# Modal Lumped Parameter Models for Representing Frequency-Dependent Impedance Functions of Soil-Foundation Systems

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**SUMMARY:**
This study verifies the applicability of a newly developed transform method to soil-foundation systems under practical conditions. With this method, the impedance function of linearly elastic systems with non-classical damping can be transformed based on conventional complex modal analysis into an exact one-dimensional spring-dashpot system (1DSD) arranged in series. In this study, the horizontal and rocking impedance functions of a shallow foundation embedded in layered soil modeled by finite element models are transformed into the 1DSDs. Numerical results show that the dynamic response of a four-story structure with inelasticity supported by the 1DSDs is compatible with that supported by the finite element models. The results also show that a marked decrease in the computational domain size and time can be achieved by using the transform method.

*Keywords: soil-structure interaction, lumped parameter models, impedance functions*

## 1. INTRODUCTION

In general, impedance functions (IFs) show frequency-dependent characteristics, such as when the soil deposit has layered strata, or the shape and structure of the foundations are complicated. It is known that IFs show the following typical frequency-dependent characteristics: (a) slight oscillation shown in soil reaction and surface rigid foundations or embedded rigid foundations (e.g. Baranov, 1967; Beredugo & Novak, 1972; Novak, 1974; Novak et al., 1978; Veletsos and Dotson, 1988; Gazetas, 1991; Saitoh, 2004); (b) multiple oscillations typically exhibited in pile groups (e.g. Kaynia & Kausel, 1982, Makris & Gazetas, 1993; Mylonakis & Gazetas, 1998); and (c) cut-off frequency below which the damping is negligible and above which the damping increases rapidly (e.g. Novak & Nogami, 1977; Kausel & Roesset, 1975; Elsabee & Morray, 1977; Takemiya & Yamada, 1981). On the one hand, a number of constitutive models of materials and structural members have been proposed, allowing the inelastic behavior of structural systems during earthquakes to be estimated appropriately. Recently, various methods that are ready to use in practice have been proposed to consider the frequency dependent IFs into the inelastic structural analysis. One of the powerful tools is to use a lumped parameter model (LPM). An LPM consists of springs, dashpots, and masses having frequency-independent coefficients. A particular combination of these elements can simulate a frequency-dependent impedance characteristic. The advantage of LPMs is that they can be easily incorporated into a conventional numerical analysis in the time domain, even under nonlinear conditions of superstructures. From the viewpoint of construction schemes in LPMs, the existing LPMs can be categorized into three types: a) semi-empirical LPMs (e.g. Meek & Veletsos, 1974; Wolf & Somaini, 1986; de Barros & Luco, 1990; Jean et al., 1990; Wolf & Paronesso, 1992; Wolf, 1997; Wu & Chen, 2001; Wu & Chen, 2002; Saitoh, 2007; Taherzadeh, 2009; Khodabakhshi, 2011); b) systematic LPMs (e.g. Wolf, 1991a and 1991b; Wu & Lee, 2002; Wu & Lee, 2004; Zhao & Du, 2008); and c) modal LPMs (Saitoh, 2010 and 2012a).

In general, LPMs need to approximate the target IFs by using specific functions. This approximation procedure does not always achieve a satisfactorily good match with the target IFs. Recently, a new transform method, which is categorized as modal LPMs, for constructing an exact LPM from the original systems has been developed in the field of computational mechanics (Saitoh, 2010). In this method, the IF in general linearly elastic systems with non-classical damping is transformed on the basis of a conventional complex modal analysis into an exact one-dimensional spring-dashpot system (1DSD) comprising units arranged in series. Each unit, which is directly related to each vibrating mode of the original system, is a parallel system consisting of a spring, a dashpot, and a unit having a spring and a dashpot arranged in series. The properties of the elements comprising the 1DSDs are automatically determined through the proposed procedure by using complex modal quantities. Furthermore, a transform method for the IF in general linearly elastic systems with classical damping was also proposed by Saitoh, 2012a.

