A Finite Strip Formulation for Nonlinear Free Vibration of Plates

S. Hashemi
Yasooj Branch, Islamic Azad University, Iran

E. Jaberzadeh
Khomeini Shahr Branch, Islamic Azad University, Iran

SUMMARY:
A solution for non-linear vibration of rectangular plates is presented. The finite strip method, an effective method for solving plate problems, is applied to establish an eigenvalue problem. In order to obtain plate free vibration frequencies and mode shapes, combined trigonometric and polynomial functions that satisfy the boundary conditions are used. The method is programmed, and several numerical examples are presented to illustrate the scope and efficacy of the procedure. The obtained results are compared with those of other investigators. The results which consider only the standard stiffness matrix, without considering the geometric stiffness matrix, are validated with the published results of other researchers and excellent agreement is observed. Moreover, for the first time, the effect of the geometric stiffness matrix is investigated. This method is more accurate and simpler than the previous methods.

Keywords: Plate, Non-linear, Vibration, Finite strip.

1. INTRODUCTION

Plates are intrinsically associated with many structures designed to undergo extreme dynamic conditions, which may lead to large amplitude vibrations causing failure. To understand the stresses induced by large amplitude dynamic responses, it is very desirable to carry out a free nonlinear vibration analysis.

Design practices based on the linear free vibration analysis of plates have proven to be extremely effective in reducing the noise and the fatigue failures. However, if the amplitude of vibration increases, the frequency of vibration also increases. This is because of the fact that, at large amplitude, the stiffness properties depend upon the deformed shape. Therefore, nonlinearity due to large deflections must be considered.

The large amplitude vibration of plates has drawn considerable attention from researchers. Sathyamoorthy (1987) published a paper which provides an excellent overview of the publications on nonlinear vibration analysis of plates. He also published a book (1998) on this subject. The book lists a very large number of papers from the open literature at the end of each chapter.

During the 1950s and 1960s, the method of analysis for this type of problem was based on the solution of the nonlinear differential equation of motion. Chu and Herrmann (1956) investigated free flexural vibrations of a rectangular elastic plate, supported by immovable hinges along all edges and solved nonlinear equations approximately. Prabhakara and Chia (1977) presented an analytical investigation of large amplitude vibrations of orthotropic plates with all-clamped and all-simply supported stress free. In the 1970s, researchers started using numerical methods to study the free nonlinear vibration of plates. Mei and his co-investigators (1973, 1997) have since then published a number of papers on free nonlinear vibration of isotropic and laminated plates, using the finite element method. Ganapathi et al. (1991) studied the large amplitude free flexural vibrations of laminated plates by finite element method.

The work of Rao et al. (1993) was based on an approximate linearization of the strain-displacement relations, where they used the iterative method to obtain the frequencies.

The geometrically nonlinear vibration of isotropic and laminated rectangular plates was investigated by Han and Petyt (1997) using the hierarchical finite element method. They used Legendre polynomials in formulating the element.
Singh and Lan (2002) employed a variational approach for the nonlinear free vibration of shallow shells having a quadrilateral boundary.
The objective of this paper is to employ a classic finite strip method, based on the geometrically nonlinear formulation for the analysis of large amplitude free vibration of rectangular plates. The overall objective of the research represented in this paper is to quantify the nonlinear free vibration of rectangular plates. The effects of standard and geometric stiffness matrices are examined, separately. The results obtained in the present paper are useful in the design of plates undergoing extreme dynamic conditions.

2. FORMULATION OF FREE VIBRATION

2.1. Finite strip discretization

A typical finite strip division of a rectangular plate is shown in Fig. 1. The global rectangular co-ordinates system *xyz* coincides with the mid-surface of the plate. Boundary conditions at x=0 and x=a are known priori and will be considered in the choice of displacement functions in x direction. Any two finite strips are connected along a nodal line, and in the assembly process, the nodal variables are matched along such lines for compatibility. Constant thickness and width are assumed along the length of each strip.

