Fractional chaotic oscillators

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SUMMARY:
In this paper some physics and mathematics aspects of chaos are highlighted. In particular the effect of the appearance of fractional derivatives in the differential equations that model the dynamic behavior of oscillators is analyzed. This is determined by comparing the phase diagrams of classical damped oscillators (derivative of integer order), with fractional damped oscillator. So, information about the physical nature of chaos and some of the mathematical tools used is presented, in particular, fundamentals of fractional calculus and fractional differential equations is provided.

Keywords: Chaotic dynamic, Fractional calculus, Fractional differential equations, Dissipative systems

1. INTRODUCTION

In this paper some physics and mathematics aspects of chaos are highlighted. In particular the effect of the appearance of fractional derivatives in the differential equations that model the dynamic behavior of oscillators is analyzed. This is determined by comparing the phase diagrams of classical damped oscillators (derivative of integer order), with fractional damped oscillator. So, information about the physical nature of chaos and some of the mathematical tools used is presented, in particular, fundamentals of fractional calculus and fractional differential equations is provided.

Therefore in this paper dynamic systems modeled by ordinary and fractional differential equations are presented. It emphasizes the qualitative description of long-term recurring movements governed by nonlinear differential equations for which analytical solution is not feasible. Dissipative systems exhibit a typical initial transient behavior, after which the motion tends to a stable behavior, so solutions converge to attractors. The simplest attractor is an equilibrium point, secondly the periodic attractors, and the most recent are chaotic attractors, whose unexpected features have generated great interest, because perfectly deterministic systems characterized by nonlinear differential equations converge to these attractors.

To understand the effect of the appearance of fractional derivatives in differential equations, graphical method is used in phase space, by which one can verify that despite the apparent randomness, topological characteristics are well defined (sometimes fractal) for different types of equations. A comparison is made of the behavior of three types of chaotic oscillators: a Duffing, Van der Pol and Chua oscillators. This is done in each case using different fractional derivatives (eg, orders of 0.2, 0.5). Finally some conclusions on the results, focused on its application in earthquake engineering, are presented. In particular it is worth noting that in chaotic systems, for a deterministic periodic input, the system can have a random-looking output, but with behavior patterns.

2. FRACTIONAL CALCULUS

The concept of fractional calculus is not new (Weisstein, 2007). It is a generalization of the differentiation and the integration (of entire order) to not entire orders (fractional, real and even complex). Various mathematicians have contributed in this theme. Among them Liouville, Riemann, and Weyl did important contributions to the theory of the fractional calculus. The developments in this matter culminated in two fractional calculus owed at work of Riemann and Liouville (RL) on the one hand and on the other hand in the work of Grunwald and Letnikov (GL); both formulations are connected. The expression of Grunwald and Letnikov is:

$$ {}_a^G D_t^{\alpha} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha+1)} \int_a^t (t-\tau)^{n-\alpha} f^{(m+1)}(\tau) d\tau \tag{2.1} $$

and the expression of Riemann and Liouville is:

$$ {}_a^L D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^{\alpha-n+1}} , \tag{2.2} $$

It can be told that, the fractional calculus is based on the fractional integral definition given by

$$ D^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(\xi) d\xi \tag{2.3} $$

Where $\Gamma(\nu)$ it is the gamma function, defined as an extension of the factorial function (where ν is entire), in which ν is generalized to complex and real numbers.

2.1 Fractional derivative

The fractional derivative of order $\mu > 0$ (if exists) can be defined in terms of the fractional integral as:

$$ D^{\mu} f(t) = D^m \left[D^{-(m-\mu)} f(t) \right] \tag{2.4} $$

Where m is an integer number.

3. FRACTIONAL DIFFERENTIAL EQUATIONS

In these equations, the order of the derivatives is fractionary, and therefore appear in it fractional terms. Like an example an equation of this type is presented and its solutions for different cases:

$$ \left[D^{2\nu} + \alpha D^{\nu} + \beta D^0 \right] y(t) = 0 \tag{3.1} $$

And its solutions are:

$$ y(t) = \begin{cases} e_{\alpha}(t) - e_{\beta}(t) & para \quad \alpha \neq \beta \\ te^{\alpha t}, \quad \sum_{k=-(q-1)}^{q-1} \alpha^k (q - |k|) D^{1-(k+1)\nu} \left(te^{\alpha^k t} \right) & para \quad \alpha = \beta \neq 0 \\ \dfrac{t^{2\nu-1}}{\Gamma(2\nu)} & para \quad \alpha = \beta = 0 \end{cases} \tag{3.2} $$

Where: $q = 1/\nu$, $\quad e_{\beta}(t) = \sum_{k=0}^{q-1} \beta^{q-k-1} E_t\left(-k\nu, \beta^q\right)$ (3.3)

3.1 Fractionary finite differences

Weighted average finite difference methods for solving fractional diffusion equations are discussed and different formulae of the discretization of the Riemann-Liouville derivative are considered (Magaña, 2008). The stability analysis of the different numerical schemes is carried out by means of a procedure close to the well-known von Neumann method of ordinary diffusion equations. The stability bounds are easily found and checked in some representative examples.

