Analytical Approach to Eliminate High Frequency Instability Caused by Multi Transmitting Formula

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SUMMARY:
To solve near-field wave problems by finite element method, absorbing boundary conditions are needed to simulate the unbound field robustly and efficiently. Taking numerical simulation of SH wave propagation in waveguide as an example, high-frequency instability caused by transmitting boundary is discussed. Directly from discrete model, mechanism of such instability is found by analyzing matching relationship of scheme of interior nodes and nodes on absorbing boundary. Results show that instability is caused by high frequency waves, which are allowed by numerical scheme, entering continuously from the absorbing boundary into the computational region without excitation. Thus an approach of eliminating such instability is proposed by adjusting the equations of motion of interior nodes to change the matching relationship. Theoretical and numerical results demonstrate that approach is efficient.

Keywords: Absorbing boundary condition, Numerical instability, Wave motion, Finite element, Dispersion curve

1. INTRODUCTION

Absorbing boundary conditions (ABCs) are needed in numerical simulation of exterior wave problems aroused in various fields of application such as geophysics, civil engineering, earthquake engineering and etc. Although various important ABCs have been proposed(Bérenger 1994; Bayliss & Turkel 1980; Clayton & Engquist 1977; Liao *et al*. 1984; Cerjan *et al*. 1985; Higdon 1986; Sochaki *et al*. 1987; Givoli 1992; Grote 1995; Rowley & Colonius 2000; Komatitsch & Martin 2007; Hagstrom *et al*. 2008; Du & Zhao 2010; Meza-Fajardo & apageorgiou 2010) during last three decades of research, it is remarkable that there is still not an ABC is perfect in terms of stability, efficiency, accuracy and wide applicability. Among various ABCs, the multi transmitting formula(MTF) based on transmitting theory proposed and developed by Chinese scholars in 1980s can yet be regarded as one of widely applied schemes(Liao *et al*. 1984). Except for its stability problem, MTF is excellent in aforementioned first three aspects. Thus stable implementation of MTF is the key problem for its practical use. The main concern of this paper is how to eliminate one of main instability phenomena caused by MTF: high-frequency instability.

Based on 1-D elastic bar model, mechanism of the instability is firstly clarified by Liao and Liu(1992), as follows: the high frequency instability results from reflection and amplification of high-frequency waves, which are meaningless for numerical simulation of wave motion based on finite element or finite difference techniques, and from repeated amplifications of multiple reflection of the waves within the finite computational region. Thus numerical simulation of wave motion can be realized by eliminating the instability occurred in the high frequency band through some filtering approaches.

However, numerical results show those approaches are not always effective. Instabilities can still occur, starting from the boundary of the computational region, especially when duration of the incident wave is long and the wave form is extremely irregular. The reason is the mechanism obtained from 1-D model cannot explain another type of high frequency instability aroused in high dimensional ones. It is caused by the locally improper coupling between scheme of interior nodes and MTF. Noticed that in finite element formulation, coupling of schemes of nodes on fixed or free boundary and interior nodes will not cause instability, Guan and Liao(1996) proposed to use traditional spectral analysis to study the mechanism of high frequency instability in two-dimensional model. They used an exterior SH wave model with rectangular set up. Three Dirichelet boundary conditions and MTF were separately set long the four truncated boundary. They found high frequency instability will arise with any choice of CFL number when uniform distributed square linear finite element grids were used. Also based on the same model but with MTF applied on all the truncated boundary, Jing *et al.*(2002) presented another kind of mechanism of high frequency instability using Trefethen's(1983) explanation of normal mode analysis based on group velocity theory. In this paper, we further clarify such mechanism as follows: high frequency instability was caused by improper matching of scheme of interior nodes and MTF. Also we present a novel way of stable implementation of MTF, in which we adjusted the interior scheme to get the proper matching that do not allow the instability.

Following is the outline of the rest of this paper. In Section 2, we present the SH guide model and its discretization. In section 3, we clarify the mechanism of high frequency instability and provide a novel way of elimination. Verification of that through numerical examples is given in Section 4. We conclude with some remarks in Section 5.

