

TIME-DELAY IN SPRING-BLOCK MODEL FOR APERIODICITY IN EARTHQUAKES

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SUMMARY:

We introduce the effect of time delay in one-dimensional spring-block model with Dieterich-Ruina's rate and state dependent friction law. Our model represents a generalization of Burridge-Knopoff model, which features are sufficiently realistic to relate its dynamic behavior to stiffness of the rock, velocity dependence of friction force and speed of tectonic plates. Standard bifurcation analysis of delay-differential equations is performed, including determination of bifurcation curves. We found that the system undergoes a Hopf bifurcation to periodic orbit only due to effect of time-delay. Increasing values of time-delay, with constant values of all other parameters, introduce chaotic behaviour of the model, which is achieved through quasiperiodicity, recognized as the broadband noise in the power spectrum. We show that even for a single-block model, with time-delay, a transition to chaos can be observed. So, this time-delay could be a potential source for the observation of aperiodicity in earthquake dynamics.

Keywords: spring-block model, time delay in friction, bifurcation curves, quasiperiodicity, earthquake analogy

1. INTRODUCTION

Although Reid (1910) originally suggested that there is a periodic recurrence of large earthquakes, this idea lacks empirical evidence (Scholz, 2002). Today, it is commonly considered that earthquakes are not periodic phenomena, and their size distribution obeys the power law, the Gutenberg–Richter law. It is known that the Gutenberg–Richter law is reproduced by spring-block model between two plates, one of which is driven with a constant velocity. In this paper we use the Burridge-Knopoff model (Burridge and Knopoff, 1967), which is today recognized as a common model for earthquake nucleation mechanism. It consists of one block of a certain rock type, connected through harmonic spring to a moving plate and driven along the rough surface, which causes the whole system to move in a stick-slip fashion (Fig. 1). Stick-slip as a possible source of earthquake process was firstly proposed by Brace and Byerlee (1966). According to their research, the earthquake is the "slip", and the "stick" is the interseismic period of the elastic strain accumulation. Later on, it was suggested that many aspects of earthquake phenomena result from the nature of the friction between the moving block and the rough surface, which was already proposed by seismologists, as a natural way of interaction at the transform or transcurrent fault (Dieterich, 1992; Marone, 1998). Subsequently, some specific constitutive laws for rock friction has been developed based on laboratory studies. These laws have been successfully used to explain various aspects of stable and unstable sliding between elastic solids as observed in the laboratory (Ruina, 1983; Rice and Ruina, 1983; Tullis and Weeks, 1986). Also, they have been used to model earthquake phenomena, including nucleation, ductile and brittle crustal slip regions, spatio-temporal slip complexities, and aftershocks (Tse and Rice, 1986; Dieterich, 1992; Boatwright and Cocco, 1996). In this paper we use Dieterich-Ruina rate- and state dependent friction law (Dieterich, 1997; 1981), which was developed to incorporate experimental observations. This friction law is of the following form:

$$\mu = \mu_0 - A \ln\left(\frac{v_0}{v} + 1\right) + B \ln\left(\frac{\theta}{b} + 1\right) \quad (1.1)$$

$$\frac{d\theta}{dt} = 1 - \frac{v\theta}{D_c} \quad (1.2)$$

where μ is the shear stress and μ_0 is the steady-state friction at $v=v_0$. In the bracketed friction term, v is slip velocity, v_0 a reference velocity, A and B are material properties, D_c is the critical slip distance, θ is the state variable and b is constant.

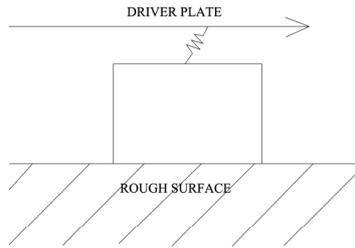


Figure 1. The Burridge-Knopoff block and spring model, represented by a slider coupled through a spring to a loader plate.

