a straight cylinder of height H_s and submerged in water of depth $d \leq H_s$ as shown Figure 1. The circular cross-section is characterized by area A, radius r_0 and diameter D. The cylinder is idealized as a one-dimensional linear elastic structure governed by the beam theory. Since the cylinder is considered as slender structure, the classical Euler-Bernoulli hypothesis is introduced to ensure a handy theory which describes effectively the deformations and by neglecting shear deformations.

The structural properties of the beam are defined by the uniform mass per unit length $A_\rho = A \cdot \rho_s$ and the uniform flexural stiffness EI, where E is the Young modulus, I is the moment of inertia and ρ_s is the density of solid material used to fabricate the cylinder. The nonlinear material effects associated to ductility, cracking, etc. are not included and thus linearity of the structural behaviour is assumed.

Given the geometry of the problem, a cylindrical coordinate system (r,θ,z) is introduced as shown in Figure 1. The z-axis coincides with the axis of the cylinder and it points vertically upwards from the origin on the clamped end at the geometric centre of the structure. Coordinate r is measured radially from the z-axis and θ from the positive x-axis. The plane x-z defines the plane of vibration of the cantilever and only the flexural displacement of the beam axis $u(z,t)$ measured with respect to the base, is considered as variable in the analysis. The flexural displacement is assumed function of both the spatial coordinate z and the time t, and thus is can be expressed in the form

$$u(z,t) = f(z) \cdot q(t) \qquad (2.1)$$

where $f(z)$ is chosen as a sufficiently smooth function defining the height wise shape of deflection, whereas $q(t)$ is known as generalized coordinate and determines the time-variation scaling factor.

The fluid occupies the domain Ω_f which extends infinitely in horizontal direction, and is bounded by the sea bottom Γ_b, the free surface Γ_s on the top and the interface with the wetted surface of the structure Γ_s. The fluid, initially considered at rest, is assumed to be perturbed by the flexural motion of the cantilever. If the horizontal dimension of the structure is large compared to the amplitude of structure motion, flow separation and wake development are not expected to occur and the problem may be treated by potential flow theory (Isaacson et al. [1990]). Therefore, any drag force is neglected. In addition, as the propagation of vorticity in the fluid domain is a slow diffusion process compared to the rapid excitation imposed by earthquakes, small-amplitude irrotational motion is expected to take place in the fluid domain as consequence of the interaction between the oscillating cylinder and the fluid particles. This enables the problem to be further simplified by a linearization of the governing equations and allows using the method of potential theory to describe mathematically the behaviour of the fluid.

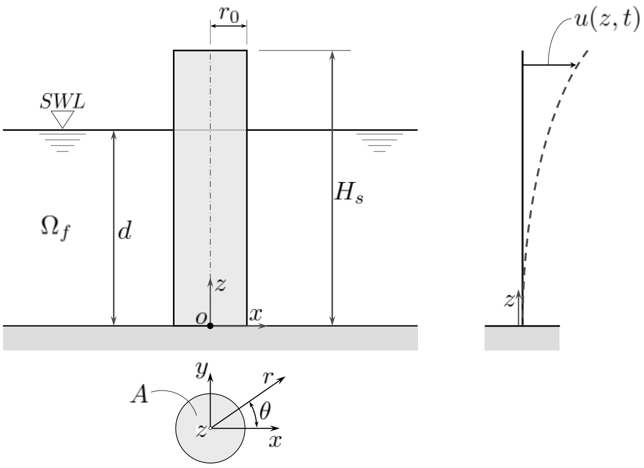

Figure 1. *On the left, the view in elevation and in plan of a circular cross-section cylinder partially submerged in water. On the right, the cylinder idealization and its deflected shape.*

Under the assumption of a homogeneous, incompressible and inviscid fluid characterized by irrotational motion, the response in the fluid domain Ω_f is governed by the Laplace equation, which in cylindrical coordinates reads as