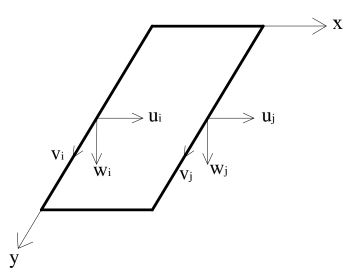

Figure 1. A finite strip

The present analysis includes both bending and membrane effects, and therefore both in-plane and out-of-plane displacements are included as nodal variables. An isolated strip with the local co-ordinates system "*xyz*" is shown in Fig.1. Displacement nodes are placed at the middle of each side, marked 1 and 2 in Fig. 1, with the three displacements and one rotation taken as the degrees of freedom, where *u* and *v* are the in-plane displacements, *w* is the out-of-plane displacement and $\theta = \partial w / \partial x$ is the rotation about the *y* axis.

2.2. Displacement functions

The displacement functions in the *y* direction, i.e. the longitudinal direction, are chosen so as to satisfy specific boundary conditions at the ends of the strip. The variation of the displacements in the "*y*" direction, i.e. across the strip, is taken as linear for *u* and *v*, and cubic for *w*. These assumptions ensure slope and displacement compatibility between adjacent strips.
The *u*, *v* and *w* displacement interpolations of a single strip can be written in terms of the nodal displacements by combining the shape functions. Assuming simply supported end conditions for strips in the longitudinal direction, these expressions for a strip of size $a \times b$ are given by

$$u=[(1-\xi)u_{1m}+\xi u_{2m}]\sin(m\pi\eta) \tag{2.1}$$

$$v=[(1-\xi)v_{1m}+\xi v_{2m}]\cos(m\pi\eta) \tag{2.2}$$

$$\begin{aligned} w &= [(1-3\xi^2+2\xi^3)w_{1m}+(\xi-2\xi^2+\xi^3)b\,\theta_{1m}+(3\xi^2-2\xi^3)w_{2m} \\ &+(\xi^3-\xi^2)b\,\theta_{2m}]\sin(m\pi\eta) \end{aligned} \tag{2.3}$$

where, u_{1m}, u_{2m} etc. are the nodal variables, $\xi = x/a$, $\eta = y/b$ and the summation convention is used for the repeated index *m*.

2.3. Strain-displacement relations

The well known large deflection strain-displacement relations for plate bending are

$$\varepsilon_x=\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}(\frac{\partial w}{\partial x})^2 \tag{2.4}$$

$$\varepsilon_y=\frac{\partial v}{\partial y}-z\frac{\partial^2 w}{\partial y^2}+\frac{1}{2}(\frac{\partial w}{\partial y})^2 \tag{2.5}$$

$$\gamma_{xy}=2\varepsilon_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-2z\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{2.6}$$

where ε_x, ε_y and ε_{xy} are the components of strain in two dimensions, and γ_{xy} is the engineering shear strain.

2.4. Stress-strain relations

The material is assumed to be elastic isotropic, therefore the constitutive relations given by

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\frac{E}{(1-\nu^2)}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=D_T\varepsilon \tag{2.7}$$

where *E* is the modulus of elasticity and *v* is the Poisson's ratio of the material.

3. STIFFNESS FORMULATION

The stiffness formulation for the present case follows the standard one for finite elements and only the most pertinent points are given in the following. The displacement functions of equations (2.1), (2.2) and (2.3) are given in shape function form as

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N]\{\hat{u}\} \tag{3.1}$$

where [N] is the matrix of shape functions and $\{\hat{u}\}$ is the vector of nodal variables.

$$\{\hat{u}\}^T = \{u_{11}, v_{11}, w_{11}, \theta_{11}, u_{21}, ...\} \tag{3.2}$$