The advantage of 1DSDs is that the 1DSD transformation offers compatibility with the merit of complex modal analysis: a large number of units associated with high modes beyond a target frequency region can be removed from the 1DSDs as an approximate expression of IFs. Accordingly, a marked decrease in the computational domain size and time with the use of the 1DSDs can be achieved. The 1DSDs transform procedure provides an exact LPM at the initial step: we can adjust the number of degrees of freedom (the number of units) in the reduced LPM by taking a balance with the accuracy from the exact LPM.

The main aim of this study is to verify the applicability of the transform method of 1DSDs to soil-foundation systems under practical conditions. This study deals with an application example of a shallow foundation embedded in layered soil resting on rigid bedrock. An adjacent building and an underground structure such as a tunnel are considered in layered soil as practical conditions. The soil-foundation system is modelled using two-dimensional isoparametric finite elements.

## 2. SYSTEM STUDIED

The total system is shown in Fig. 1. A shallow foundation of width 10m, length 50m, and depth 2m is embedded in layered soil up to the middle height of the foundation. The elastic modulus of the foundation is assumed to be rigid, imposing unique displacements $u_f$ and $\theta_f$ at the centre of gravity in the horizontal and rotational directions, respectively. The mass and the mass moment of inertia of the foundation are $m_f = 1000\,\mathrm{t}$ and $J_f = 8500\,\mathrm{tm}^2$, respectively. A four-storey building supported by the foundation is represented by a four-degree-of-freedom system. In this study, the inelasticity in each story is taken into account. The soil-foundation system is modelled using conventional two-dimensional rectangular isoparametric elements, where each element has eight degrees of freedom. The soil strata consist of two soil layers resting on rigid bedrock. The bottom of the layered soil is fixed in the vertical and lateral directions, whereas viscous boundary proposed by Lysmer and Kuhlemeyer, 1969 is applied to the sidewalls of the soil as a fictitious boundary that dissipates energy toward infinite region of soil. The moduli of elasticity and the damping ratios of the soils shown in the figure are assumed to approximately account for appreciable levels of strain during ground shaking. As adjacent structures, a six-story building with the height of 16m and the bay of 10m; and a tunnel with the height of 5m and the width of 8m are modelled by using the isoparametric elements. The unit weight, the modulus of elasticity, and Poisson's ratio of the elements for the adjacent structures are $\rho_c = 2.5\,\mathrm{t/m}^3$, $E_c = 2.5\times10^7\,\mathrm{kN/m}^2$, and $\nu_c = 0.25$ respectively. The total numbers of nodes comprising the isoparametric elements for the total system are 1707, whereas the degrees of freedom subtracting the fixed degrees of freedom are 4074. The thickness of the elements is the same as the length of the foundation (50m) under the plane-strain condition.