4. GENERAL CONCEPTS OF THE CHAOS THEORY

The chaos theory (Magaña, 2006) deals with the unpredictability and the predictability of the systems, even of the most unstable. The chaotic systems are not random, although they seem to be, and they possess the following characteristic that define them:

- They are deterministic in the sense that an equation that governs its behavior exists.
- They are very sensitive to the initial conditions, since a small change in the start point can cause a vastly different result, which makes them be unpredictably.
- They are disordered and fortuitous, although certainly they are not, because under such a random behavior, there it exist some order sense and pattern.

Accordingly, true random systems are not chaotic in the true sense because random implies nonexistence any relation between cause and effect for a given phenomenon.

The chaos theory is plagued of strange attractors that produce a seemingly random and unpredictable behavior. For example, in meteorology you don't know exactly what climatic pattern will exist within hundred days, but there are patterns that are likely to develop and others that are not. The group of possible patterns is known as attractor because it attracts toward itself the evolution of the system toward certain states and this concept is critical for the understanding of the chaos theory.

5. ATTRACTORS

One of the basic concepts within the field of dynamical systems theory is the concept of attractor. Typical dissipative dynamic systems, initially exhibit a transient behavior, after which the movement tends to a recurrent long term behavior. Movements with initial conditions close to each other tend to converge to stable solutions which are attractors (Magaña, 2011). The simplest attractor is a steady point at which all motion disappears. The pendulum is the archetypal example of this type of attractor, and was studied experimentally by Newton. Second, we have the periodic attractor. This for instance occurs in a thin steel plate driven by an electromagnet which carries an alternating current, this will settle to a constant vibration in resonance with the forcing frequency. After a small perturbation, transitory effects which fade slowly occur, after that the fundamental oscillation is reset. Thirdly we have a chaotic attractor (recently discovered), whose unexpected features have generated an explosion of interest and comes from the numerical solution of an equation perfectly deterministic and well defined. Leading to a perpetual chaos, in which the motion history has a broadband random power spectrum.

6. CHAOTIC OSCILLATORS

6.1. Chaotic dynamics in the Duffing oscillator

Let us now be more concrete, consider the behavior of the specific equation

$$\ddot{x} + 0.05\dot{x} + x^3 = 7.5\cos t \qquad (6.1)$$

In mechanical engineering, such equation models, for example, the sinusoidal motion of a structure subjected to force and large elastic deformation. An example of the solution of a time series x vs. t

obtained by numerical integration of Eqn. 6.1 is presented in Figure 6.1. This chaos has an irregular appearance, which persists for as long time meanwhile integrations are performed. Despite their recurrent nature is evidenced by the fact that certain patterns in the waveform are repeated at irregular intervals, never exact repetition, and movement is not really periodic (Ueda, 1980).
In practice, chaotic attractors can be identified as stable structures in long-term trajectories in a limited region of phase space, beam whose trajectories are bent back on itself, resulting in the mixture and the divergence of neighboring states.

6.2. Relaxed and cardiac pulse oscillations

The second order equation of the oscillator:

$$\ddot{x} + F(\dot{x}) + \omega^2 x = 0 \tag{6.2}$$

A nonlinear damping function which exerts a force always opposite to the speed direction will cause a qualitative behavior similar to a linear damping system. However, the behavior of solutions of Eqn. 6.2 is qualitatively different from a linear damped oscillator if F acts sometimes in the same direction that the speed, which in this case indicates the presence of an energy source. The final behavior for Eqn. 6.2 can then be a limit cycle in the autonomous case, as discussed below.

Differentiating Eqn. 6.2 with respect to time, and substituting υ by $\dot{x}$ so we have

$$\ddot{v} + F'(v)\dot{v} + \omega^2 v = 0 \tag{6.3}$$

Choosing $F(v) = \alpha(v^3/3 - v)$ so we obtain the equation of Van der Pol

$$\ddot{v} + \alpha(v^2 - 1)\dot{v} + \omega^2 v = 0 \tag{6.4}$$

This equation was extensively studied by Van der Pol (Van der Pol, 1928) with analog simulation using vacuum tube circuits, where the function F corresponds to the nonlinear characteristic of a triode.