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{2.2}$$

where symbol $\nabla^2(\cdot)$ denotes the Laplace operator, and the scalar function $\Phi(r,\theta,z,t)$ represents the velocity potential with the stipulation that the velocity vector in any point of the fluid field is defined as the gradient of $\Phi(r,\theta,z,t)$ (see Clauss et al. [1992]). In view of the specific case depicted in Figure 1, the following set of boundary conditions must be introduced:

1) Impermeable and rigid seabed able to prevent any fluid flow across the boundary Γ_b requires

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = 0 \tag{2.3}$$

2) Zero pressure at the free surface boundary Γ_f implies

$$\Phi = 0 \tag{2.4}$$

3) Impermeable and motion consistent boundary condition at the fluid-structure interface Γ_s, which implies that the fluid cannot flow through the cylinder and the fluid particles in contact with the cylinder surface has the same displacement, velocity and acceleration in radial direction as that of the solid particles composing the cylinder (kinematic compatibility), so no gap occur between the two, thus

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=r_0} = \frac{\partial u_r}{\partial t} \tag{2.5}$$

4) Radiation condition in the far field applies as follows

$$\lim_{r\to\infty}\Phi = 0 \tag{2.6}$$

5) Symmetry condition about $\theta=0$ plane, namely

$$\left.\frac{\partial \Phi}{\partial \theta}\right|_{\theta=0} = \left.\frac{\partial \Phi}{\partial \theta}\right|_{\theta=\pi} = 0 \tag{2.7}$$

The hydrodynamic pressure in the fluid field is related to the velocity potential through the linearized unsteady Bernoulli equation (see Clauss et al. [1992])

$$p = -\rho \frac{\partial \Phi}{\partial t} \tag{2.8}$$

where ρ is the mass density of the fluid. As depicted in Figure 2, the hydrodynamic pressures p at the interface are distributed symmetrically with respect to the zx plane, and act normally to the cylinder surface, since no shear stress is admitted in the fluid. With the aid of Figure 2, the sectional hydrodynamic resultant (i.e. resultant per unit length in the direction of vibration x) of the hydrodynamic pressure acting on the outer wetted surface of the cylinder at height z is calculated by the following circumferential integral

$$P_x(z) = \int_0^{2\pi} p_c r_0 \cos^2 \theta d\theta \tag{2.9}$$

where $p_c(z)$ denotes the pressure for $\theta=0$. Term "sectional" stands here to denote a property which is defined per unit length. Despite the presence of the hydrostatic pressure acting on the interface, its self-equilibrated nature, contrarily to the hydrodynamic force, does not induce any deflection in the beam, and its effect can therefore be discarded if the dynamics of the structure is mainly concerned.

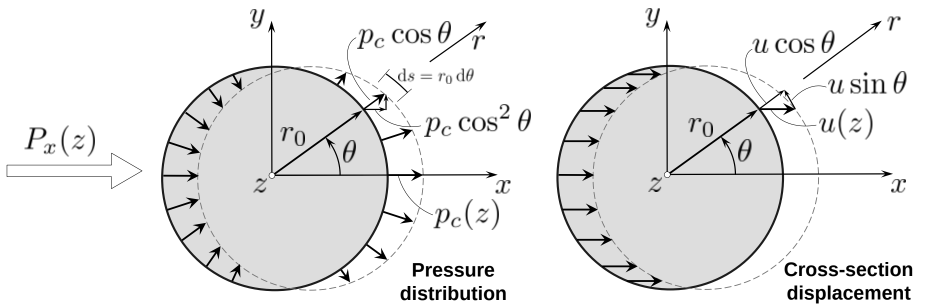

Figure 2. *Representation of pressure distribution on the wet surface of cylinder (on the left), and the displacement of generic cross-section (on the right), at a given time instant.*