Substitution of equation (3.2) into the strain-displacement relations of equations (2.4-6) gives the strains as functions of the nodal variables which can be written in matrix form

$$\{\varepsilon\} = [B]\{\hat{u}\} + [C_0(\hat{u})] \tag{3.3}$$

where [B] is the usual strain-displacement matrix in which its elements are independent from nodal displacement and given as following equation

$$[B] = \begin{bmatrix} \frac{\partial N_1^u}{\partial x} & 0 & -z\frac{\partial^2 N_1^w}{\partial x^2} & -z\frac{\partial^2 N_2^w}{\partial x^2} & \frac{\partial N_2^u}{\partial x} & 0 & -z\frac{\partial^2 N_3^w}{\partial x^2} & -z\frac{\partial^2 N_4^w}{\partial x^2} \\ 0 & \frac{\partial N_1^v}{\partial y} & -z\frac{\partial^2 N_1^w}{\partial y^2} & -z\frac{\partial^2 N_2^w}{\partial y^2} & 0 & \frac{\partial N_2^v}{\partial y} & -z\frac{\partial^2 N_3^w}{\partial y^2} & -z\frac{\partial^2 N_4^w}{\partial y^2} \\ \frac{\partial N_1^u}{\partial y} & \frac{\partial N_1^v}{\partial x} & -2z\frac{\partial^2 N_1^w}{\partial x \partial y} & -2z\frac{\partial^2 N_2^w}{\partial x \partial y} & \frac{\partial N_1^u}{\partial y} & \frac{\partial N_2^v}{\partial x} & -2z\frac{\partial^2 N_3^w}{\partial x \partial y} & -2z\frac{\partial^2 N_4^w}{\partial x \partial y} \end{bmatrix} \tag{3.4}$$

$[C_0]$ contains the quadratic non-linear terms, which can be written as

$$[C_0(\hat{u})] = \begin{Bmatrix} \frac{1}{2}\frac{\partial N_i^w}{\partial x}\frac{\partial N_j^w}{\partial x} w_i w_j \\ \frac{1}{2}\frac{\partial N_i^w}{\partial y}\frac{\partial N_j^w}{\partial y} w_i w_j \\ (\frac{\partial N_i^w}{\partial x}\frac{\partial N_j^w}{\partial y} + \frac{\partial N_i^w}{\partial y}\frac{\partial N_j^w}{\partial x}) w_i w_j \end{Bmatrix} = [C]\{\hat{u}\} \tag{3.5}$$

For plate strips, the non-linear terms related to in-plane displacements have a marginal effect compared to non-linear terms related to out-of-plane displacements (w) (Zienkiewicz and Taylor, 1989).

The [C] matrix of equation (3.5) is given symbolically as

$$[C] = [[C_1],[C_2]] \tag{3.6}$$

in which

$$
[C_1]=\begin{bmatrix}
0 & 0 & \frac{1}{2}\frac{\partial N_1^w}{\partial x}\frac{\partial N_j^w}{\partial x}w_j & \frac{1}{2}\frac{\partial N_2^w}{\partial x}\frac{\partial N_j^w}{\partial x}w_j \\
0 & 0 & \frac{1}{2}\frac{\partial N_1^w}{\partial y}\frac{\partial N_j^w}{\partial y}w_j & \frac{1}{2}\frac{\partial N_2^w}{\partial y}\frac{\partial N_j^w}{\partial y}w_j \\
0 & 0 & (\frac{\partial N_1^w}{\partial x}\frac{\partial N_j^w}{\partial y}+\frac{\partial N_1^w}{\partial y}\frac{\partial N_j^w}{\partial x})w_j & (\frac{\partial N_2^w}{\partial x}\frac{\partial N_j^w}{\partial y}+\frac{\partial N_2^w}{\partial y}\frac{\partial N_j^w}{\partial x})w_j
\end{bmatrix} \tag{3.7}
$$