The Dieterich-Ruina friction law (1.1) includes a state variable θ , which is, in effect, an evolving time scale as indicated by its dimensions. This dynamic time scale represents the delayed reaction of the friction to instantaneous changes in velocity. The “state” of the friction is merely a dynamically evolving scale that in turn may influence the systems dynamics. It is this parameter that describes the memory effect in the model, or the so-called “aging” (Scholz, 2002; Szkutnik, Kawecka-Magiera and Kulakowski, 2003). In our paper we model this effect by including the time-lag when observing the change of the state variable, which was not explicitly considered in previous papers.

Another reason of modelling the earthquake nucleation mechanism by using the delay differential equations lies in their resemblance with the excitable systems, such as neurons. It is known that a neuron represents an excitable medium, which has the capacity to propagate an electric impulse, and which cannot support the passing of another impulse until a certain amount of time has passed (the refractory time). It is this refractory stage of the neuronal activity, that is similar to the stick phase in Burridge-Knopoff model. Appreciating the apparent, phenomenological analogy between the models from theoretical neurology and Burridge-Knopoff model, the time-delay effect is involved in equations of motion of the system under study. The basis of our model is represented by the system of equations proposed by R. Madariaga, already used in the work of Erickson, Birnir and Lavallee (2008), where we introduce the time-lag in the change of the state variable. The results were obtained through standard bifurcation analysis and confirmed by using the software package DDE-BIFTOOL, due to extreme stiffness in the numerical sense. DDE-BIFTOOL represents a collection of Matlab routines for numerical bifurcation analysis of systems of delay differential equations with several constant and state-dependent delays (Engelborghs, Lamire, Belair and Roose, 2001; Luzyanina, et al., 2001; Pieroux, et al., 2001).

2. THE EARTHQUAKE MODEL

Our numerical simulations of a spring-block model are based on the system of equations proposed by Madariaga (Erickson, Birnir and Lavallee, 2008). These equations of motion coupled with Dieterich-Ruina rate and state dependent friction law are originally given by:

$$\begin{aligned}
\dot{\theta} &= -(v/L)(\theta + B \log(v/v_0)) \\
\dot{u} &= v - v_0 \\
\dot{v} &= (-1/M)(ku + \theta + A \log(v/v_0))
\end{aligned} \tag{2}$$

where the parameter M is the mass of the spring block and u represents displacement of the block, while the other parameters are the same as in the equation (1.1). In the context of seismology, the spring-block model illustrated in Figure 1 can be understood as a representation for one-dimensional earthquake motion. In this context, the spring stiffness k corresponds to the linear elastic properties of the rock mass surrounding the fault (Scholz, 2002). According to Dieterich and Kilgore (1994), the parameter L corresponds to the critical sliding distance necessary to replace the population of asperity contacts. The parameters A and B are empirical constants, which depend on material properties. For convenience, system (2) is non-dimensionalized by defining the new variables θ' , v' , u' , v' and t' in the following way: $\theta = A\theta'$, $v = v_0 v'$, $u = Lu'$, $t = (L/v_0)t'$. This non-dimensionalization puts the system into the following form:

$$\begin{aligned}
\dot{\theta} &= -v(\theta(1 + \varepsilon)\log(v)) \\
\dot{u} &= v - 1 \\
\dot{v} &= -\gamma^2 [u + (1/\xi)(\theta + \log(v))]
\end{aligned} \tag{3}$$

where $\varepsilon = (B-A)/A$ measures the sensitivity of the velocity relaxation, $\xi = (kL)/A$ is the nondimensional spring constant, and $\gamma = (k/M)^{1/2}(L/v_0)$ is the nondimensional frequency (Erickson, Bimir and Lavalée, 2008).

The system has only one stationary solution, namely $(\theta, u, v) = (0, 0, 1)$, which corresponds to steady sliding. We will now expand our study by adding the time delay term τ in the first equation of the system (2) which describes the nature of temporal change of the state variable. More precisely, we introduce the delay effect concerning the retardation time between the movement of the upper plate and the block, due to complex friction force between the block and the rough lower plate. In this way, we obtain the following system of delay differential equations:

$$\begin{aligned}
\dot{\theta} &= -v(\theta(1 + \varepsilon)\log(v - \tau)) \\
\dot{u} &= v - 1 \\
\dot{v} &= -\gamma^2 [u + (1/\xi)(\theta + \log(v))]
\end{aligned} \tag{4}$$

We shall proceed in the standard way to determine and analyze the characteristic equation of (4) around a stationary solution $(0, 0, 1)$.

