Inverse Multiquadric (IMQ) function as radial basis function for plane dynamic analysis using dual reciprocity boundary element method

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SUMMARY:
This paper presents a new radial basis function (RBF) for the boundary element method in the analysis of plane transient elastodynamic problems. The dual reciprocity method (DRM) is reconsidered by using the Inverse Multiquadric (IMQ) function as a new generation of RBFs to approximate the inertia term. The particular solution kernels of the proposed RBF corresponding to displacement and traction, with no singular terms, have been explicitly derived. Furthermore, the limiting values of the particular solution kernels have been evaluated. To illustrate the validity and accuracy of the present RBFs, some numerical examples are examined and compared to the results of analytical and other RBFs reported in the literature. In comparison with other RBFs, IMQ RBFs represent more accurate results, using a smaller degree of freedom, and hence they are more efficient.

Keywords: Boundary elements, Dual reciprocity method, IMQ radial basis function, 2D elastodynamics

1. INTRODUCTION

Boundary elements have been developed as a versatile and powerful alternative to finite elements especially in situations where higher accuracy is required for problems such as stress concentration or infinite domains (Brebbia et al., 1984). Early elastodynamics formulations of BEM may be assigned to the researches of Friedman and Shaw (1962), Banaugh and Goldsmith (1963), and Cruse and Rizzo (1968). Three main elastodynamics formulations of BEM have been studied in the literature: the time domain, the Laplace transform, and the domain integral techniques. The first two methods suffer from the mathematical complications involved in their formulations, and the third one needs domain integrations (see for example: Dominguez, 1993; Mansur, 1983; Karabalis and Beskos, 1984, and Antes, 1985). In order to resolve these problems, Brebbia and Nardini in their applications, Nardini and Brebbia (1983), Brebbia and Nardini (1983), and Nardini and Brebbia (1985) developed the well-known dual reciprocity method (DRM). In this method, the integral equation of the domain is expressed in terms of boundary integrals as well a domain integral related to the domain inertia terms. By means of a new collocation method to approximate the inertia terms (or, domain accelerations), they transformed the domain integral to the boundary integral to eliminate the domain terms from formulations. In this collocation method, various classes of approximation functions, which are called the radial basis functions (RBFs), may be used (Golberg and Chen, 1994). As the RBFs directly influence the accuracy of the results of the DRM, the choice of suitable RBFs play a significant role in the solution. From a general point of view, it seems that the RBFs may be categorized into two main classes, namely globally and locally based RBFs. Among various RBFs, the commonly used conical functions (Nardini and Brebbia, 1983; Brebbia and Nardini, 1983; Nardini and Brebbia, 1985, and Golberg and Chen, 1994), the thin plate splines (Golberg and Chen, 1994; Agnantiaris et al., 1996; Bridges and Wrobel, 1994; Golberg, 1995; Chen, 1995; Karur and Ramachandran, 1995, and Mehraeen and Noorzad, 2001), the Gaussian functions (Agnantiaris et al., 1996, and Rashed, 2002a), multiquadrics (Golberg and Chen, 1994; Agnantiaris et al., 2001; Samaan and Rashed, 2007, and Samaan and Rashed, 2009), Sinusoidal (Rashed, 2008), Fourier (Hamzeh Javaran et al., 2011a), and J-Bessel (Hamzeh Javaran et al., 2011b, and Fornberg et al, 2006) belong to the globally based class, while compact supported RBFs (Wendland, 1995; Chen et al., 1999; Golberg et al., 2000; Rashed, 2002b, and Samaan et al., 2007) are locally based ones. The authors (Hamzeh Javaran et al., 2011a, and Hamzeh Javaran et al., 2011b) proposed two types of oscillatory RBF, Fourier and J-Bessel, for the analysis of 2D transient elastodynamic problems.

In this study, the application of DRM in solving 2D transient elastodynamic problems has been extended by employing a set of globally RBFs, called Inverse Multiquadric RBFs in the literature (Fornberg et al., 2006). The particular solution kernels of IMQ RBFs corresponding to displacement and traction are derived. These solutions are presented in closed-form expressions that involve no singular terms. When the field point coincides to the source point, the limiting case emerges. Two numerical examples have been tested. To demonstrate the accuracy and validity of the proposed method, the results have been compared to the results obtained from analytical solution and other RBFs reported in the literature.

2. DUAL RECIPROCITY METHOD (DRM) FORMULATION

A reciprocal relation can be obtained between an elastodynamic state and an elastostatic one. For any general point $\boldsymbol{x}$ of a two-dimensional (2D) body Ω with boundary Γ, the equilibrium equation for an elastodynamic state, in the absence of body forces, is given in the following form (Dominguez, 1993)

$$\mu u_{k,jj} + (\lambda + \mu) u_{j,jk} - \rho \ddot{u}_k = 0 \tag{2.1}$$

where tensor notation ($k = 1,2$) is used: $u_k = u_k(\boldsymbol{x})$ represents the displacement field, μ and λ are the Lame's constants, ρ is the mass density, the over dots indicate differentiation with respect to time, and comma is derivative with respect to spatial coordinates of the domain point. The solution of Eqn. 2.1 may be assumed as $u_k = u_k^c + u_k^p$; where, u_k^c denotes the complementary/homogeneous solution, and u_k^p represents the particular one.