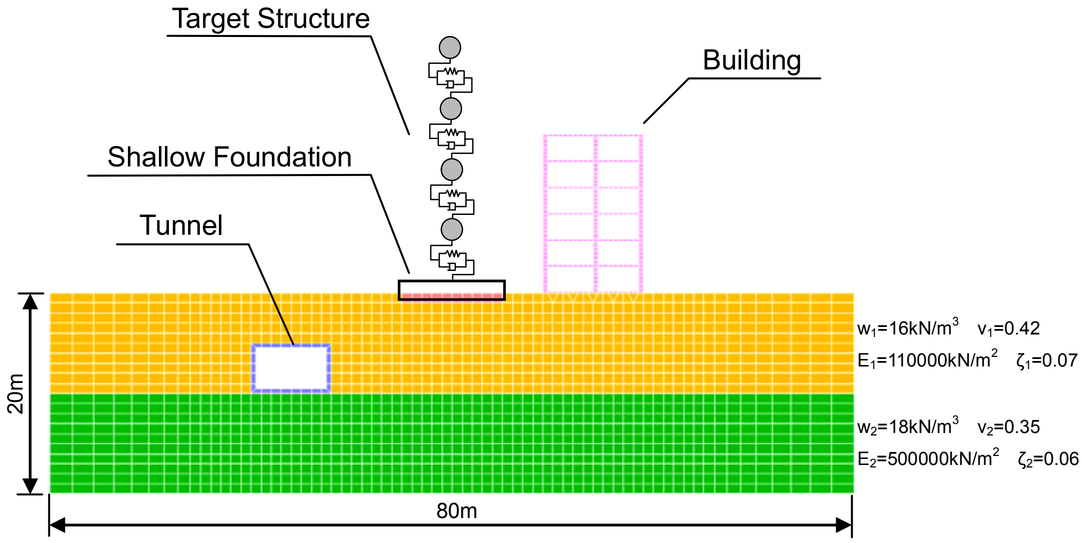


**Figure 1.** Two-dimensional finite element model for soil and shallow foundation system supporting a four-storey building. The unit weight, the modulus of elasticity, Poisson's ratio, and the damping ratio of the *i*-th soil layer are denoted as $w_i$, $E_i$, $\nu_i$, and $\zeta_i$, respectively.

In this model, the soil-foundation system consists of conventional isotropic elements, whereas the structural system comprising the superstructure and the mass of the foundation are discretized by springs, dashpots, and masses. Therefore, the global mass matrix, stiffness matrix, and damping matrix in the equations of motion of the total system are obtained by superimposing local matrices in both equilibrium equations.

## 3. TRANSFORMING ORIGINAL SOIL-FOUNDATION SYSTEM INTO 1DSD

### 3.1. Overview of Transform Method (Saitoh, 2010, 2012b)

In this method, the equations of motion of general linearly-elastic structural systems comprising $N$ DOFs are considered and are expressed by the following form:

$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+[K]\{u\}=\{p\} \tag{3.1}$$

where $[M]$, $[C]$, and $[K]$ are the mass matrix, damping matrix and stiffness matrix respectively, of the original structural systems. Each matrix has the order $N \times N$; $\{u\}$ and $\{p\}$ are the response displacements and the external forces at the nodes, respectively, and each vector has the order $N$. The dots denote partial derivatives with respect to time $t$. In this study, the damping matrix $[C]$ is assumed to be based on non-classical damping.

In complex modal analysis, the following $2N$ first-order equations are considered instead of $N$ second-order equations of Eqn. 3.1:

$$[R]\{\dot{z}\}+[S]\{z\}=\{f\} \tag{3.2}$$

where

$$[R]=\begin{bmatrix}[C] & [M]\\ [M] & [0]\end{bmatrix},\ [S]=\begin{bmatrix}[K] & [0]\\ [0] & -[M]\end{bmatrix},\ \{z\}=\begin{Bmatrix}\{u\}\\ \{\dot{u}\}\end{Bmatrix},\ \{f\}=\begin{Bmatrix}\{p\}\\ \{0\}\end{Bmatrix}.$$

According to the conventional complex modal procedure, the complex eigenvalues and eigenvectors can be obtained. Each complex eigenvalue $\lambda_n$ is known to have an eigenvalue $\overline{\lambda}_n$ that is the complex conjugate of $\lambda_n$; the corresponding vector $\{\phi_n\}$ has a vector $\{\overline{\phi}_n\}$ whose components are complex conjugates of those of $\{\phi_n\}$. The eigenvectors are assembled compactly into a matrix using diagonal matrices $[\Omega]$ and $[\overline{\Omega}]$ comprising the eigenvalues $\lambda_n$ and $\overline{\lambda}_n$, respectively, as

$$[\Psi] = \begin{bmatrix} [\phi] & [\overline{\phi}] \\ [\phi][\Omega] & [\overline{\phi}][\overline{\Omega}] \end{bmatrix} \tag{3.3}$$

where

$$[\phi] = \left[ \{\phi_1\} \quad \{\phi_2\} \quad \cdots \quad \{\phi_N\} \right] \tag{3.4}$$

$$[\overline{\phi}] = \left[ \{\overline{\phi}_1\} \quad \{\overline{\phi}_2\} \quad \cdots \quad \{\overline{\phi}_N\} \right] \tag{3.5}$$

$$[\Omega] = \left[ diag\, \lambda_n \right], \;\; n = 1, 2, \cdots, N \tag{3.6}$$

$$[\overline{\Omega}] = \left[ diag\, \overline{\lambda}_n \right], \;\; n = 1, 2, \cdots, N \tag{3.7}$$

The matrix $[\Psi]$ is called the modal matrix. In general, $[\Psi]^T [R][\Psi]$ becomes a diagonal matrix owing to the orthogonality relations. Here, the upper *N* components of the matrix are denoted as $\alpha_n$, whereas the lower *N* components are complex conjugates of $\alpha_n$, denoted as $\overline{\alpha}_n$.