$$
[C_2]=\begin{bmatrix}
0 & 0 & \frac{1}{2}\frac{\partial N_3^w}{\partial x}\frac{\partial N_j^w}{\partial x}w_j & \frac{1}{2}\frac{\partial N_4^w}{\partial x}\frac{\partial N_j^w}{\partial x}w_j \\
0 & 0 & \frac{1}{2}\frac{\partial N_3^w}{\partial y}\frac{\partial N_j^w}{\partial y}w_j & \frac{1}{2}\frac{\partial N_4^w}{\partial y}\frac{\partial N_j^w}{\partial y}w_j \\
0 & 0 & (\frac{\partial N_3^w}{\partial x}\frac{\partial N_j^w}{\partial y}+\frac{\partial N_3^w}{\partial y}\frac{\partial N_j^w}{\partial x})w_j & (\frac{\partial N_4^w}{\partial x}\frac{\partial N_j^w}{\partial y}+\frac{\partial N_4^w}{\partial y}\frac{\partial N_j^w}{\partial x})w_j
\end{bmatrix} \tag{3.8}
$$

where $j = 1,2,3,4$.

Substitution of equation (3.3) into equation (3.5) results in

$$\{\varepsilon\} = \left[[B]+\left[C\right]\right]\{\hat{u}\} \tag{3.9}$$

3.1. Hamilton's principal

The equilibrium equations are obtained from Hamilton's principle.

$$\delta\int_{t_i}^{t_f}(T-P)dv = 0 \tag{3.10}$$

in which P and T are the strain and kinetic energy of the plate strip, respectively. The strain energy can now be written for a single finite strip as

$$P = \frac{1}{2}\int_v \sigma\varepsilon dv \tag{3.11}$$

The kinetic energy of the plate strip can be written as

$$T = \frac{1}{2}\iint_A \overline{m}(\xi,\eta)([\frac{\partial U(\xi,\eta,t)}{\partial t}]^2+[\frac{\partial V(\xi,\eta,t)}{\partial t}]^2+[\frac{\partial W(\xi,\eta,t)}{\partial t}]^2)dxdy \tag{3.12}$$

in which $\overline{m}(x,y)$ is the mass per unit area of plate. The nodal displacements are periodic functions of time with frequency ω. Generally the displacement functions can be written as:

$$\begin{cases} U(\xi,\eta,t) = u\sin\omega t \\ V(\xi,\eta,t) = v\ \sin\omega t \\ W(\xi,\eta,t) = w\sin\omega t \end{cases} \tag{3.13}$$

where u, v and w are given in equations (2.1-3).

Substituting equations (3.13) into equation (3.12) gives the kinetic energy as the following:

$$T = \frac{\omega^2}{2}\cos^2\omega t\iint_A \overline{m}(x,y)(u^2(x,y)+v^2(x,y)+w^2(x,y))dxdy \tag{3.14}$$

Substitution of equations (3.1,3.9, 3.11, 3.14) into equation (3.10) results in the equilibrium for a single strip

$$\int_v [[B]+[C]]^T\{\sigma\}dv - \rho\omega^2\int_v([N]^T[N]\{\hat{u}\})dv = 0 \tag{3.15}$$

where ρ is the mass density of plate material.

3.2. Newton-Raphson

The equilibrium equation for a single strip can also be written as

$$\{\varphi(\delta)\} = 0 \tag{3.16}$$

where $\{\phi\}$ is the left hand side of equation (3.16). $\{\phi\}$ can be expanded in aTaylor series, around a known solution $\{\delta_0\}$, as

$$\{\phi(\delta_0)\} + \left.\frac{\partial\{\phi\}}{\partial\{\delta\}}\right|_0 (\{\delta\}-\{\delta_0\}) + \ldots = 0 \tag{3.17}$$

This equation can be rearranged, after neglecting higher order terms, to get

$$([k]-\omega^2[m])(\Delta\delta) = 0 \tag{3.18}$$

The above equation has the standard form of an eigenvalue problem and its solution will provide the natural frequencies and corresponding mode shapes, in which [k] is the tangent stiffness matrix given by

$$[k] = [k_S]+[k_G] \tag{3.19}$$

$$[k_S] = \int_v \{[[B]+[C]]^T[D_T][[B]+[C]]\}dv \tag{3.20}$$

$$[k_G] = \int_v \{[U]\}dv \tag{3.21}$$

In the above equations k_s and k_G are the standard and geometric stiffness matrices, respectively.