3. LOCAL STABILITY AND BIFURCATIONS OF THE STATIONARY SOLUTION

Linearization of the system (4) and substitution $\theta = Ae^{\lambda t}$, $u = Be^{\lambda t}$, $v = Ce^{\lambda t}$ and $v(t-\tau) = Ce^{\lambda(t-\tau)}$ results in a system of algebraic equations for the constants A , B and C . This system has a nontrivial solution if the following is satisfied:

$$-\lambda^3 - \lambda^2 \left(\frac{\gamma^2}{\xi} + 1 \right) - \lambda \gamma^2 \left(\frac{1}{\xi} + 1 \right) - \gamma^2 + \lambda(1 + \varepsilon) \frac{\gamma^2}{\xi} e^{-\lambda\tau} = 0 \quad (5)$$

The equation (5) is the characteristic equation of the system (4). Infinite dimensionality of the system (4) is reflected in the transcendental character of (5). However, the spectrum of the linearization of the equations (4) is discrete and can be divided into infinite dimensional hyperbolic and finite dimensional non-hyperbolic parts (Hale and Lunel, 1993; Belair and Campbell, 1994). As in the finite dimensional case, the stability of the stationary solution is typically, i.e. in the hyperbolic case, determined by the signs of the real parts of the roots of (5). Exceptional roots, equal to zero or with zero real part, correspond to the finite dimensional center manifold where the qualitative features of the dynamics, such as local stability, depend on the nonlinear terms.

Further, we substitute $\lambda = i\omega$ in equation (5) to obtain:

$$\frac{i\omega^3 + \omega^2 \left(\frac{\gamma^2}{\xi} + 1 \right) - i\omega\gamma^2 \left(\frac{1 + \xi}{\xi} \right) - \gamma^2}{i\omega(1 + \varepsilon) \frac{\gamma^2}{\xi}} = -(\cos \omega\tau - i \sin \omega\tau) \quad (6)$$

The resulting two equations for the real and imaginary part of (6) after squaring and adding give an equation for each of the parameters, ε and ξ in terms of the other parameters, ω and γ , and after division for τ in terms of the parameters ω , γ and ξ . In this way, one obtains parametric representations of the relations between τ and the other parameters, which correspond to the bifurcation values $\lambda = i\omega$. The general form of such relations is illustrated by the following formulas: for ε as a function of ω :

$$\varepsilon = -1 + \sqrt{C} \quad (7)$$

where:

$$C = \frac{\left[\omega^3 - \omega\gamma^2 \left(\frac{1 + \xi}{\xi} \right) \right]^2 + \left[\omega^2 \left(\frac{\gamma^2}{\xi} + 1 \right) - \gamma^2 \right]^2}{\left(\omega \frac{\gamma^2}{\xi} \right)^2} \quad (8)$$

On the other hand, for ξ as a function of ω :

$$\xi = \frac{[\omega\gamma^2 \sin(\omega\tau) + \omega^2\gamma^2 \cos(\omega\tau)]}{(\omega^3 - \omega\gamma^2)\sin(\omega\tau) + (\gamma^2 - \omega^2)\cos(\omega\tau)} \quad (9)$$

For τ as a function of ω :

$$\tau = \tau_c = \frac{1}{\omega} \left[\operatorname{arctg} \left(\frac{-\omega^2 \left(\frac{\gamma^2}{\xi} + 1 \right)}{-\omega^3 + \omega \gamma^2 \left(\frac{1 + \xi}{\xi} \right)} \right) + 2k\pi \right] \quad (10)$$

where k is any nonnegative integer such that $\tau_k \geq 0$.