6.3. Phase Diagram of chaotic oscillators

6.3.1. Chaotic system Duffing

This system describes a chaotic motion, which is governed by the differential equation (Eqn. 7.1) and for whom $f(t) \neq 0$ and $\alpha = 1$. For this system, initial conditions are $y(0) = 3.0$ y $\dot{y}(0) = 4.0$ with values of $m = 1.0$, $C = 0.05$ and $K = 1.0$. (Magaña, 2011).

The time evolution of this system is shown in figure 6.1, corresponding to the graphic depicting the accelerogram. The phase diagram $x - \dot{x}$ is presented in figure 6.2.

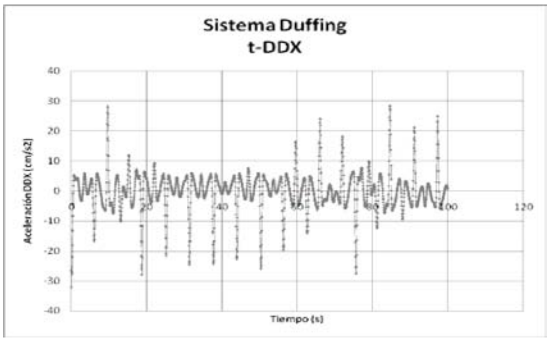

Figure 6.1. Graphic $t - \ddot{x}$

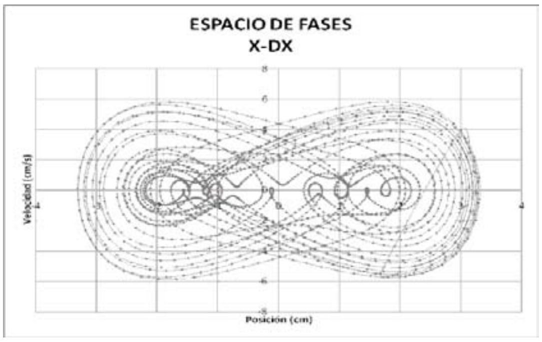

Figure 6.2. Phase diagram $x - \dot{x}$

6.3.2. Van Der Pol oscillator

This system describes a chaotic motion, which is governed by the differential equation (Eqn. 7.2) and for whom $f(t) \neq 0$ and $\alpha = 1$ (Magaña, 2011).

For this system, initial conditions are $y(0) = 2.0$ y $\dot{y}(0) = 4.0$ and values of $m = 1.0$, $\mu = 8.53$, $K = 1.0$, $A = 1.2$ and. $\omega = 2\pi/10$.

The time evolution of this system is shown in figure 6.3, corresponding to the graphs depicting the accelerogram. In figure 6.4 the phase diagram $x - \dot{x}$ is presented .