In equations (3.29) and (3.23), [U] and [m] (mass matrix) are given as following equations.

$$U_{ij} = \sum_{k=1}^{3} \frac{\partial C_{ki}}{\partial \delta_j} \sigma_k \tag{3.22}$$

$$[m] = \rho \int_{v} \{[N]^T [N]\} dv \tag{3.23}$$

3.3. Process of problem solution

The solution can be obtained iteratively as described below.
Step 1. Initialize vector $\{\delta_0\}$ by setting zero values for all of its components. Calculate the stiffness and mass matrices and enforce the geometric boundary conditions. Save the mass matrix separately, as it does not change during the iteration. Only the stiffness matrix is calculated at each step.
Step 2. Solve equation (3.18) for the lowest natural frequency and the corresponding mode shape. This provides the natural frequency and mode shape for the free linear vibration of the plate. Using the calculated eigenvalues (mode shape values), obtain the value of maximum out-of-plane displacement (w_{max}). Divide all terms of mode shape values by w_{max} and then multiply those by a factor between 0.001 to 0.02. The thickness of the plate is assumed to be 0.01 width of the plate. With the new values of mode shape values, recalculate the stiffness matrix, enforce the geometric boundary conditions, and calculate the eigenvalue and corresponding eigenvector. Modify the eigenvector as before.
Step 3. Repeat step 2 a few times until the eigenvalue and eigenvector converge to a desired accuracy. Normally, depending upon the matrix size, 10–15 iterations are sufficient to have convergence.

4. NUMERICAL RESULTS AND VERIFICATION

4.1. Linear analysis

Linear analysis is studied first. Szilard (1974) has derived equation (4.1) for determining natural modes of flexural vibration in a rectangular plate.

$$\omega_{mn} = \pi^2 \left[\frac{m^2}{A^2} + \frac{n^2}{B^2} \right] \sqrt{\frac{D}{\overline{m}}} \tag{4.1}$$

where m,n=1,2,3,… also A, B, $D = Et^3 / 12(1-\upsilon^2)$ and $\overline{m}$ represent width, length, flexural rigidity and the mass per unit area of the plate. Table 4.1 shows the results of the present study and exact solution (Szilard, 1974) are in good agreement.

Table 4.1. Comparison of results for linear analysis

A (m)	**B (m)**	**E (N/m²)**	**ν**	**t (m)**	**ρ (kg/m³)**	**ω [exact] (1/s)**	**ω [present] (1/s)**
0.4	0.4	2.1e11	0.3	0.0025	7850	482.74	482.71
0.4	1.6	2.1e11	0.3	0.0025	7850	256.46	256.45

4.2. Nonlinear analysis

A large amplitude free vibration is carried out for square plates. In this paper, the effect of different boundary conditions that may arise from different combinations of clamped (C), simply supported (S) and free (F) edges of a plate are studied. The boundary condition of the plate is specified, through its edge conditions (C, S or F), in anticlockwise direction starting from the edge x=0. The numerical study is carried out for a square steel plate having *E=2.1e11* (Pa), $\rho=7850$ (kg/m^3) and $\upsilon=0.3$.

Figure 2 presents the period ratio (T_{NL}/T_L) of a simply supported square plate for the fundamental mode with amplitude to thickness ratio (w_0/t) of 0-1.5 with 4 strips. The results of the present study with and without considering geometric stiffness matrix are shown and compared with that of other studies (Chu and Hermann, 1956, Mei, 1973, Singh and Lan, 2002, Wah, 1963). The results without considering geometric stiffness matrix in plates are in better accordance with other references. So in later examples, the geometric stiffness matrix is not considered.