Assuming a continuous weight function (or, so-called fundamental solution) defined over Ω, writing a weighted residual form of Eqn. 2.1, and using Green's identity, the following equation is obtained (Brebbia et al., 1984)

$$c_{lk}(\xi) u_k(\xi) + \int_\Gamma p_{lk}^* u_k \,\mathrm{d}\Gamma = \int_\Gamma u_{lk}^* p_k \,\mathrm{d}\Gamma - \int_\Omega u_{lk}^* \rho \ddot{u}_k \,\mathrm{d}\Omega \tag{2.2}$$

where $c_{lk}(\xi) = \delta_{lk}$ if $\xi \in \Omega$, $c_{lk}(\xi) = 0$ if $\xi \notin \Omega$, and $c_{lk}(\xi) = \frac{1}{2}\delta_{lk}$ if $\xi \in \Gamma$ and the boundary is smooth at ξ. In addition, for the source point ξ and the field point $\boldsymbol{x}$, $u_{lk}^* = u_{lk}^*(\xi, \boldsymbol{x})$ and $p_{lk}^* = p_{lk}^*(\xi, \boldsymbol{x})$ represent the two-point displacements and tractions fundamental solutions (Brebbia et al., 1984), respectively. Moreover, p_k indicates the surface stress (or, traction).

Since the last term of Eqn. 2.2 is a domain integral, it is not possible to develop a boundary integral equation formulation unless the domain integral is taken to the boundary. This acceleration domain term may be transformed to the boundary by means of the dual reciprocity method (DRM) as the following relation (for more details, see Dominguez, 1993):

$$-\int_\Omega u_{lk}^* \rho \ddot{u}_k \,\mathrm{d}\Omega = c_{lk}(\xi) u_k^p(\xi) + \int_\Gamma p_{lk}^* u_k^p \,\mathrm{d}\Gamma - \int_\Gamma u_{lk}^* p_k^p \,\mathrm{d}\Gamma \tag{2.3}$$

where u_k^p and p_k^p concerns to the particular solution for an infinite domain with no boundary conditions. These particular solutions may be analytically achieved from Eqn. 2.1, $\mu u_{k,jj}^p + (\lambda + \mu) u_{j,jk}^p - \rho \ddot{u}_k = 0$, if the inertia term $\rho \ddot{u}_k$ is a known function. In practice, this expression is an unknown function for which, an approximate alternative consisting of radial basis functions is proposed in the next section.

3. PROPOSED RADIAL BASIS FUNCTIONS

In order to find the mentioned particular solutions, the inertia term $\rho \ddot{u}_k = \rho \ddot{u}_k(\boldsymbol{x})$ over the domain Ω is approximated employing a set of unknown coefficients $\alpha_k^m = \alpha_k(\boldsymbol{y}^m)$ and a class of RBFs denoted by $f^m = f(\boldsymbol{x}, \boldsymbol{y}^m)$, where $m = 1, 2, ..., M$ indicates the number of class member, and $\boldsymbol{x}$ and $\boldsymbol{y}^m$ denote the new source and field points, respectively (Dominguez, 1993). Therefore, we can write

$$\rho \ddot{u}_k = \sum_{m=1}^{M} f^m \alpha_k^m \tag{3.1}$$

The main question may arise here: which type of functions are the most appropriate one to be used as RBFs? In this paper, one is selected to be a class of global RBFs namely inverse multiquadric (IMQ). Altogether IMQ RBF is defined as the following form

$$f(r) = \frac{1}{\sqrt{1 + (\varepsilon r)^2}} \tag{3.2}$$

The general set of IMQ RBFs can be represented by the following equation

$$f^m = \frac{1}{\sqrt{1 + (\varepsilon r^m)^2}}, \quad 0 \le r^m \le r_{\max}^m \tag{3.3}$$

where $r^m = \left| \boldsymbol{x} - \boldsymbol{y}^m \right|$ is the distance between $\boldsymbol{x}$ and $\boldsymbol{y}^m$ points.

The displacement and traction fields of this particular solution may be respectively introduced via new fictitious displacement and traction kernels as the following form (Brebbia and Nardini, 1983)

$$u_j^p = \sum_{m=1}^{M} \psi_{jl}^m \alpha_l^m \tag{3.4}$$

and

$$p_j^p = \sum_{m=1}^{M} \eta_{jl}^m \alpha_l^m \tag{3.5}$$

where $\psi_{jl}^{m}=\psi_{jl}(\boldsymbol{x},\boldsymbol{y}^{m})$ and $\eta_{jl}^{m}=\eta_{jl}(\boldsymbol{x},\boldsymbol{y}^{m})$ are appropriate fictitious kernel functions, which will be introduced in the following paragraph. Furthermore, $l=1,2$ indicates the direction of the corresponding load.