The previous parametric equations for ε , ξ and τ give the Hopf bifurcation curves that are illustrated in Fig. 2 and 3. Fig.2 illustrates bifurcation curves $\tau(\varepsilon)$ for the fixed values of the parameters ξ (0,5) and γ (0.8). Diagram in fig.3 represents bifurcation curves $\xi(\varepsilon)$ for the fixed values of τ (12) and γ (0.8). We adopted the value of $\gamma=0,8$, because for commonly used values of $\gamma = 10^4-10^{12}$ according to Madariaga (Erickson, Birnir and Lavallee, 2008) the system under study becomes very stiff in numerical sense. Another reason for taking the constant value of γ is that, in this case, the trajectory for a Hopf bifurcation depends only on ε and ξ . In that way, we can observe a transition to chaos by simply increasing the value of those two parameters.

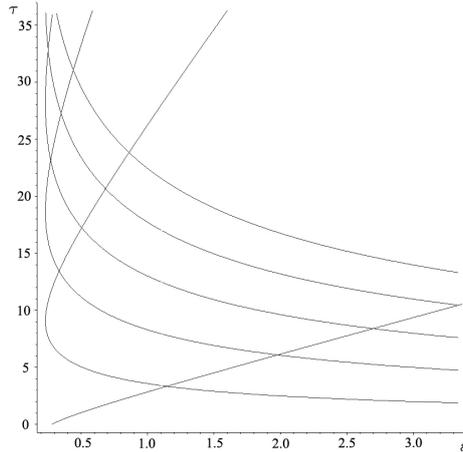


Figure 2. Hopf bifurcation curves $\tau(\varepsilon)$, for the fixed values of parameters $\xi = 0.5$, and $\gamma = 0.8$.

As we can see, as the time delay τ is varied, the fixed point undergoes a supercritical Hopf bifurcation, i.e. it is changed from a stable into an unstable fixed point, and a limit cycle is born. The critical value of the time delay τ_c , when the bifurcation from the stable into unstable fixed point occurs, is given by the relation (10).

The effect of the introduced time-delay is of specific importance, because, as it can be seen in figure 2, only by increasing the time-lag, the system under study exhibits chaotic behavior. In other words, our system exhibits quasiperiodic (Ruelle-Takens-Newhouse) route to chaos (Ruelle and Takens, 1971; Newhouse, Ruelle and Takens, 1978). So, by means of two supercritical Hopf bifurcations, regular motion become highly unstable and replaced by a strange attractor.

The derivatives of the solutions of the characteristic equation (5) with respect to τ at the bifurcation values in the fig. 2 are not always positive, so that for $\tau \in (\tau_c, \tau_{c+1})$, by crossing the bifurcation curve, there are two new unstable directions created or two already existing unstable directions lost in a sufficiently small

neighbourhood of $(0,0,1)$. Also, it is possible that, by crossing the bifurcation curve, only enlarging the time-delay, the dynamic state of the system changes from oscillatory behavior to stable equilibrium state, which is known as time-delay induced amplitude death. This phenomenon has been the subject of numerous research papers (Yamaguchi and Shimizu, 1984; Aronson, Ermentrout and Kopell, 1990).

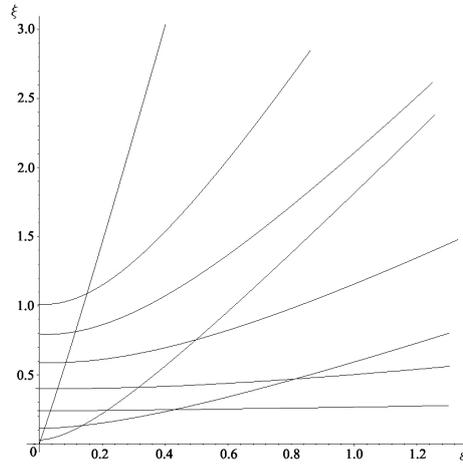


Figure 3. Hopf bifurcation curves $\zeta(\epsilon)$, for the fixed values of parameters $\tau = 12$, and $\gamma = 0.8$.

Fig. 4 and 5 represent temporal evolution of variable v for the fixed values of ϵ , ξ , γ and τ .