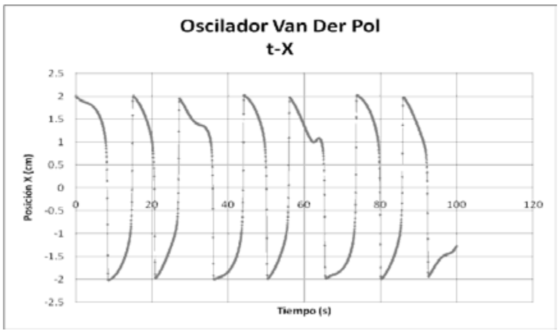

Figure 6.3. Graphic $t - x$

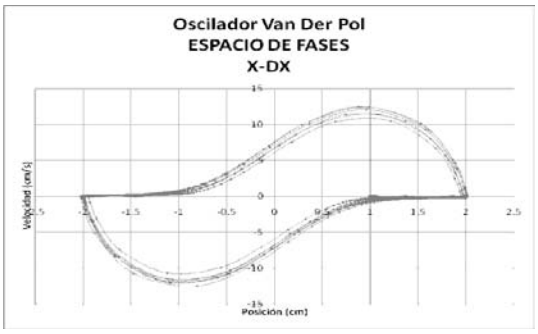

Figure 6.4. Phase diagram $x - \dot{x}$

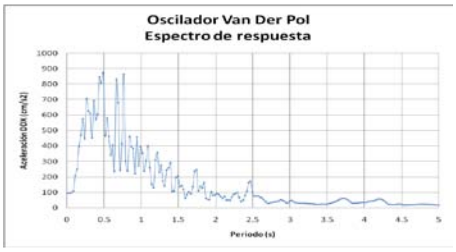

Figure 6.5. Van der Pol response spectrum

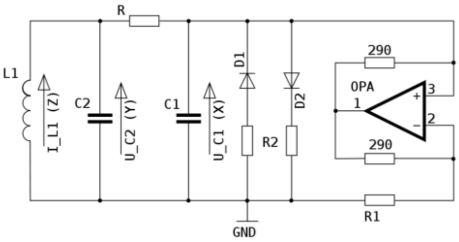

Figure 6.6. Chua's circuit

6.3.3. CHAOTIC CHUA'S SYSTEM

Chua's circuit (also known as a Chua circuit, figure 6.6) is a simple electronic circuit that exhibits classic chaos theory behaviour (Kuznetsov et.al, 2011). It was introduced in 1983 by Leon O. Chua, who was a visitor at Waseda University in Japan at that time. The ease of construction of the circuit has made it a ubiquitous real-world example of a chaotic system, leading some to declare it "a paradigm for chaos". By means of the application of the laws of electromagnetism, the dynamics of Chua's circuit can be accurately modeled by means of a system of three nonlinear ordinary differential equations in the variables *x(t)*, *y(t)* and *z(t)*, which give the voltages across the capacitors *C1* and *C2*, and the intensity of the electrical current in the inductor *L1*, respectively. These equations are:

$$\begin{aligned} \frac{dx}{dt} &= \alpha[y - x - f(x)] \\ \frac{dy}{dt} &= x - y + z \\ \frac{dz}{dt} &= -\beta y \end{aligned} \tag{6.5}$$

7. FRACTIONAL ANALYSIS OF CHAOTIC SYSTEMS

In this chapter a series of fractional chaotic systems analysis are presented. For such dynamic analysis the package Matlab-Simulink was used. The behavior of the Duffing (7.1) and the Van der Pol (7.2) oscillators, and a dependent electrical system of three variables called Chua (7.3) were simulated. For all models, the classical solutions and with fractional exponent are presented.

$$m\frac{d^2x(t)}{dt^2} + C\frac{d^\alpha x(t)}{dt^\alpha} + x(t)^3 = 7.5\cos(t) \qquad (7.1)$$

$$m\frac{d^2x(t)}{dt^2} + \mu(1 - x(t)^2)\frac{d^\alpha x(t)}{dt^\alpha} + K \cdot x(t) = A \cdot sen(\omega t) \qquad (7.2)$$

$$\left.\begin{aligned} \frac{d^\alpha x(t)}{dt^\alpha} &= A \cdot (y(t) + \frac{x(t) - 2 \cdot x(t)^3}{7}) \\ \frac{d^\beta y(t)}{dt^\alpha} &= x(t) - y(t) + z(t) \\ \frac{d^\gamma z(t)}{dt^\alpha} &= -\frac{100}{7} y(t) = -By(t) \end{aligned}\right\} \qquad (7.3)$$

The fractional derivative is denoted by: $D^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha}$ (7.4)

For values of $\alpha = 1$ the equations 7.1-7.3 coincide with the classical differential equations of each respective chaotic system. Below the chaotic analysis are presented.

7.1. Chaotic system Duffing

This system describes a chaotic movement, which is governed by the differential Eqn. (7.1) and where $f(t) \neq 0$. For this system, the initial conditions are $x(0) = 3.0$ and $D^\alpha(x(0)) = 4.0$, with values of $m = 1.0$, $C = 0.05$ y $K = 1.0$. The next is the analysis of the movement equation (7.1), corresponding to a value of $\alpha = 0.2$. The time evolution of the system is shown in figures 7.1 and 7.2, corresponding to the graphics $t - x$, $t - D^2x(t)$ which represent the history of movement and the accelerogram, respectively. It can be seen, in figures 7.3 and 7.4, the phase diagrams.