At the end of the mathematical derivation in his study (Saitoh, 2010), the configuration of 1DSD was theoretically determined as shown in Fig. 2. Moreover, the properties of the elements comprising 1DSD can be determined using the following formula:

$$k_{Tn} = \frac{\sigma_n^2 + \omega_{dn}^2}{2\left(G_n \sigma_n - R_n \omega_{dn}\right)} \tag{3.8}$$

$$c_{Tn} = \frac{1}{2G_n} \tag{3.9}$$

$$k_n = \frac{-\left(G_n^2 + R_n^2\right)\omega_{dn}^2}{2G_n^2 \left(G_n \sigma_n - R_n \omega_{dn}\right)} \tag{3.10}$$

$$c_n = \frac{-\left(G_n^2 + R_n^2\right)\omega_{dn}^2}{2G_n \left(G_n \sigma_n - R_n \omega_{dn}\right)^2} \tag{3.11}$$

where $G_n + iR_n = \dfrac{\phi_{nI}\phi_{nJ}}{\alpha_n}$ and $G_n - iR_n = \dfrac{\overline{\phi}_{nI}\overline{\phi}_{nJ}}{\overline{\alpha}_n}$.

Here, $\phi_{nI}$ and $\phi_{nJ}$ are the components of the $n$-th eigenvector at the *I*-th and *J*-th DOFs, respectively; $\overline{\phi}_{nI}$ and $\overline{\phi}_{nJ}$ are the complex conjugates of the components $\phi_{nI}$ and $\phi_{nJ}$, respectively. $\sigma_n$ is the $n$-th modal decay rate and $\omega_{dn}$ is the $n$-th damped natural circular frequency defined as

$$\lambda_n = -\sigma_n + i\omega_{dn} \tag{3.12}$$

$$\bar{\lambda}_n = -\sigma_n - i\omega_{dn} \tag{3.13}$$

Practically, over-damped modes often appear. In this case, eigenvalues $\lambda_n$ are real and negative. In Saitoh, 2010, it was mathematically derived that the impedance function associated with over-damped modes is expressed as a Kelvin-Voigt unit comprising the following spring $k_{Tn}$ and dashpot $c_{Tn}$ shown in Fig. 2.

$$k_{Tn} = \frac{\sigma_n}{G_n} \tag{3.14}$$

$$c_{Tn} = \frac{1}{G_n} \tag{3.15}$$

Note that over-damped modes generally appear with even numbers $2m$ in $2N$ modes, so the total unit number $N$ changes to $N'(= N + m)$ when over-damped modes exist.

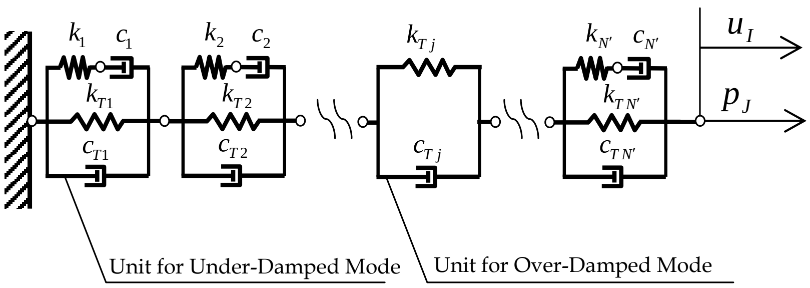


**Figure 2.** (a) One-dimensional lumped parameter model with spring and dashpot elements (1DSDs) for simulating the impedance function $S_{IJ}(\omega) = p_J / u_I$ in general structural systems. (b) Unit associated with under-damped mode and (c) unit associated with over-damped mode, proposed by Saitoh, 2010.