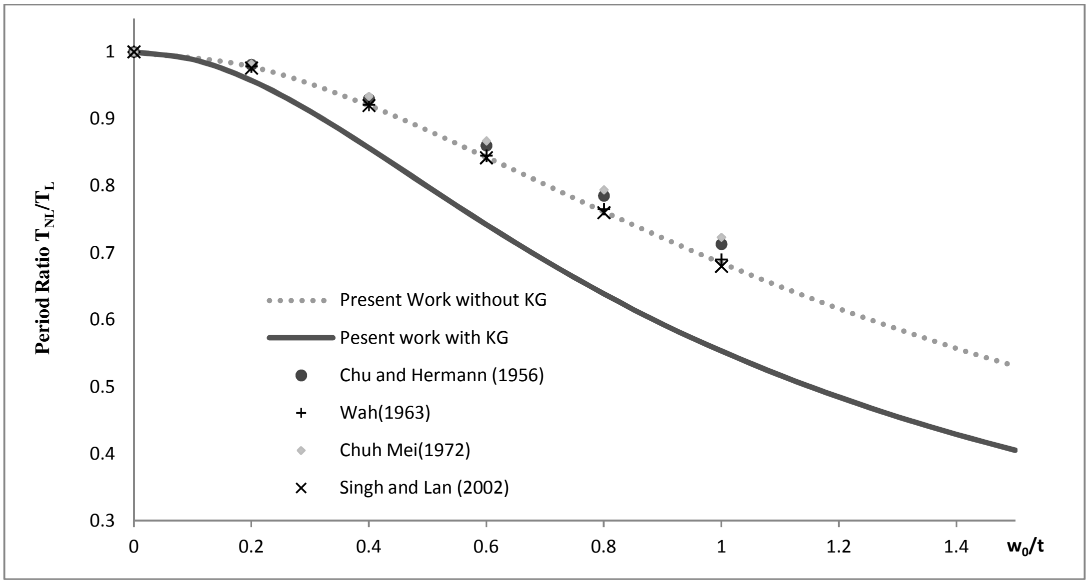

Figure 2. Period ratio of a simply supported square plate for the fundamental mode

(T_L is the value obtained with w_0/t=0)

Table 4.2. Period ratio of rectangular plates with two opposite edge simply supported and the other two clamped (C-S-C-S)

$\frac{T_{Nl}}{T_L}$	Aspect Ratio, A/B								
	0.5			1			2		
W_0 /t	Chu-Mei (1973)	**Present Work**	Wah (1963)	Chu- Mei (1973)	**Present Work**	Wah (1963)	Chu-Mei (1973)	**Present Work**	Wah (1963)
0	**0.9997**	**1.0000**	**1.0000**	**1.0004**	**1.0000**	**1.0000**	**0.9998**	**1.0000**	**1.0000**
0.2	**0.9818**	**0.9780**	**0.9838**	**0.9919**	**0.9877**	**0.9887**	**0.9931**	**0.9913**	**0.9918**
0.4	**0.9337**	**0.9195**	**0.9396**	**0.9675**	**0.9534**	**0.9570**	**0.9738**	**0.9667**	**0.9684**
0.6	**0.8677**	**0.8412**	**0.8780**	**0.9307**	**0.9036**	**0.9106**	**0.9444**	**0.9298**	**0.9330**
0.8	**0.7961**	**0.7581**	**0.8095**	**0.8858**	**0.8445**	**0.8560**	**0.9079**	**0.8852**	**0.8896**
1	**0.7268**	**0.6786**	**0.7417**	**0.8370**	**0.7854**	**0.7987**	**0.8673**	**0.8369**	**0.8420**

Table 4.2 shows reasonable agreement between the present results and those by Mei (1973) and Wah (1963) for plates having two opposite edges simply supported and the other two clamped, with aspect ratios of 0.5, 1.0 and 2.0. Figure 3 shows convergence stages of solution versus number of strips for a simply supported square plate. Figure 4 shows reasonable agreement between the present results and

those by Mei (1973) and Rao et al. (1993) for plates having three edges simply supported and another clamped. If boundary conditions of the plate are stiffened, the nonlinear period ratio must become smaller. So the results of Figures 5, 6 must be reasonable.