As stated in Isaacson et al. [1990], the sectional hydrodynamic loading $P_z(z)$ can be conveniently decomposed into two parts, one in phase with the velocity and the other in phase with the acceleration of the beam, that is

$$P_x(z,t) = -\left(\mu \frac{\partial^2 u}{\partial t^2} + \lambda \frac{\partial u}{\partial t} \right) \tag{2.10}$$

where the real-valued parameters μ and λ can be physically interpreted respectively as the sectional mass of fluid which is accelerated by the structure's motion, and the damping coefficient associated to the energy transmitted through the water to its infinite boundary by radiation and then dissipated (see Xing [2007]). Note that the negative sign in equation (2.10) is due to the reactive nature of the force.
If the characteristic cross-sectional dimension of cylinder is large compared to the length of generated waves, and assuming negligible any dissipation mechanisms caused by the fluid, such as radiation damping, vortex shedding and fluid turbulence, the nature of hydrodynamic force is primarily inertial. Thereby the pressure resultant P_x can be conveniently computed assuming solely an equivalent added mass, and Equation (2.10) reduces to

$$P_x(z,t) = -\mu(z) \frac{\partial^2 u}{\partial t^2} \tag{2.11}$$

thus, section by section, the hydrodynamic resultant can be replaced by the product of the sectional acceleration and the equivalent added mass $\mu(z)$, which reflects the inertial effects of the water on the structure when the later oscillates.

Now, focusing on the structure, one may recognized that, by equilibrium, the sectional resultant P_x represents a supplemental loading force applied to the structure. Whether the effects of shear deformation and rotary inertia are neglected, the equation governing the flexural undamped vibrations of a beam partially submerged in water can be derived following D'Alembert's principle and setting to zero the sum of elastic restoring force, the inertial force and the hydrodynamic force $P_x(z,t)$ acting on the infinitesimal element of the beam. Considering the expression (2.11), the equation of motion reads as (see Chopra [2012])

$$EI\frac{\partial^4 u}{\partial z^4}+\left(A_\rho+\mu(z)\right)\frac{\partial^2 u}{\partial t^2}=0 \tag{2.12}$$

This equation represents the free vibration equation of motion since no pure external loads are applies to the structure. In fact, in lieu of equation (2.11), the hydrodynamic pressure is converted into the product of structural acceleration and the coefficient $\mu(z)$, and hence assumes the physical meaning of an added inertial term, rather than an independent external action.

3. ANALYTICAL DETERMINATION OF SECTIONAL ADDED MASS

The method of separation of variables can be applied to solve the Laplace equation (2.2). For semi-infinite region, the solution can be expanded in terms of eigenfunctions which satisfies the Laplace equation and all the boundary conditions except that on the interface Γ_s (see e.g. Zhou [1990]). The resulting velocity potential Φ turns out to be

$$\Phi(r,\theta,z,t)\big|_{r=r_0}=\dot{q}(t)\cdot\cos\theta\sum_{n=1}^{\infty}A_n K_1(\alpha_n r_0)\cos(\alpha_n z) \tag{3.1}$$

in which $n=1,2,3,\ldots$ is an integer number, $K_1(\cdot)$ is the modified Bessel function of the second kind and first order, $\alpha_n=(2n-1)\pi/(2d)$ and a superimposed dot denotes differentiation with respect to time. The coefficients A_n are determined by the remaining kinematic boundary condition on the interface Γ_s, and they assume the form

$$A_n=\frac{1}{\alpha_n\cdot K_1'(\alpha_n r_0)}\cdot\frac{2}{d}\int_0^d f(z)\cdot\cos(\alpha_n z)dz \tag{3.2}$$

where the prime denotes differentiation with respect to the argument, and $f(z)$ is the shape function. It is essential to point out that, although condition (2.5) formalizes the fluid-structure interaction condition, it remains substantially undetermined until the shape function $f(z)$ is specified. The selection of proper function $f(z)$ is the thorniest point in any simplified procedure which aims to deal with FSI from an analytical point of view. A possible approach is to solve analytically the vibration response of the cylinder-fluid coupled system and determine the mode shapes as presented in Han and Xu [1996] for the special case where $d=H_s$. In general $d<H_s$, and the analytical approach requires four additional matching conditions at the joint level between the wetted and dry portion of the beam to ensure the deflection, the rotation angle, the shear force and bending moment to be continuous (see for instance Xing et al. [1997]).