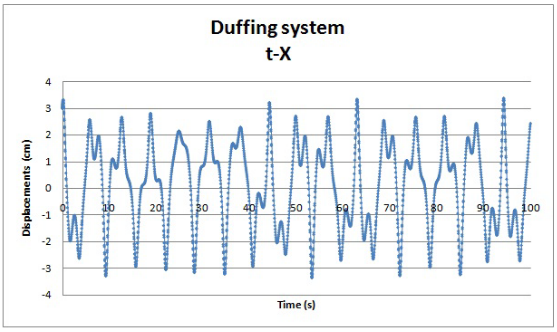

Figure 7.1. Graphic $t - x$

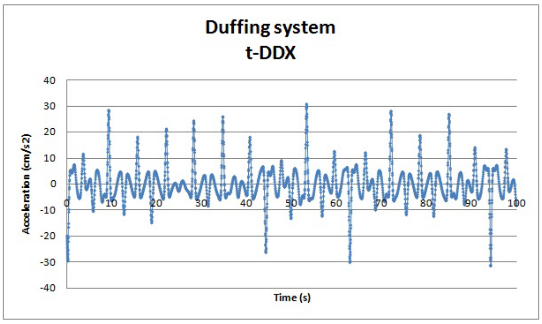

Figure 7.2. Graphic $t - D^2x(t)$

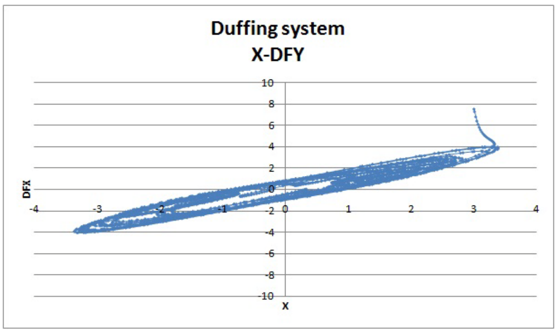

Figure 7.3. Phase diagram $x - D^\alpha x(t)$

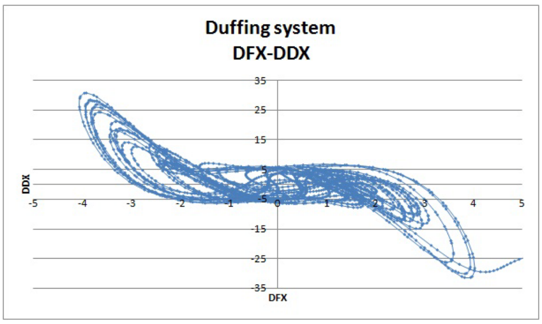

Figure 7.4. Phase diagram $D^\alpha x(t) - D^2x(t)$

The next is the analysis of the movement Eqn. 7.1, corresponding to a value of $\alpha = 0.5$. The time evolution of the system is shown in figures 7.5 and 7.6, corresponding to the graphics $t - x$, $t - D^2x(t)$ which represent the history of movement and the accelerogram, respectively. It can be seen, in figures 7.7 and 7.8, the phase diagrams.

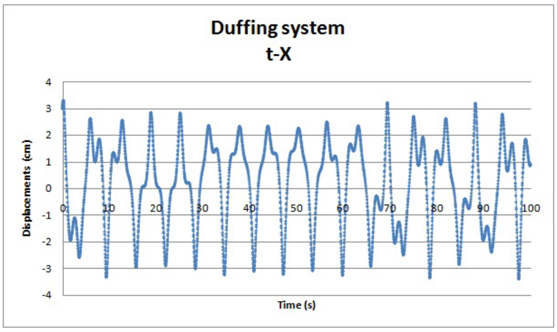

Figure 7.5. Graphic $t - x$

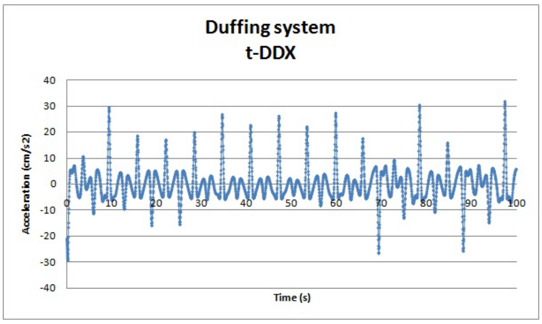

Figure 7.6. Graphic $t - D^2x(t)$

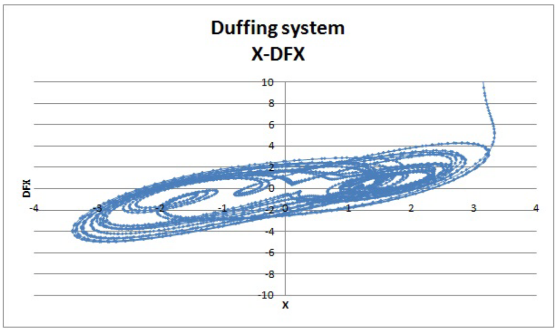

Figure 7.7. Phase diagram $x - D^{\alpha}x(t)$

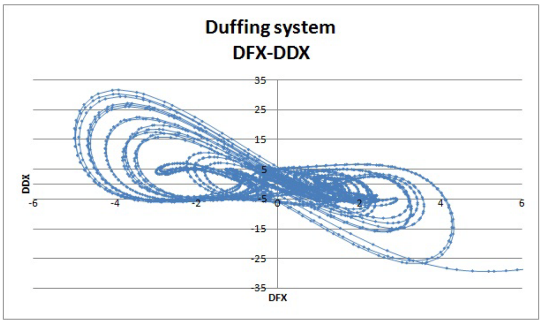

Figure 7.8. Phase diagram $D^{\alpha}x(t) - D^2x(t)$

In Figures 7.4 and 7.8 shows that the phase diagram appears flattened to the lowest value of α , as if he had presented a "rotation" on an axis in the plane of the figure.