2. SH WAVEGUIDE MODEL AND ITS DISCRETIZATION

We consider wave propagation in a two-dimensional wave guide, as described in Fig. 1. A Cartesian coordinate system (x, y) is introduced such that the wave-guide is parallel to the x direction. The width of the wave-guide is denoted by 2b. On the top and bottom boundaries Γ_U and Γ_R we specify the Dirichlet condition

$$u\left(x,\pm b,t\right)=0 \tag{2.1}$$

In the wave guide we consider the linear inhomogeneous SH equation

$$\left[\partial_t^2 - c_s^2\left(\partial_x^2+\partial_y^2\right)\right]u = f \tag{2.2}$$

Here and elsewhere we use the following shorthand for partial derivatives:

$$\partial_a^i = \partial^i / \partial a^i \tag{2.3}$$

In (2.1), u is the unknown wave field, c_s is the share wave speed, and f is the given wave source function. The c_s is allowed to be functions of location; however, it is assumed that outside a finite region, they do not depend on x but only possibly on y. (Such y-dependence corresponds to a stratified medium.) The wave source f is a function of location and time, but it is assumed to have a local support in $\left(-x_0,x_0\right)\otimes\left(-b,b\right)$.We truncate the semi-infinite domain by introducing artificial east boundaries Γ_L, Γ_R separately located on x=x_R and x=-x_R(x_R>x_0); see Fig. 1

Now we consider the discretization of above model. A uniform grid consists of rectangular element are introduced with size of Δx and Δy separately in x and y direction. Mass-lumping linear conform finite element scheme(Liao, 2002a)and Leap-frog time scheme are applied to discrete Eq. (2.2).

$$
\begin{aligned}
u_{l,m}^{p+1}-2u_{l,m}^{p}+u_{l,m}^{p-1}=\frac{c_s^2\Delta t^2}{6\beta\Delta x^2}[&(1+\beta)\left(u_{l+1,m+1}^{p}+u_{l-1,m-1}^{p}+u_{l+1,m-1}^{p}+u_{l-1,m+1}^{p}-4u_{l,m}^{p}\right)\\
&+2(2-\beta)\left(u_{l,m+1}^{p}+u_{l,m-1}^{p}-2u_{l,m}^{p}\right)-2(1-2\beta)\left(u_{l+1,m}^{p}+u_{l-1,m}^{p}-2u_{l,m}^{p}\right)]
\end{aligned}
\tag{2.4}
$$

In (2.4) $u_{l,m}^{p}=u\left(l\Delta x,m\Delta y,p\Delta t\right)$, p, l, m are integer; p=0,1,… ; l=-L+1,…,-1,0,1, …, L-1, $L\Delta x=x_R$, m=-M+1,…,-1,0,1,…,M-1, $M\Delta y=b$, $\beta=\left(\Delta y/\Delta x\right)^2$. We assume that CFL condition is satisfied in (2.4)

$$
\frac{c_s\Delta t}{\Delta x}\le\min\left\{1,\sqrt{\beta},\sqrt{3\beta/(1+\beta)}\right\}
\tag{2.5}
$$

Dirichlet condition on the top and bottom boundaries Γ_U and Γ_R are discrete as

$$
u_{l,M}^{p+1}=0;\quad u_{l,-M}^{p+1}=0
\tag{2.6}
$$

l=-L,…,-1,0,1, …, L. N_{th} order MTF is applied on absorbing boundary:

$$
\left[I-\left(t_1+t_2K+t_3K^2\right)Z^{-1}\right]^N u_{-L,m}^{p+1}=0
\tag{2.7}
$$

$$
\left[I-\left(t_1+t_2K^{-1}+t_3K^{-2}\right)Z^{-1}\right]^N u_{L,m}^{p+1}=0
\tag{2.8}
$$

Here $t_1=(s-2)(s-1)/2$, $t_2=s(2-s)$, $t_2=s(s-1)/2$, $s=c_a\Delta t/\Delta x$, c_a is artificial wave speed. The space and time shifting operator K, Z are defined by $K^{l_0}u_{l,m}^{p}=u_{l+l_0,m}^{p}$, $Z^{p_0}u_{l,m}^{p}=u_{l,m}^{p+p_0}$; l_0 and p_0 are integers. In practical numerical simulation, the effect of order of MTF on accuracy is much bigger than the value of artificial wave speed. Thus we simply set $s=1$ in Eq. (6) and (7). In this case those two equations can be written simply as:

$$
\left[I-KZ^{-1}\right]^N u_{-L,m}^{p+1}=0
\tag{2.9}
$$

$$
\left[I-K^{-1}Z^{-1}\right]^N u_{L,m}^{p+1}=0
\tag{2.10}
$$

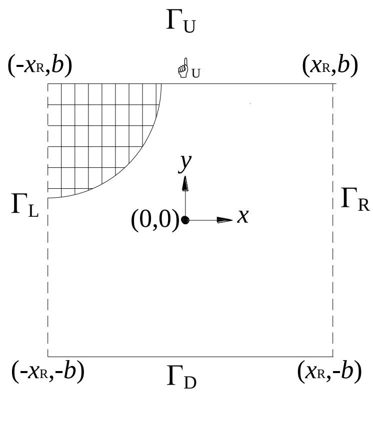


Figure 1. Schematic diagram of infinite SH waveguide

3. MECHANISM OF HIGH FREQUENCY INSTABILITY AND ELIMINATION MEASURE

In this section, we firstly clarify the local property of high frequency instability phenomena aroused in high dimensional numerical model. Then we introduce the stability analysis method based on normal mode analysis. The key idea of the method is that: stability of complex numerical model can be transformed into analysis of several sub-models obtained by decoupling of the original one. Those sub-models are consist of either schemes of interior nodes, or schemes of interior nodes and nodes on absorbing boundary, or schemes of different kinds of nodes. Finally the mechanism of high frequency instability is clarified through stability analysis of those sub-models. The mechanism can be expressed as: high frequency instability are caused by improper matching of schemes of interior nodes and nodes on absorbing boundary, the improper coupling of those schemes allows high frequency wave component radiated from absorbing boundary continuously into computational domain without any exterior stimulation on boundary. Thus a method of stable implementation of MTF is proposed through changing matching relationship between schemes of interior nodes and nodes on absorbing boundary. With proper matched schemes, high frequency waves will not be allowed thus stable implementation of MTF can be achieved.

Summarizing from previous numerical results and research work presented in various papers, we find that(Liao *et al*. 2002b; Xie, 2011; Liao & Xie 2011): once the instability occurred, if we trace the source of errors caused by instability, we can observer that errors firstly appeared on nodes of absorbing boundary. Then they are increased in terms of amplitude and distribution area. Considering that CFL condition is satisfied by scheme of interior nodes, we can conclude that instability caused by ABC are local phenomenon, which means that instability are caused by the improper matching of scheme of ABC and interior nodes adjacent to absorbing boundary, or by improper setup of ABCs' analytical formulation and their discretization.

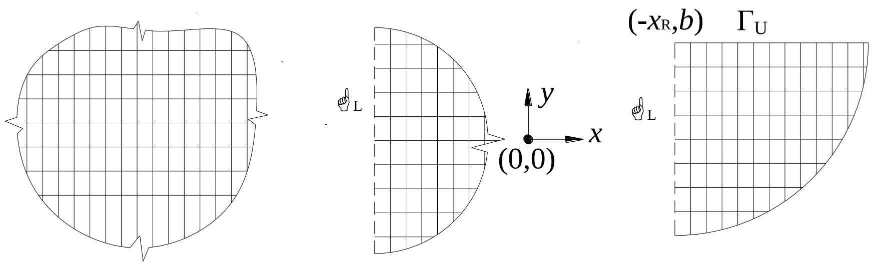


Figure 2. Schematic diagram of sub-model

In explicitly decoupled numerical simulation, the velocity of wave propagated in discrete grid is finite. Take Fig. 1 as an example, if we set force in central of the model, the wave propagation is controlled only by scheme of interior nodes when the effect of boundary is not arrived. Thus, during this time interval, the stability of discrete model is equivalent to whether scheme of interior nodes satisfy CFL condition or not. It is the same that, when force are set on the left-top corner of model, the stability of numerical model is only controlled by coupled schemes of interior nodes and nodes on absorbing boundary. Thus, the stability analysis of the discrete model can be transformed into analyzing the stability of sub-models, which consist of several different kinds of schemes of different nodes. Such a transformation enables us to discover the mechanism of stability in an analytical way, also can be very useful to find measure to eliminate the instability. The main idea of aforementioned stability analysis method is firstly present by Godunov *et al*.(1963). Then based on their work, Gustafsson，Kreiss，Sundström(1972) and Osher proved GKS theorem, in which the necessary and sufficient condition of the stability of finite difference discretation of one dimensional hyperbolic partial different initial boundary value problem is provided. The main contribute of GKS theorem is that it proved that the stability of the original numerical model is equivalent to several sub-models decoupled from the original model. For the model presented in this paper, to analyze the stability we only need to investigate three sub-models showed in Fig 2. Fig.2 show that the sub-model one is only consists of scheme of interior nodes, while the second one is consists of scheme of interior nodes and of nodes on absorbing boundary, and the last one is consists the aforementioned two schemes plus scheme of nodes top boundary.