In order to circumvent the lengthy calculations which arise in the analytical solution of integro-differential equation for partially immersed cylinder, one can specify a priori a reasonable function $f(z)$ which satisfies the boundary conditions at the ends of the beam. Because the first mode shape is expected to govern the dynamics of the "wet" column, it is assumed that the shape function $f(z)$ does not differ significantly from the first "dry" mode shape, which for practical purposes is approximated by the following expression

$$f(z)=1-\cos\left(\frac{\pi}{2}\frac{z}{d}\right) \tag{3.3}$$

It is worth to remark that, as the present study is mainly focused on steady-state vibrations, only the shape of $f(z)$ is of concern in calculating the added mass $\mu(z)$, and thus any scaling factor applied to $f(z)$ has no effects in the subsequent computation of natural frequencies. For sake of convenience, in this work the shape function was normalized in such a way that the largest value of $f(z)$ is equal to 1.

Once determined the velocity potential Φ form equation (3.1) for the shape function (3.3), the hydrodynamic pressure $p_c(z)$ exerted against the wetted contour of the column can be easily computed invoking Bernoulli equation (2.8). Finally, integrating the pressure on the interface by means of equation (2.9) to determine the sectional hydrodynamic resultant, the sectional added mass $\mu(z)$ is deduced from (2.11) as

$$\mu(z) = -\pi\rho r_0 \sum_{n=1}^{\infty} A_n K_1(\alpha_n r_0)\cos(\alpha_n z) \tag{3.4}$$

where constants A_n are set as

$$A_n = \frac{1}{\alpha_n K_1'(\alpha_n r_0)} \cdot B_n \tag{3.5}$$

$$B_n = \begin{cases} \dfrac{4}{\pi} - 1 & n = 1 \\ \dfrac{4}{\pi}\dfrac{(-1)^{n-1}}{2n-1} & n \geq 2 \end{cases} \tag{3.6}$$

4. NUMERICAL CALCULATION OF NATURAL FREQUENCIES (lumped mass model)

Natural frequencies of a partially immersed cylinder can be determined by using a semi-analytical approach based on the normal Finite Element Method and the analytical estimation of added mass. In particular, the simple mechanical analogy proposed by McCormick [1989] is partially adopted also in the present work for its simplicity. As illustrated in Figure 3, the approach consists into modelling the continuous cantilever system as a M-DOF discrete system, by lumping the structural mass at a number of nodes which are joined together by massless but flexible 2D beam elements, for convenience assumed with equal length. The inertial effect of water is introduced in the model by a set added masses rigidly mounted to the structure and lumped at the nodes lying beneath the free surface. The value of this lumped mass is calculated as suggested by McCormick [1989], by integrating the sectional added mass $\mu(z)$ over the length of the *i*-th finite element located at depth z_i ,z_{i+1}, and equally distributing the resulting total mass on the two nodes of the element. With the aid of Figure 3 we have