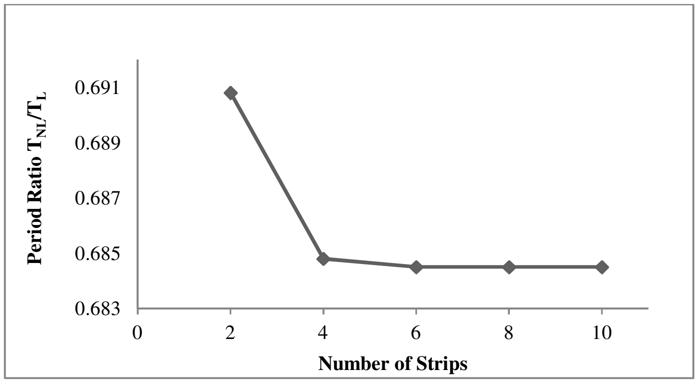

Figure 3. Convergence stages of solution versus number of strips

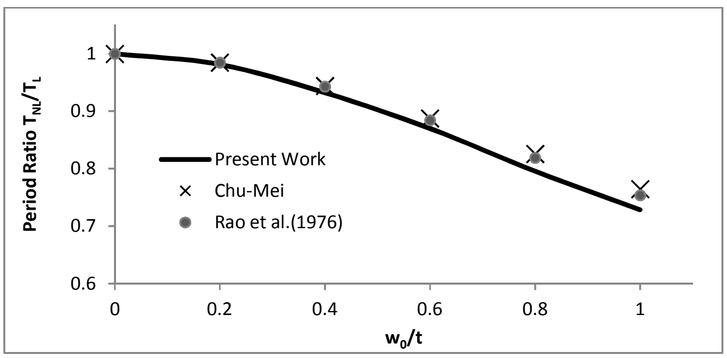

Figure 4. Period ratio of an S-S-S-C square plate for the fundamental mode

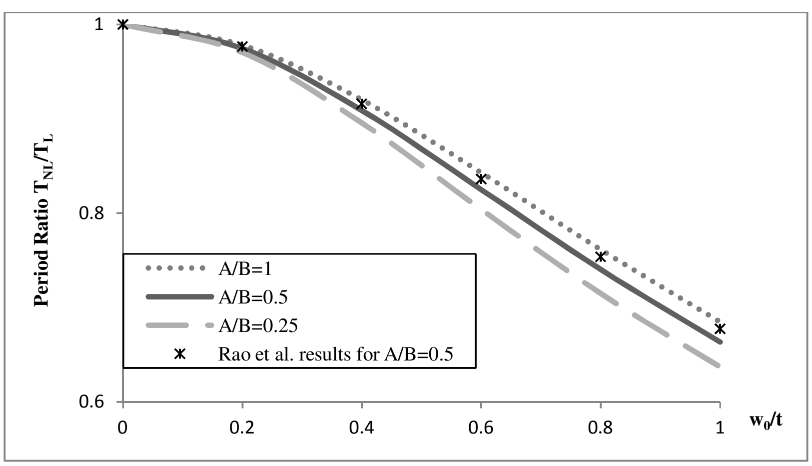

Figure 5. Period ratio of simply supported rectangular plates.