7.2. Van Der Pol oscillator

This system describes a chaotic movement, which is governed by the differential Eqn. 7.2 and where $f(t) \neq 0$. For this system, the initial conditions are $x(0) = 2.0$ y $D^{\alpha}(x(0)) = 4.0$, $m = 1.0$, $\mu = 8.53$, $K = 1.0$, $A = 1.2$ y $\omega = 2\pi/10$. The next is the analysis of the movement Eqn. 7.2, corresponding to a value of $\alpha = 0.2$. The time evolution of the system is shown in figures 7.9 and 7.10, corresponding to the graphs $t - x$, $t - D^2x(t)$ which represent the history of movement and the accelerogram, respectively. It can be seen, in figures 7.11 and 7.12, the phase diagrams.

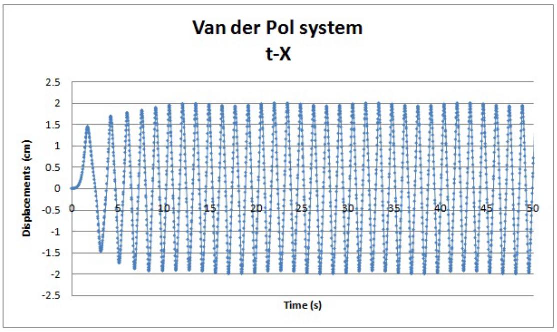

Figure 7.9. Graphic $t - x$

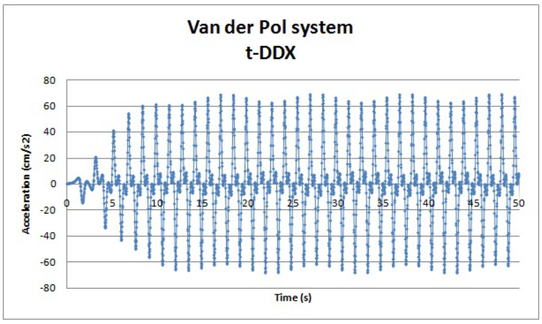

Figure 7.10. Graphic $t - D^2x(t)$

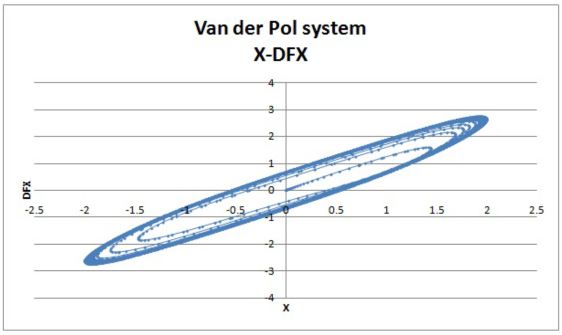

Figure 7.11. Phase diagram $x - D^{\alpha}x(t)$

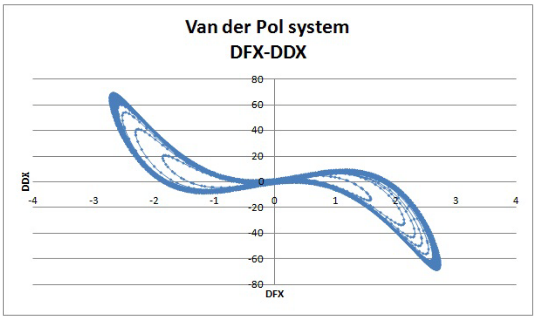

Figure 7.12. Phase diagram $D^{\alpha}x(t) - D^{2}x(t)$

The next is the analysis of the movement Eqn. 7.2, corresponding to a value of $\alpha = 0.5$. The time evolution of the system is shown in figures 7.13 and 7.14, corresponding to the graphics $t - x$, $t - d^2x/dt^2$ which represent the history of movement and the accelerogram, respectively. It can be seen, in figures 7.15 and 7.16, the phase diagrams.