### 3.2. 1DSD Transformation of FEmodel

According to the procedure shown above, the FEmodel is to be transformed into an equivalent 1DSD hereinafter. First of all, complex modal analysis is performed to obtain the fundamental quantities by which the properties of the elements in the 1DSD are determined. As described above, a great advantage of the 1DSDs is that the units comprising the 1DSDs are associated with the vibration modes of the original structural system. Therefore, a small set of units associated with modes from the lowest order can appropriately express the dynamic characteristics of structural systems without using all the units. In recent study, Saitoh, 2012c studied the influence of frequency dependency in pile-group impedance functions upon elastic and inelastic responses of superstructures. The results indicate that the important frequency range is the dominant frequency of foundation input motions that excite the inertial structural systems. Fig. 3 shows the time-history response acceleration at the ground surface calculated by using conventional one-dimensional wave propagation theory with the soil properties shown in Fig. 3. An observed earthquake record, 1940 El Centro NS is applied to the bottom soil layer. This response acceleration is to be used in the following calculations as the foundation input motion in this study, which indicates that no adjustment for the kinematic interaction effects is conducted for simplicity. The figure shows that the foundation input motion contains a wide range of frequency components. The amplitude of the acceleration ranges from 0Hz to 10Hz. So, this frequency range is considered to be the target frequency range in this study.

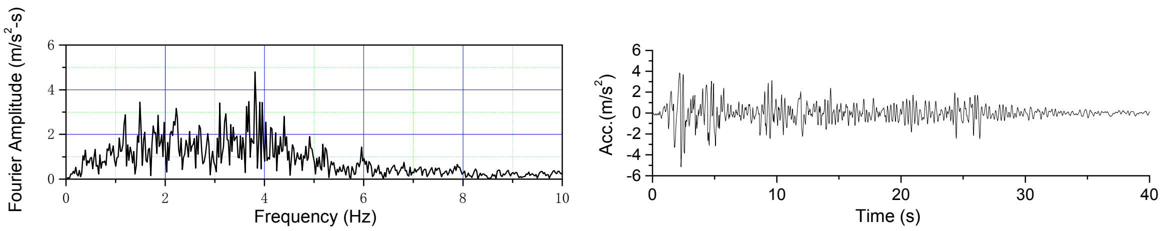


**Figure 3.** Fourier amplitude and time-history of the input motion applied to structural system.

Fig. 4 shows a comparison of the impedance functions obtained from 1DSDs with those obtained with the FEmodel. The results indicate that the impedance functions obtained from the 1DSDs agree closely with those obtained with the FEmodel within the target frequency range. In 1DSD transformation, many units contain relatively much larger spring constant than others. These units can appreciably be removed so that the impedance functions of the reduced 1DSD are in sufficient agreement with those of the original system. In this study, the reduced 1DSD for the horizontal impedance functions consists of 411 units (148 under-damped modes and 263 over-damped modes), whereas the reduced 1DSD for the rotational impedance functions consists of 83 units (69 under-damped modes and 14 over-damped modes). The 1DSDs in the horizontal and rotational directions contain the residual stiffness units representing the stiffness effect above 40Hz and 30Hz, respectively (c.f. Saitoh, 2012c).