$$\bar{\mu} = \frac{1}{2}\int_{z_i}^{z_{i+1}} \mu(z)dz \tag{4.1}$$

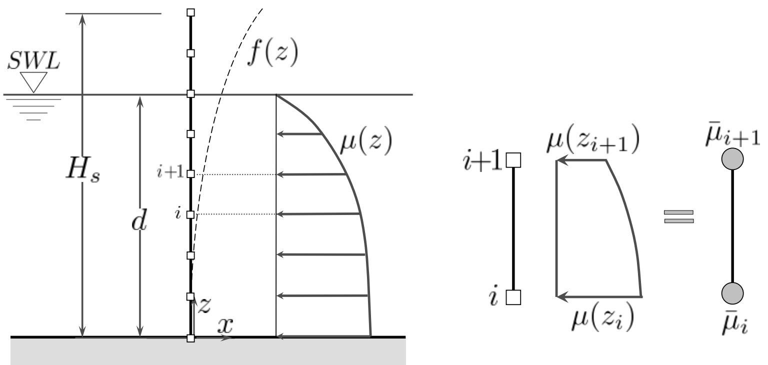

Figure 3. *Schematic representation of discrete model.*

Therefore, the inertial characteristics of the beam will depend not only on its own structural mass, but also on the added mass mobilized by effect of interaction with water. Obviously it is expected that the behaviour of the discrete model illustrated above, approaches that of the continuous system, as the number of degrees of freedom in the former is increased. The application of Finite Element Method to the structure illustrated in Figure 3 allows one to turns the differential problem (2.12) into an algebraic problem suitable to be solved numerically. The process of seeking the natural frequencies of the coupled column-water system involves the solution of a generalized eigenvalue problem

$$[K]\{\phi\} = \omega^2 [M]\{\phi\} \tag{4.2}$$

where matrix *[K]* and *[M]* are respectively the global stiffness and mass matrices associated to the structure, ϕ is the mode shape vector and ω determine the circular natural frequency.

5. PARAMETRIC STUDY ON THE "WET" FUNDAMENTAL NATURAL FREQUENCY

By using the semi-analytical method presented above, a parametric study was conducted on a set of circular cantilever beams, partially immersed in water and rigidly fixed at the base. For a more general discussion of the results, the selected varying non-dimensional parameters are the immersion ratio $\beta=d/H_s$ ($0 \le \beta \le 1$) which defines the ratio of the water depth to structure height, and the slenderness $\lambda=H_s/D$ ($5 \le \lambda \le 30$). In total 286 cases were examined. It is worth noting that the limit case with $\beta=0$ identifies "dry" conditions, whereas the opposite case with $\beta=1$ indicates the situation of cantilever submerged to the head.

The analyses are performed discretizing the structure with 512 linear elastic beam elements, and assigning only translational lumped mass at the nodes. Solutions for lumped added mass are obtained assuming flexible cylinder vibrating according to the shape provided in equation (3.3), and truncating the series expansion (3.4) to the first 100 terms. The density of water and the density of the column are $\rho=1000$ kg/m^3 and $\rho_s=2450$ kg/m^3, respectively; the elastic modulus of the beam $E=29{,}4$ GPa is assumed constant; the acceleration of gravity is $g=9.81$ m/s^2. Beam dimensions through all the study are such that Euler-Bernoulli theory can be applied.

Ignoring the effect of water compressibility and surface waves, the values of fundamental "wet" natural frequencies are plotted in Figure 4 as function of immersion ratio β and slenderness λ. Frequencies are normalized such that, for a given slenderness, the "dry" natural frequency is always the unity. It is also reported the "exact" solution found by Han and Xu [1996] for cantilever immersed to the height (case with $\beta=1$). Results indicate that:

(i) For a given slenderness λ, fundamental "wet" natural frequencies are always lower than "dry" natural frequencies. The higher the water level, the significant the effect of the water on dynamic characteristics of the coupled column-water system, i.e. the lower the natural frequency of the coupled system. This result should not surprise since it comes as direct consequence of modeling the effect of water on structure through the inclusion of an added mass in the model. This conclusion agrees with the mathematically proven results of Xing et al. [1997].

(ii) The variation of the ratio $\omega_{wet}/\omega_{dry}$ is markedly nonlinear with the immersion ratio β for the fundamental mode. The reduction of frequency in the first mode is rather limited for low values of parameter β, while it results more pronounced for values $\beta>0.5$. This reflects the influence of water for high immersion ratios, regardless the value of slenderness.

(iii) The maximum reduction in frequency is obtained for column completely immersed in water, and its value is on the order of 10%, with little variation between high and low slenderness structure.

(iv) For $\beta=1$ there exist differences between the ratio $\omega_{wet}/\omega_{dry}$ calculated with the present method and that proposed by Han and Xu [1996]. Approximately the error is in the order of 3.5% and it may be ascribed to the assumed shape function *f(z)* in which the structure is constrained to vibrate.