Next we analyze the stability of sub-models. According to sub-model one is consists of schemes of interior nodes, it is stable because the CFL condition is assumed being satisfied. Then we analyze the stability of sub-model two and clarify the mechanism of high frequency instability. For convenience, we present the formulation of sub-model two:

$$
\begin{aligned}
u_{l,m}^{p+1} - 2u_{l,m}^{p} + u_{l,m}^{p-1} = \frac{c_s^2 \Delta t^2}{6\beta \Delta x^2}[(1+\beta)\left(u_{l+1,m+1}^{p} + u_{l-1,m-1}^{p} + u_{l+1,m-1}^{p} + u_{l-1,m+1}^{p} - 4u_{l,m}^{p}\right) \\
+ 2(2-\beta)\left(u_{l,m+1}^{p} + u_{l,m-1}^{p} - 2u_{l,m}^{p}\right) - 2(1-2\beta)\left(u_{l+1,m}^{p} + u_{l-1,m}^{p} - 2u_{l,m}^{p}\right)]
\end{aligned}
\tag{3.1}
$$

Here p=0,1,… ; l=-L+1,…,-1,0,1, …, L-1, m=-∞,…,-1,0,1,…,∞. MTF are applied on absorbing boundary

$$
\left[I - KZ^{-1}\right]^{N} u_{-L,m}^{p+1} = 0 \tag{3.2}
$$

In (3.2), p and m take the same value as in (3.1)

According to Trefethen’s work(1983), the stability condition of sub-model two can be summarized as: 1, the model do not support exponentially increased solution such as

$$
u_{l,m}^{p} = z^{p} \kappa_x^{l} \left(e^{ik_y \Delta y}\right)^{m} \tag{3.3}
$$

Here $|z|>1$, $|\kappa_x|<1$; 2, the model do not support plane wave solution, of which the group velocity pointed direct into interior computational domain

$$
u_{l,m}^{p} = e^{ip\omega\Delta t} e^{ilk_x \Delta x} e^{imk_y \Delta y} \tag{3.4}
$$

Substituting (3.3) into (3.1) and (3.2), we can easily verified that the condition 1 is satisfied. Thus to analysis the stability of that model, we only have to verified the second condition. First we substitute (3.4) into (3.1) and (3.2) we get

$$
\sin^2 \frac{\omega \Delta t}{2} = \frac{c_s^2 \Delta t^2}{\Delta y^2}[\sin^2 \frac{k_y \Delta y}{2} + \beta \sin^2 \frac{k_x \Delta x}{2} - \frac{2(1+\beta)}{3} \sin^2 \frac{k_y \Delta y}{2} \sin^2 \frac{k_x \Delta x}{2}] \tag{3.5}
$$

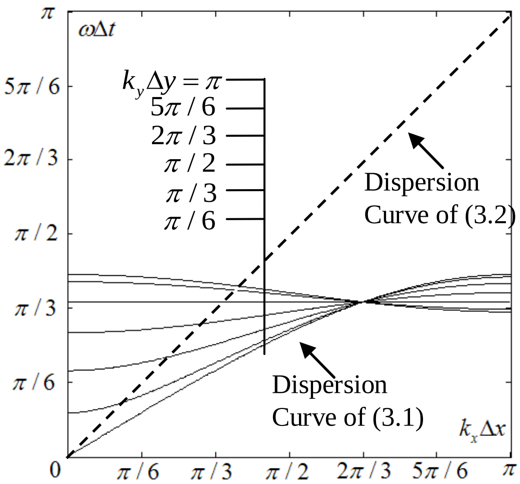
$$
\omega \Delta t = k_x \Delta x \tag{3.6}
$$