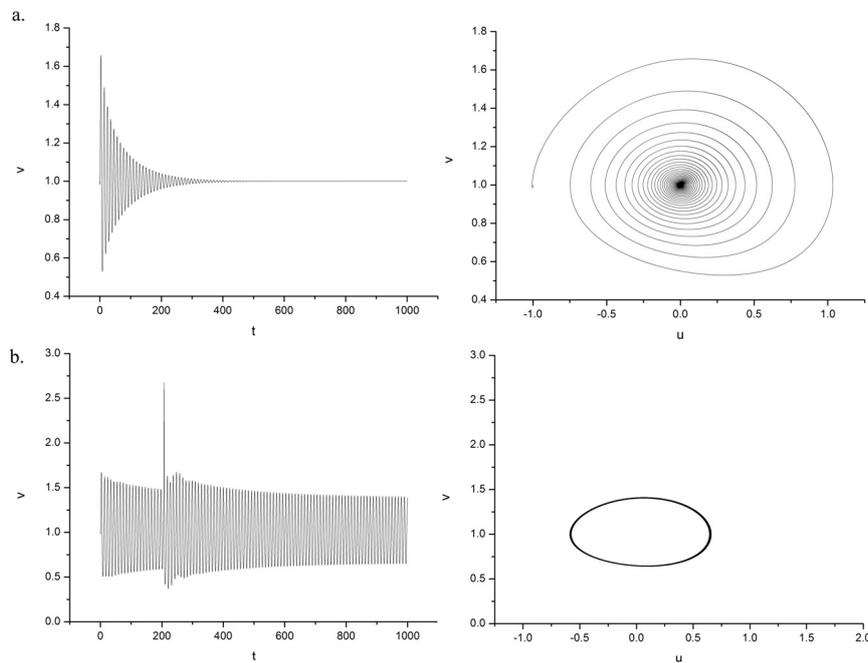


Figure 4. Temporal evolution of variable v and appropriate phase portrait for a. $\tau=0$, $\epsilon = 0.3$, $\xi = 0.8$ and $\gamma = 0.8$ (fixed point); b. $\tau=10$, $\epsilon = 0.35$, $\xi = 0.8$ and $\gamma = 0.8$ (oscillations).

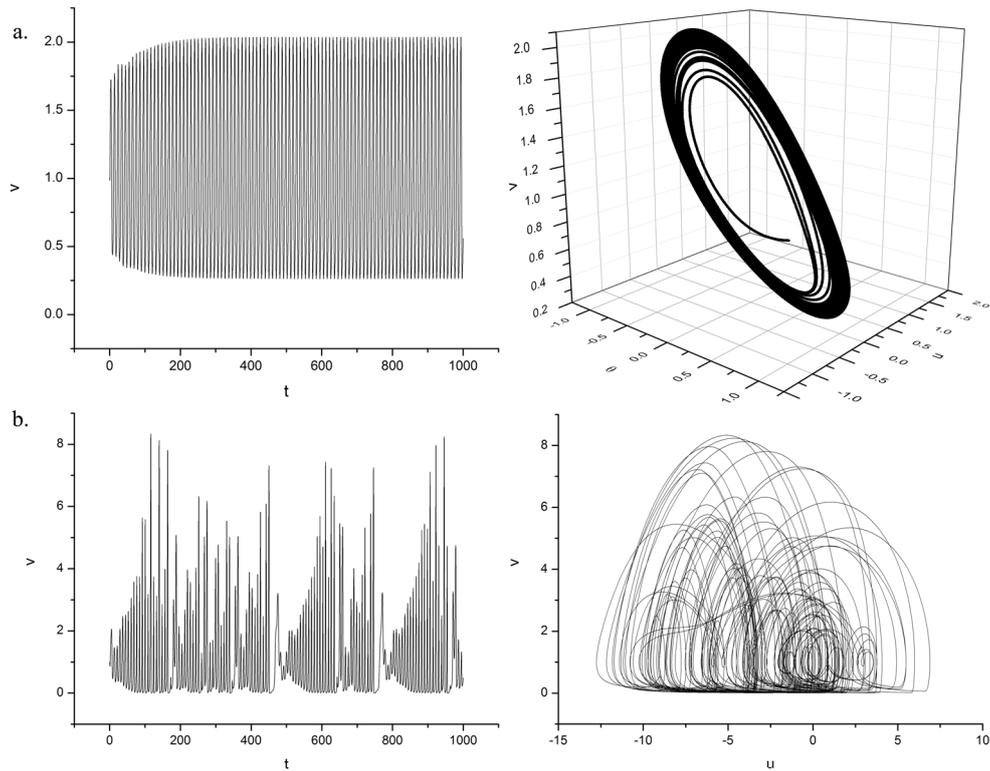


Figure 5. a. Temporal evolution of variable v and appropriate phase portrait for a. $\tau=10$, $\varepsilon = 0.5$, $\xi = 0.8$ and $\gamma = 0.8$ (torus); b. $\tau=7$, $\varepsilon = 0.7$, $\xi = 0.5$ and $\gamma = 0.8$ (chaos).