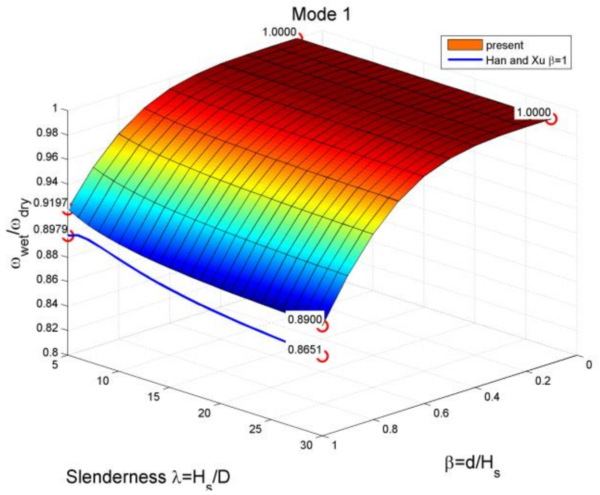

Figure 4. *Distribution of the ratio $\omega_{wet}/\omega_{dry}$ for different value of immersion ratio β. The surface represents the present results, whereas the line denotes the result obtained by Han and Xu [1996] for β=1.*

6. APPROXIMATION OF ADDED MASS COEFFICIENT

From a practical point of view, it is highly desirable by side of practitioner to have a disposal a simple and reasonably accurate formula for calculating natural frequencies of cylinder partially immersed in water. Starting from the assumption that, when a submerged structure vibrates it induces acceleration in the surrounding fluid, the extra hydrodynamic effect can be accounted for by a hypothetical mass of fluid which is considered moving rigidly with the structure. The added mass concept, successfully adopted by Han and Xu [1996] to facilitate the computation of "wet" natural frequencies of cantilever systems, is here extended to the case where the immersion ratio $\beta=d/H_s$ is free to varies between 0 and 1.

Consider a cantilever beam of height H_s oscillating in air, and characterized by flexural stiffness EI and structural mass per unit length ρA. The exact "dry" fundamental frequency for a circular beam is given by (Chopra [2012])

$$\omega_1^a = \frac{\alpha_1^2}{2}\frac{r_0}{H_s^2}\sqrt{\frac{E}{\rho_s}} \tag{6.1}$$

where $\alpha_1=1.8751$. In light of equation (2.12), for an oscillating cantilever partially submerged in water it is assumed that the "wet" fundamental frequency is equivalent to the fundamental frequency of a cantilever oscillating in air and having virtual density $\rho_s+C_m\cdot\rho_w$, i.e.

$$\omega_1^w = \frac{\alpha_1^2}{2}\frac{r_0}{H_s^2}\sqrt{\frac{E}{\rho_s + C_m\rho_w}} \tag{6.2}$$

where C_m is the added mass density coefficient, defined as the ratio of the added density to the density of water. Solving equation (6.2) for the unknown C_m, we have

$$C_m = \left[\frac{\alpha_1^4}{4}\frac{r_0^2}{H_s^4}\frac{E}{\left(\omega_1^w\right)^2} - \rho_s\right]\frac{1}{\rho_w} \tag{6.3}$$

By determining the "wet" frequency through the semi-analytical method described earlier, and substituting it into equation (6.3), one can compute C_m for any value of parameter E, ρ_w, ρ_s, H_s and r_0. It is important to note that the added density coefficient also depends on the immersion ratio β, even though it is not explicitly evident in equation (6.3). In order to test the different influence of these parameters on C_m, a parametric study was conducted by varying the immersion ratio in the range $0.1\leq\beta\leq1$, the slenderness $5\leq\lambda\leq 30$, the elasticity modulus $10GPa\leq E\leq 60GPa$ and the column height $10m\leq H_s\leq 100m$. It was found that C_m only varies with the slenderness λ and the immersion ratio β, whereas it results independent on other two parameters. Figure 5 shows the typical behavior of surface function C_m with respect to β and λ for the fundamental "wet" mode.