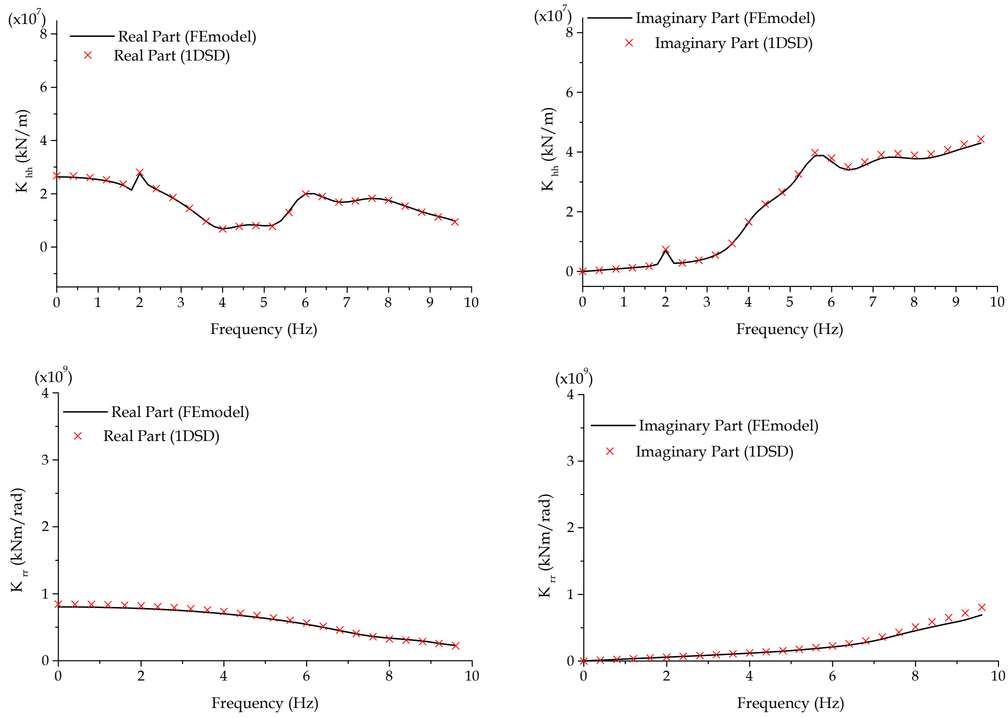


**Figure 4.** Impedance functions of soil and shallow foundation system using reduced 1DSDs [ $K_{hh}$ in horizontal direction and $K_{rr}$ in rotational direction]. Results obtained from the original FEmodel are shown for comparison.

## 4. PERFORMANCE VERIFICATION OF 1DSD

### 4.1. Dynamic Response of the Structural System in Frequency Domain

In this section, the dynamic response of the structural system computed by using the reduced 1DSDs in the frequency domain is verified by comparing it with the dynamic response obtained with the original FEmodel. The total structural system using the 1DSDs in both horizontal and rotational directions is shown in Fig. 5. The properties of the superstructure are summarized in Table.1. The reduced 1DSDs obtained above are connected respectively with each degree of freedom in the foundation as shown in the figure. The equations of motion of the structural system can easily be constructed with conventional spring-dashpot matrices expressing the reduced 1DSDs (details are described in Saitoh, 2012a). The resultant equilibrium equations of the total system can be formulated as:

$$[M_T]\{\ddot{u}\}+[C_T]\{\dot{u}\}+[K_T]\{u\}=\{0\} \tag{4.1}$$

where

$$\{u\}=\left[u_s\ u_f\ \ u_1\ u_2\cdots u_i\cdots u_m\ \theta_f\ \theta_1\ \theta_2\cdots\theta_i\cdots\theta_n\right]^T \tag{4.2}$$

where the mass matrix $[M_T]$, the damping matrix $[C_T]$, and the stiffness matrix $[K_T]$ are the resultant matrices formed by superimposing the partial matrices. $u_i$ and $\theta_i$ are the displacements at the degrees of freedom in the reduced 1DSDs. $m$ (=559) and $n$ (=152) are the maximum degrees of freedom in both directions, respectively.