As it could be seen, only by increasing the time-lag τ (0, 7 and 10) and with slightly changes of the other parameter values, dynamical state of block changes from fixed point, across the oscillations (first Hopf bifurcation) and torus (second Hopf bifurcation) to chaos.

The presence of chaos was confirmed by calculation the Fourier power spectrum for oscillations, torus and chaotic orbits, shown in fig. 6 and 7. The single peak in power in fig.6a indicates the oscillatory behavior of the system under study, while the second peak in fig.6b indicates a presence of torus. The broadband noise in fig. 7 indicates that the attractor is strange.

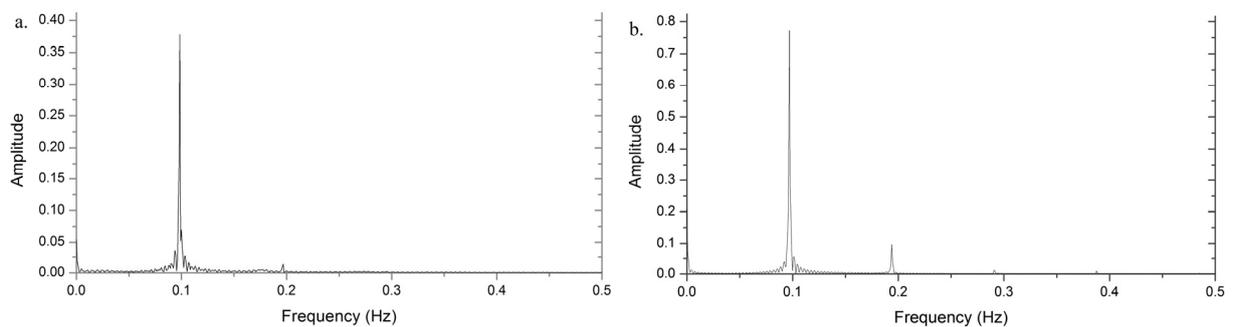


Figure 6.a. Single peak in power indicates the oscillatory behavior of the model. **b.** Two peaks in power indicate the appearance of torus (second Hopf bifurcation).

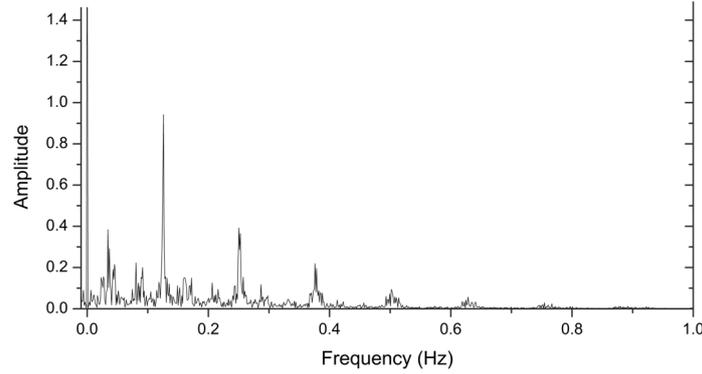


Figure 7. The broadband noise in the Fourier power spectrum indicate the chaotic behavior of the system.

4. SEISMOLOGICAL INTERPRETATION

Concerning the fact that each of the observed parameters represents one feature of the model, it is possible to give a qualitative interpretation of the obtained results. In our analysis, parameter γ is held constant. So, the trajectory for a Hopf bifurcation depends on ε and ζ . In other words, a transition to chaos is observed by simply increasing the value of the parameters A and B , independent of the values of parameters k and L . In an earthquake analogy to the single block, illustrated in Fig. 1, this suggests that potential transitions to chaos are essentially controlled by the ratio of the stress parameters $(B-A)$ and A (fig. 8) and independent of the elastic property of the medium surrounding the fault (idealized by the parameter k) and the critical length L of the friction law.