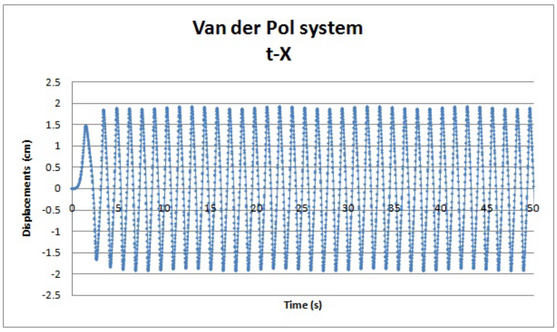

Figure 7.13. Graphic $t - x$

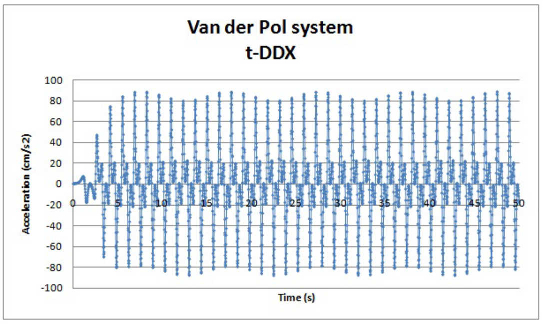

Figure 7.14. Graphic $t - D^{2}x(t)$

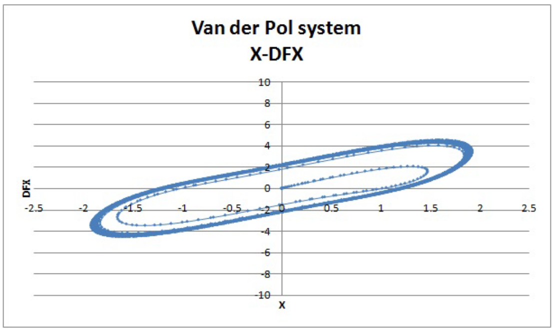

Figure 7.15. Phase diagram $x - D^{\alpha}x(t)$

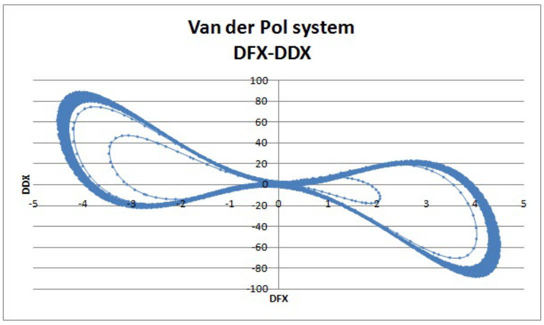

Figure 7.16. Phase diagram $D^{\alpha}x(t) - D^{2}x(t)$

In Figures 7.12 and 7.16 also shows the phase diagram flatter for the lower value of α .

7.3. Chua's system

This system describes a chaotic movement, which is governed by the system of differential equations 7.2. For this system, the initial conditions are $x(0) = 0.1$, $y(0) = 0.1$y $z(0) = 0.0$. The next is the dynamic analysis of Eqn. 7.3, corresponding to a value of $\alpha = \beta = \gamma = 0.0$, $A = 10.0$. The time evolution of the system is shown in figures 7.17 and 7.18, corresponding to the graphics $t - x$, $t - z$ Which represent the time histories of x and z , respectively. It can be seen, in figures 7.19 and 7.20, the phase diagrams $x - y$,. $y - z$.

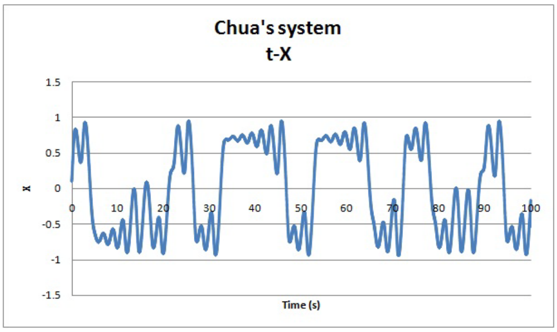

Figure 7.17. Graphic $t-x$

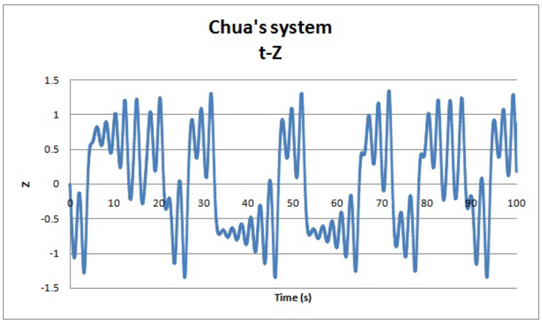

Figure 7.18. Graphic $t-z$

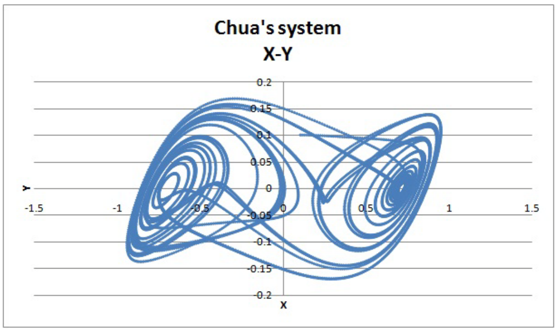

Figure 7.19. Phase diagram $x-y$

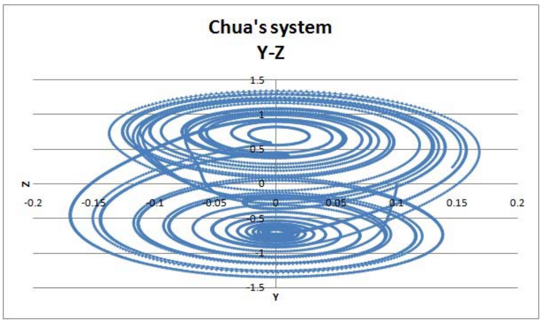

Figure 7.20. Phase diagram $y-z$

The next is the dynamic analysis of Eqn. 7.3, corresponding to a value of $\alpha = 0.2$, $\beta = 0.1$ $\gamma = 0.0$, $A = 8.5$. The time evolution of the system is shown in figures 7.21 and 7.22, corresponding to the graphs $t-x$, $t-z$ which represent the history of movement and the accelerogram, respectively. It can be seen, in figures 7.23 and 7.24, the phase diagrams.