According the aliasing effect in discrete grid, $\omega\Delta t$, $k_x \Delta x$, $k_y \Delta y$ are valued in the domain $[0,\pi]$. (3.5) and (3.6) are separately the dispersion equation corresponding to (3.1) and (3.2). Dispersion curve of (3.5) and (3.6) can be drawn with the dispersion equation. Apparently if the plane wave solution (3.4) can be supported by both (3.1) and (3.2), then the two dispersion curve must have at least one intersection. According to its definition, the group velocity of (3.4) is the slope of the tangent at the point (ω, k_x, k_y) of dispersion curve. Because that the slope of dispersion curve of (3.2) is always positive, the stability condition two is equivalent to the product of two slopes of the tangent of each dispersion curve at the point on the intersection point is positive or not. If the product is negative, plane wave with (ω, k_x, k_y), of which the group velocity has group velocity directed into computational domain, valued at intersection point are supported by both schemes of interior nodes and MTF. Otherwise, if it is positive, then none of those waves are supported, which means the sub-model two are stable.

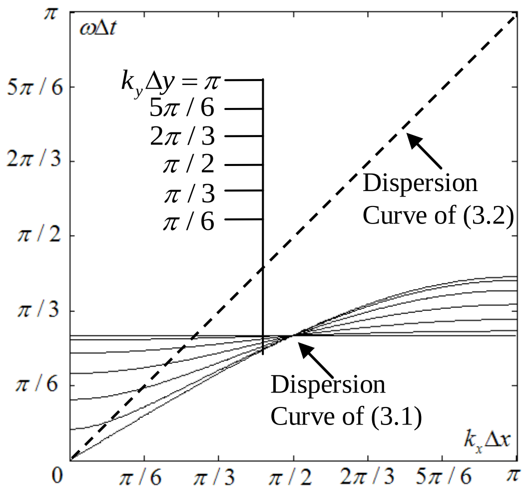
Next we analyze sub-model two satisfy the condition two or not. Take $c_s \Delta t / \Delta y = 3/5$ as an example, Fig. 3 shows the dispersion curve of (3.1), (3.2). In Fig. 3a, in the case of $\Delta y / \Delta x = 1$, we can see that those two curve exist intersection with negative product of slopes of the tangent of each dispersion curve. According to the intersection are placed in high frequency domain $\omega \Delta t > 1$, which means the high frequency waves are supported, thus high frequency instability will arise in sub-model. However in fig. 3b, in the case of $\Delta y / \Delta x = 2$, none of those intersections exist. Further, we can show from the expression of group velocity of plane wave solution:

$$C_{g,x} = -\frac{\partial \omega(k_x, k_y)}{\partial k_x} = -\frac{c_s^2 \Delta t}{\Delta x} \frac{\left[1-(2+2\beta)/3\beta \sin^2(k_y \Delta y/2)\right] \sin(k_x \Delta x/2)}{\sin(\omega \Delta t/2)} \tag{3.7}$$

We can further prove that sub-model two satisfy the stable condition two under if $\Delta y \geq \sqrt{2}\Delta x$. Thus the model two is stable. Furthermore, we can prove sub-model three are also stable under the condition $\Delta y \geq \sqrt{2}\Delta x$. Considering the above mentioned theory that stability of original model is equivalent to sub models, we can conclude that the original model is stable with $\Delta y \geq \sqrt{2}\Delta x$ and with CFL condition satisfied by scheme of interior nodes. Then numerical simulation of wave motion with grid consist of element with $\Delta y \geq \sqrt{2}\Delta x$ and with CFL condition satisfied by scheme of interior nodes can be seen as the new method for eliminating the high frequency instability.