As it can be seen in fig. 2 and 3, bifurcation diagrams are given in function of the parameter ε , which represents the sensitivity of the observed system to the applied force: $\varepsilon = (B-A)/A$. In other words, the higher the value of parameter ε , the smaller the frictional force on the contact between the block and the rough surface. In the context of seismology, this kind of behavior could appear along the pre-existing faults, with a certain amount of gouge on the contact between two tectonic plates, or blocks of rock mass.

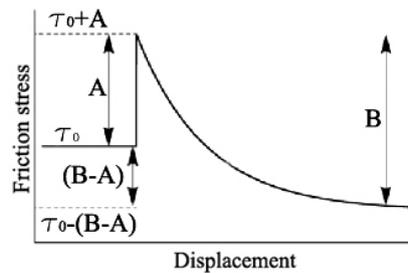


Figure 8. A schematic illustration of the response to a step change in the imposed velocity v of the spring-block model illustrated in figure 1. The imposed velocity, initially maintained constant at v_0 , is suddenly incremented by Δv and subsequently held constant at $v_0 + \Delta v$ (Erickson, Birnir and Lavallee, 2008).

In this paper we showed that a transition to chaos can be observed for a single block with a nonlinear friction law, including the time-lag between the motion of the driving plate and the block. The analogy with earthquake motion suggests that the friction law can be a potential source for the observation of aperiodicity in earthquake dynamics. If we assume that the friction law is the main physical process regulating the frequency of earthquakes, then the presence of a strange attractor suggests that earthquakes

are typically aperiodic. Thus aperiodic orbits on the strange attractor may exhibit dynamics analogous to the dynamics during an earthquake. Furthermore, it is important to note that aperiodic behavior observed in the system under study may be partially responsible for irregular ground motion during the earthquakes.

5. CONCLUDING REMARKS

We have analyzed the properties of a system of equations, describing the Burridge-Knopoff model coupled with Dieterich-Ruina friction law, with the time-delay introduced in the state variable. The time lag describes the memory effect that commonly appears during the stick-slip motion of the block along the rough surface. In our paper, standard bifurcation analysis is used to indicate the presence of the bifurcation from the fixed point to the periodic limit cycle and from the periodic limit cycle into the quasi-periodic oscillations in the model. As the amplitude of the perturbation parameters is increased the torus with the quasi-periodic orbits breaks up into a complicated invariant set with chaotic orbits.

We have obtained exact parametric representation of the Hopf bifurcation curves in the general case of the introduced time-lag. These bifurcation relations provide, for a given value of the parameters, the time-lag that will lead to destabilization of the stationary solution. The corresponding Hopf bifurcations are of the sub-or super-critical type.

On the other hand, even though we examined the model with only one block, which could be considered as the simplest case of Burridge-Knopoff model, we believe that our results are important for several reasons. Firstly, they reveal some characteristics of the general physics that is behind the generation of the earthquakes in terms of a chaotic process. Secondly, they confirm the fact that the memory effect included in this model represents the natural feature of the real earthquakes, and that it should certainly be involved in the further research. Also, the qualitative resemblance of the stick phase in a spring-block model and a refractory phase in a neuronal activity, together with the similar chaotic behavior when modeled by the means of delay differential equations, assures us that a Burridge-Knopoff model could be considered as a relaxation oscillator under certain conditions and restraints. Finally, the strategy applied here can be replicated for other models, i.e. with greater number of blocks, in order to better understand the chaotic behavior of earthquakes in general.

The obtained results are encouraging, since they strongly support the opinion, besides the Gutenberg-Richter and Omori-Utsu scaling laws, that mechanism of earthquake nucleation could be represented by the spring-block model, which also exhibits chaotic behavior. We believe that this new detailed approach concerning the involvement of time-delays in the equations for motion for Burridge-Knopoff model that we used in this work will help in the future to improve our knowledge of the earthquake physics.

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