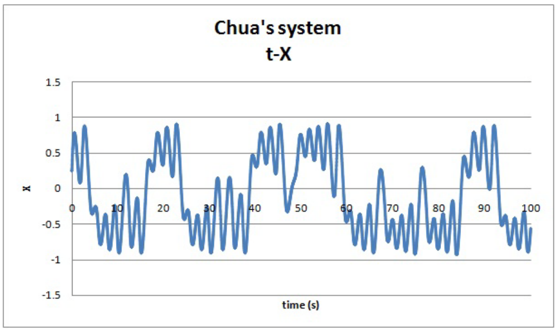

Figure 7.21. Graphic $t-x$

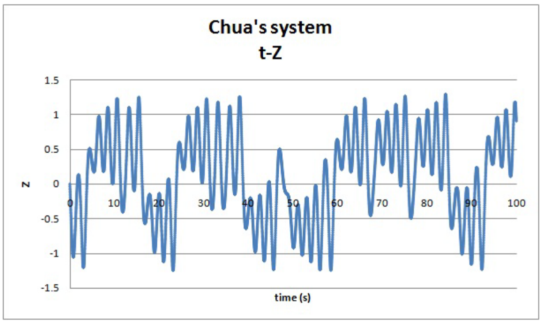

Figure 7.22. Graphic $t-z$

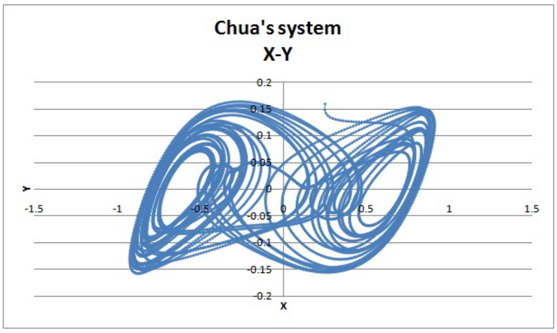

Figure 7.23. Phase diagram $x-y$

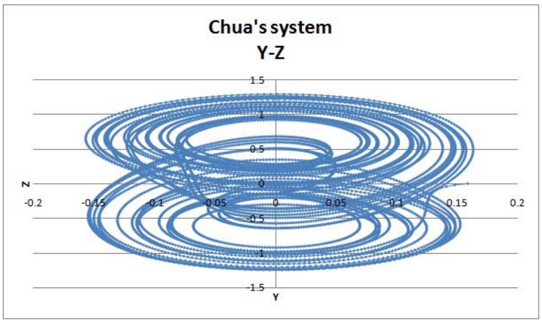

Figure 7.24. Phase diagram $y-z$

In this case also seen in Figures 7.20 and 7.24 the "rotation" of the phase diagram about a horizontal axis in the plane of the figure.

8. CONCLUSIONS

From the above we can make the following conclusions (both non-linearity, and the fractional order of derivation):

Conclusions on the non-linearity:

- The phase diagram is helpful to detect the geometric characteristics of different types of attractors.
- The existence of multiple attractors induces a large irregularity in the movement history
- The waveform is altered by nonlinearity ceasing to be sinusoidal and the projection of motion in the phase diagram can be very complex.
- In a chaotic oscillator a periodic input generates an output irregular (random looking), but has patterns of behavior.
- The response spectra have several peaks, although the oscillator is a degree of freedom (ie are broadband in Van der Pol response spectrum, figure 6.5).
- The phase diagram in a chaotic oscillator shows some geometric structure, which does not exist in a pure random signal. This is one of the differences between both types of signals (random and chaotic).
- The study in this work shows that the Duffing oscillator, the Van der Pol, and Chua phase diagrams are geometrically very different. Also, their movement histories have different patterns.

Conclusions about the order of derivation:

- In all analyzed chaotic oscillators, the effect of changing the value of the fractional order of derivation in the phase diagram is showed, as in the motion history.
- The response spectrum of several peaks appears to be due only to the non-linearity of the parameters M, C and K, remaining this feature whatever the fractional order of derivation.
- It is noted that with fractional derivatives there is a greater ability to simulate different motion histories.

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