(a): $\Delta y / \Delta x = 1$



(b): $\Delta y / \Delta x = 2$

Figure 3. Dispersion curve of Eq. 15 and Eq. 16 when using finite element of different length-width Ratio

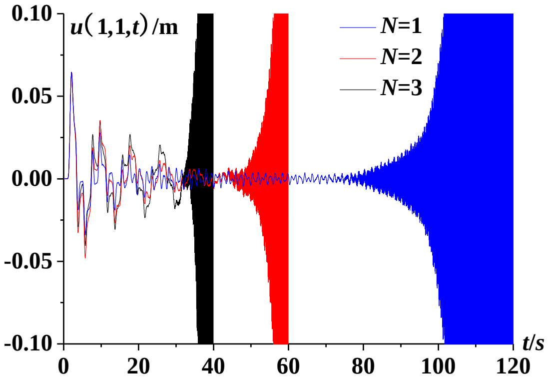
4. NUMERICAL EXAMPLES

We apply the new method to a number of test problems in a wave guide as described in Section 1 and illustrated in Fig. 1. We set b=2m and c_s=1m/s. The initial conditions are zero throughout the domain. On the walls Γ_U and Γ_D we use the Dirichlet boundary condition (2.1) and introduce force on line y=0m, $f(x,0,t) = F_S F_x(x) F_t(t)$. Here $F_S = 1.0 N/m$, $F_x(x)$ is approximate δ pulse function, $F_t(t)$ is triangular pulse function

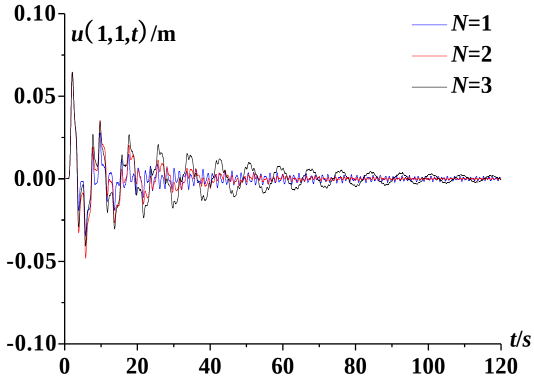
$$F_x(x)=\begin{cases}0 & x\le -1\\ 2(x+1)^3 & -1<x\le -\frac{1}{2}\\ 1-6(x^3+x^2) & -\frac{1}{2}<x\le 0\\ 1+6(x^3-x^2) & 0<x\le \frac{1}{2}\\ -2(x-1)^3 & \frac{1}{2}<x\le 1\\ 0 & x>1\end{cases},\quad F_t(t)=\begin{cases}0 & t\le 0\\ 2t & 0<t\le \frac{1}{2}\\ 2(1-t) & \frac{1}{2}<t\le 1\\ 0 & t>1\end{cases} \tag{4.1}$$

We introduce the artificial boundary at *x*=2*m* and *x*=-2*m*。 We discretize the SH wave equation using interior scheme (2.4). On Γ_L and Γ_R we impose the NRBC (2.9) and (2.10). In Ω we use two different uniform grids. First kind of grid: $\Delta x=0.04m$， $\Delta y=0.04m$， $\Delta t=0.02s$; Second kind of grid: $\Delta x=0.02m$， $\Delta y=0.04m$， $\Delta t=0.02s$

Fig. 4 shows the displacement time history at point (1, 1). For the first kind of grid, we can see clearly the high frequency instability in Fig 4a. It is clearly that the instability appears more quickly with higher order absorbing boundary condition. However when computed with second kind of grid, no instability phenomenon was observed. Thus the method of eliminate instability caused by MTF is effective. In Fig. 5 reference solution is compared the numerical solution gotten with different order MTF and second kind of grid. The reference solution is obtained by solving the problem in a longer domain, thus during the computation the solution in domain are not affected by the wave reflected back from absorbing boundary. The picture on top of each subgraph of Fig 5 is the displacement in domain $(x,y)\in[0,2]\otimes[0,2]$。 It is clearly showed by Fig. 5, High order MTF can absorb the outward propagating wave with high accuracy.



(a): The first calculation scheme



(b): The second calculation scheme

Figure 4. Displacement time history of observation point(1,1)