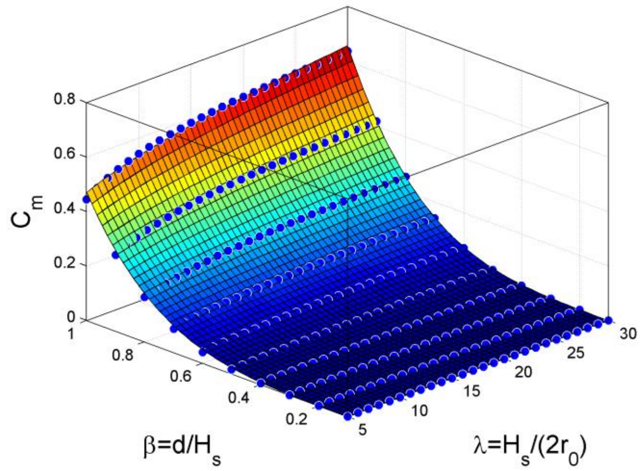

Figure 5. *Added mass coefficient for the first mode. Dots indicate samples of parametric study, whereas the surface is representative of the fitting model.*

The dataset obtained as output of equation (6.3) are fitted by using the following model

$$C_m = \left[a \cdot \left(\frac{1}{2\lambda} \right)^2 - b \cdot \left(\frac{1}{2\lambda} \right) + c \right] \cdot \beta^d \tag{6.4}$$

which results a good compromise between the necessity to have an easy-to-apply model and an accurate fit. Coefficients *a, b, c, d* are estimated based on the nonlinear least squares method as

$$a = 12.01; \quad b = 3.811; \quad c = 0.7023; \quad d = 4.282 \tag{6.5}$$

Thus, equation (3.10) can be used to predict ω_1 for any value of immersion ratio β and slenderness λ.
In order to evaluate the adequacy of the fitting model, in first place a visual screening is done by plotting residuals and visually inspect their magnitude and distribution on the studied parametric region. In second place, two statistics were adopted as criterion to evaluate effectiveness of the model, namely the sum of squares due to error (SSE) and the R-square number (R^2). The former measures the total deviation of the response values from the fit to the available response values calculated from equation (6.4). A value close to 0 indicates that the model has globally a small error and that the fit is useful for prediction. The latter measures how successful the fit is in explaining the variation of the data, and it varies between 0 and 1, with values close to 1 indicating that a greater proportion of variance is accounted for by the model.
Table 6.1. compares the fundamental frequencies of “dry” beam, denoted by ω_a, and “wet” beam, denoted by ω_w for various slenderness. Values refer to a cantilever with $E=29.4$ *GPa*, $d=20$ m and $\beta=1$.

Table 6.1. Fundamental frequencies of “dry” beam and “wet” beam.

$\lambda=H_s/D$	ω_a	ω_w - Han and Xu [1996]	ω_w - Present	Error
-	[rad/s]	[rad/s]	[rad/s]	%
5	30.4495	27.340065	28.029540	2.52
10	15.2247	13.577711	13.777426	1.47
15	10.1498	8.936509	9.113889	1.98
20	7.6124	6.647492	6.806448	2.39
25	6.0899	5.288923	5.430688	2.68
30	5.0749	4.390395	4.517344	2.89
35	4.3499	3.752403	3.866892	3.05
40	3.8062	3.276104	3.380136	3.18
45	3.3833	2.906995	3.002199	3.27
50	3.0449	2.612578	2.700264	3.36

7. CONCLUSIONS

The semi-analytic method described herein offers an approach which greatly reduces the computational effort associated to more refined FSI models for calculating the fundamental natural frequency of a flexible circular cantilever beam partially immersed in water. The solution is compared to closed-form solution available in literature for the special case where $\beta=1$. The agreement of the approximated numerical results is observed satisfactory for practical purposes, even though a note of caution should be given about the lack of perfect matching of the values. This depends on the shape function in which the structure is constrained to vibrate and consequently on the added mass mobilized during oscillations. The first natural frequency is computed in a wide range of possible combinations of the immersion ratio β and the slenderness λ, and finally a simplified formula is proposed for the calculation of added density, which should be useful to designers working with structures where the correct modelling of FSI is an issue.

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