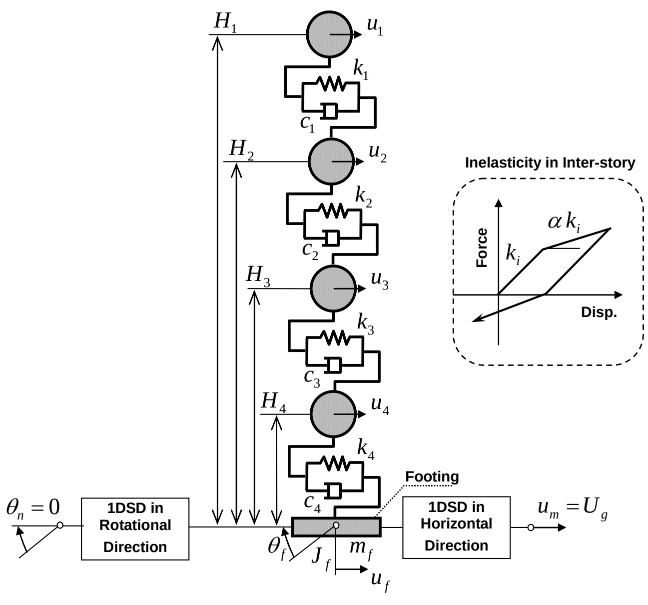


**Figure 5.** Mathematical models for non-linear response history analysis (Saitoh, 2012b)

**Table 1.** Properties of Four-Storey Building

| Story No. | Units | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Mass $m_i$ | t | 750 | 750 | 750 | 750 |
| Stiffness $k_i$ | kN/m | 2000000 | 2000000 | 2000000 | 2000000 |
| Height $H_i$ | m | 12 | 9 | 6 | 3 |
| Yield Strength $p_i$ | kN | 5000 | 10000 | 11000 | 14000 |

Fig. 6 shows the real part and imaginary part of the transfer functions (TFs) of the structural systems (the inelasticity in the superstructure is not considered here). Here, the TF of the absolute acceleration at the top of the superstructure with respect to the foundation input motion is defined as $T_{sa}$. Furthermore, the TFs of the footing are also computed. They are defined as $T_{ha}$ and $T_{ra}$ for the absolute acceleration associated with the horizontal and rotational motions, respectively. The figure shows that the TFs obtained with the reduced 1DSDs are in good agreement with those of the original FEmodel. This implies that the 1DSDs comprising almost 17% of the degrees of freedom in the original system can properly represent the IFs and the TFs in the target frequency region.

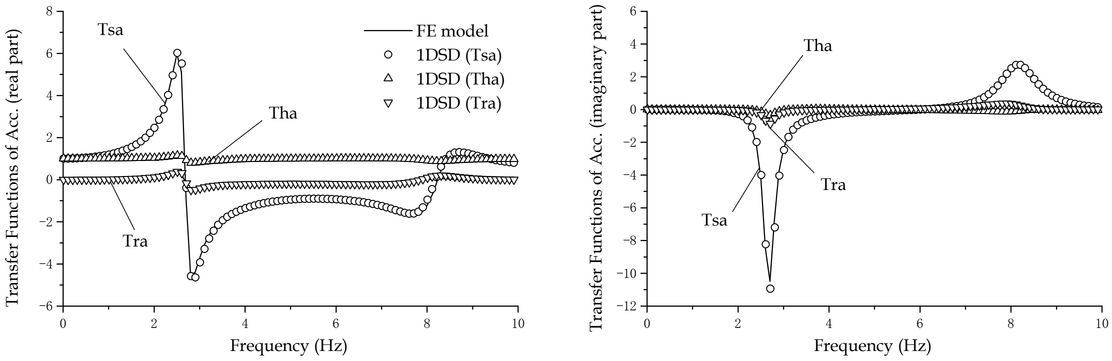


**Figure 6.** Comparisons of the transfer functions of the structural system computed by using 1DSDs and original FEmodel [real part and imaginary part].

### 4.2. Dynamic Response of the Structural System with Inelasticity in Time Domain

In this part, it is attempted to compute the time-history response of the structural system with the inelasticity in the superstructure employing the 1DSDs when subjected to the foundation input motion shown in Fig. 3 and to be compared with that with the original FEmodel.

Time-history analysis is performed by using Newmark's $\beta$ method ($\beta = 1/4$) as a numerical integration scheme, where the time interval $\Delta t$ is $0.001$ s. The inelasticity of the superstructure is assumed to be the Clough model (Clough & Johnson, 1966), which is generally used to model reinforced concrete members. The spring of the superstructure has a bi-linear skeleton curve where the ratio of the tangent stiffness to the initial stiffness is assumed to be 0.1, as shown in Fig. 5. The yield strength $p_i$ in each story is presented in Table.1. In this study, the modified Newton–Raphson method is applied to calculate the nonlinear response of the system.