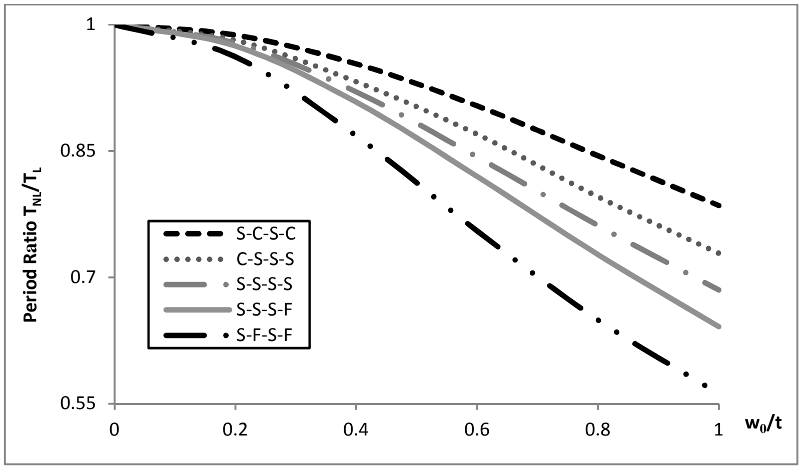

Figure 6. Period ratio of a square plate with different support conditions.

5. CONCLUSIONS

Large amplitude free vibration analysis of thin square and rectangular plates with different boundary conditions is investigated. The study is carried out by finite strip method. The results without considering the geometric stiffness matrix are in better accordance with other references. So, in later examples, the geometric stiffness matrix is not considered; and further investigation about its effect is suggested. The results of nonlinear analyses are validated with the published results of other researchers and excellent agreement is observed.

REFERENCES

Sathyamoorthy, M. (1987). Nonlinear Vibration Analysis of Plates: A Review and Survey of Current Developments, *Appl. Mech. Rev.* **40:11**, 1553-1562.

Sathyamoorthy, M. (1998), Nonlinear analysis of structures, CRC Press.

Chu, H.N., Herrmann, G. (1956). Influence of large amplitude on free flexural vibrations of rectangular elastic plates, *Journal of Applied Mechanics and Technical Physics*. **41:12**, 532-540.

Prabhakara, M.K., Chia, C.Y. (1977). Nonlinear flexural vibrations of orthotropic rectangular plates, *Sound Vibration* **52:4**, 511–518.

Mei, C. (1973). Finite element displacement method for large amplitude free Flexural vibration of beams and plates, *Computers & Structures* **3:1,** 163-174.

Shi, N. Lee, RYY, Mei, C. (1997). Finite element method for non-linear free vibrations of composite plates, *AIAA Journal* **35:1,** 159-166.

Ganapathi, M., Varadan, T.K., Sharma, B.S. (1991). Nonlinear flexural vibrations of laminated orthotropic plates, *Computers & Structures* **39:6**, 685-688.

Rao, S.R., Sheikh, A.H., Mukhopadhyay, M. (1993). Large-amplitude finite element flexural vibration of plates/stiffened plates, *J. Acoust. Soc. Am.* **93:6,** 3250– 3257.

Han, W., Petyt, M. (1997). Geometrically nonlinear vibration analysis of thin, rectangular plates using the hierarchical finite element method—I: the fundamental mode of isotropic plates, *Computers & Structures* **63:2,** 295–308.

Han, W., Petyt, M. (1997). Geometrically nonlinear vibration analysis of thin, rectangular plates using the hierarchical finite element method—II: 1st mode of laminated plates and higher modes of isotropic and laminated plates, *Computers & Structures* **63:2,** 309–18.

Singh, A.V., Lan, K. (2002). Numerical Analysis for the Nonlinear Free Vibration of Shallow Shells, *Journal of Engineering Mechanics* **128:12,** 1259-1266.

Zienkiewicz, O. C., Taylor, R. L. (1989). The Finite Element Method, Mc-Graw Hill.

Wah, T. (1963). Large amplitude flexural vibration of rectangular plates, *Inter. Journal of Mech. Sci.* **5:6**, , 425-438.

Szilard, R. (1974). Theory and analysis of plates, Prentice-Hall.