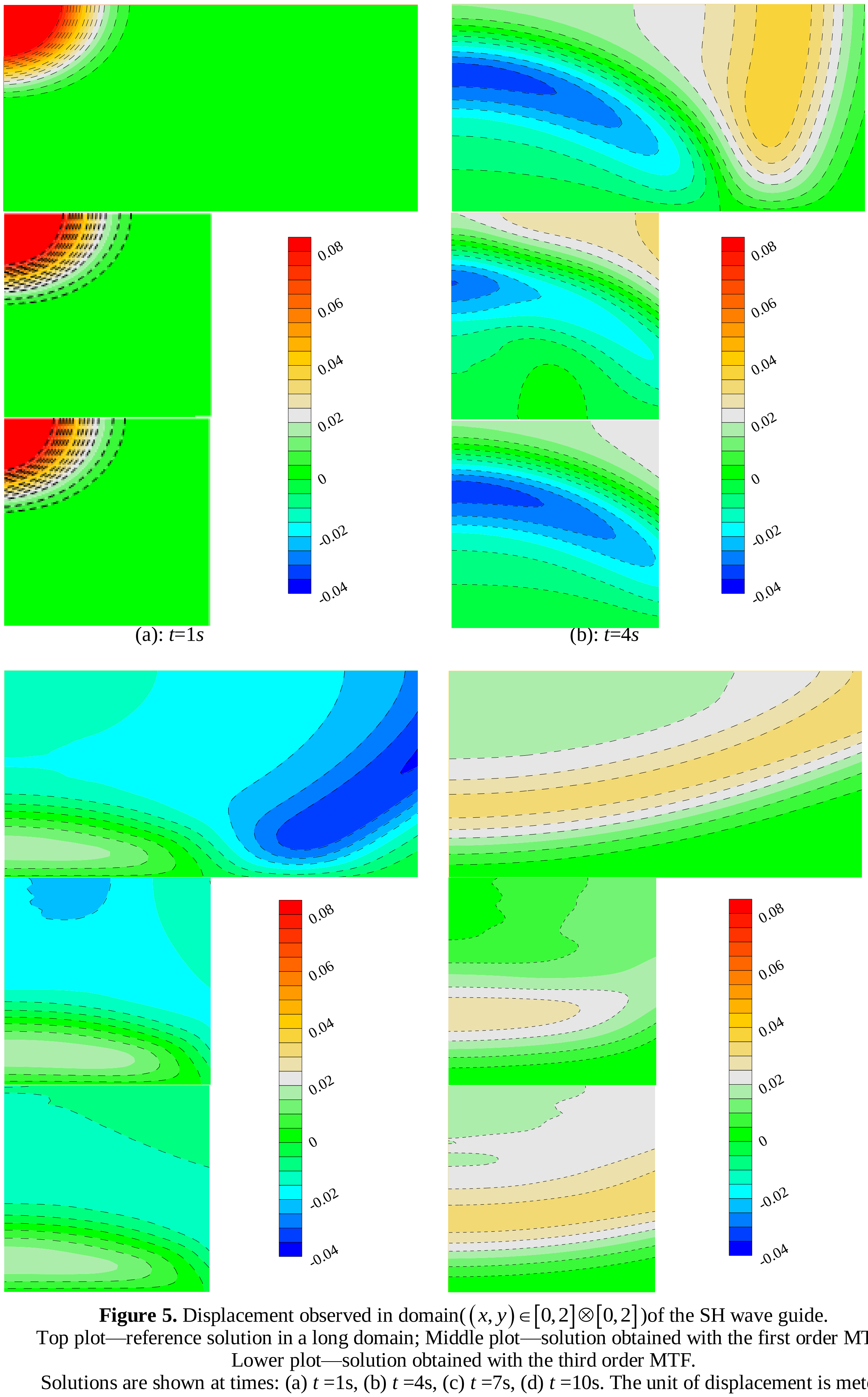


Figure 5. Displacement observed in domain($(x, y) \in [0,2] \otimes [0,2]$)of the SH wave guide.
Top plot—reference solution in a long domain; Middle plot—solution obtained with the first order MTF;
Lower plot—solution obtained with the third order MTF.
Solutions are shown at times: (a) t =1s, (b) t =4s, (c) t =7s, (d) t =10s. The unit of displacement is meter.

5. CONCLUSION REMARKS

In this paper, we discuss one important type of High frequency instability problem caused by transmitting boundary condition with SH model. According to local property of instability, we note that instability is caused by improper match of scheme on interior node and nodes on absorbing boundary. Through analyzing the dispersion curve, we show the mechanism of instability and method of eliminating instability is proposed by changed matching relationship between scheme of interior node and nodes on absorbing boundary. Differing from previously posed method, we justify the method in term of both numerical verification and analytical proof. Numerical tests show that high order MTF when implemented stably can absorb outward propagating wave with high accuracy. In future, further research on extending this method to eliminate the instability caused by MTF in more complex model is worthy to carry out.

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