Fig. 7 shows shear force and inter-storey drift relation in each story. The results indicate that the inelastic responses obtained with the 1DSDs show sufficiently close agreement with those obtained from the original FEmodel. According to a rough measurement, when using the authors' PC (CPU 3.40GHz, RAM 4.00GB), the inelastic responses shown above were obtained in about 36000sec with the original system, while those with the 1DSDs were obtained in about 300sec. Therefore, the 1DSD transform method can be a new option for efficient computation in SSI problems.

## 5. CONCLUSIONS

This study verifies the applicability of the transform method using a so called "one-dimensional spring-dashpot system (1DSD)" to soil-foundation systems under practical conditions. This study deals with an application example of a multi-story building supported by a shallow foundation embedded in layered soil resting on rigid bedrock. An adjacent building and an underground structure such as a tunnel are considered in layered soil as practical conditions. The impedance functions obtained with the 1DSDs properly simulates those with the original FEmodel. The transfer functions of the structural systems in the frequency domain using the 1DSD show fairly good agreement with those obtained with the FEmodel. The time-history responses of structures with the inelasticity in the superstructure are properly simulated by using the 1DSDs. The results indicate that the 1DSDs markedly decreases the computational time taken for the results. Therefore, it may be concluded that the 1DSD transformation is effective and efficient for the numerical computation in SSI problems under practical conditions.

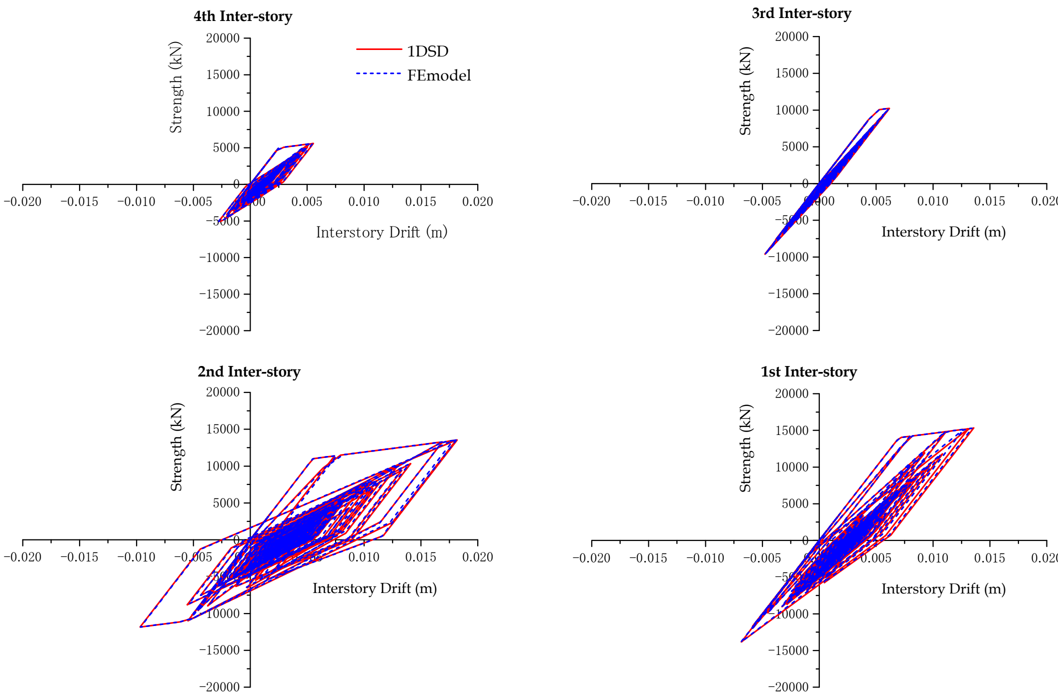


**Figure 7.** Shear force and inter-storey drift relation in each story when subjected to ground motion associated with 1940 El Centro NS by using 1DSDs and the original FEmodel.


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