Substituting from Eqns. 3.1 and 3.4 into Eqn. 2.1 and using the Galerkin vector decomposition method, the following expressions may be derived after some algebraic manipulations (for more details, see Hamzeh Javaran et al, 2011a):

$$\nabla^4 G=\frac{1}{\mu}f^{m} \tag{3.12}$$

$$\psi_{kl}^{m}=\delta_{kl}\left[\frac{G'}{R}+G''\right]-\frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{G'}{R}(\delta_{kl}-R_{,k}R_{,l})+G''R_{,k}R_{,l}\right] \tag{3.13}$$

$$\begin{aligned}\eta_{kl}^{m}=(n_l R_{,k}+\delta_{kl}R_{,n})\mu\left[\frac{\lambda}{\lambda+2\mu}\left(\frac{G'}{R^2}-\frac{G''}{R}\right)+G'''\right]+\\ n_k R_{,l}\mu\left[\left(\frac{G'}{R^2}-\frac{G''}{R}\right)+\frac{\lambda}{\lambda+2\mu}G'''\right]-R_{,k}R_{,l}R_{,n}\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\left[3\left(\frac{G'}{R^2}-\frac{G''}{R}\right)+G'''\right]\end{aligned} \tag{3.14}$$

where δ_{kl} indicates the Kronecker delta symbol, G presents the Galerkin method parameter, $R\ (=r^{m})$ is chosen for simplicity, n_l indicates the components of the normal vector for which $R_{,n}=R_{,l}n_l$. In addition, G',G'' and G''' are the derivatives of G with respect to R that will be derived in section 4.

4. PARTICULAR SOLUTION OF BI-HARMONIC EQUATION

Since we assume that fictitious body is an infinite domain, each point of this domain may be supposed to be the center of a circle of infinite radius. Consider an elastostatic state in the infinite domain, in the absence of boundary conditions. Therefore, we adopt a polar coordinate system, whose derivatives with respect to angular variable (θ) vanishes owing to cylindrical symmetry. Neglecting the θ variable in polar coordinate system, the bi-harmonic operator may be written as follows:

$$\nabla^4=\nabla^2\nabla^2=\left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}\right)\left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}\right)=\frac{\partial^4}{\partial R^4}+\frac{2}{R}\frac{\partial^3}{\partial R^3}-\frac{1}{R^2}\frac{\partial^2}{\partial R^2}+\frac{1}{R^3}\frac{\partial}{\partial R} \tag{4.1}$$

Whereas, G function do not use in calculation of ψ_{kl}^{m} and η_{kl}^{m} we can sort differential equation 3.12 by G' and straightly it is obtained from solving differential equation.

$$\frac{\partial^3}{\partial R^3}(G')+\frac{2}{R}\frac{\partial^2}{\partial R^2}(G')-\frac{1}{R^2}\frac{\partial}{\partial R}(G')+\frac{1}{R^3}(G')=\frac{1}{\mu\sqrt{1+(\varepsilon R)^2}}. \tag{4.2}$$

Homogeneous and particular solutions of aforesaid differential equation can be obtained as follows

$$G'^{c} = C_1 R + \frac{C_2}{R} + C_3 R \ln R \tag{4.3}$$

where

$$\begin{aligned} G'^{p} = \frac{1}{12\mu\varepsilon^4 R}\{&-3\varepsilon^2 R^2 + 4\varepsilon^2 R^2\sqrt{1+\varepsilon^2 R^2} + 6\varepsilon^2 R^2 \ln R \\ &-6\varepsilon^2 R^2 \ln\left(\frac{1+\sqrt{1+\varepsilon^2 R^2}}{2}\right) - 2\sqrt{1+\varepsilon^2 R^2}\} \end{aligned} \tag{4.4}$$

C_i in complementary solution is selected as singularity problem is not generated in continuance.

$$G' = G'^{c} + G'^{p} \tag{4.5}$$

where

$$C_1 = 0, \quad C_2 = \frac{1}{6\mu\varepsilon^4}, \quad C_3 = -\frac{1}{2\mu\varepsilon^2} \tag{4.6}$$

After simplification, the first derivative of Galerkin function is obtained as the following form:

$$G' = \frac{1}{12\mu\varepsilon^4 R}\{4Z^3 - 3Z^2 - 6Z - 6(Z^2-1)\ln\left(\frac{Z+1}{2}\right) + 5\} \tag{4.7}$$

The second and third derivative of G function may be easily expressed as

$$G'' = \frac{1}{12\mu\varepsilon^2 (Z+1)}\{8Z^2 - Z - 6(Z+1)\ln\left(\frac{Z+1}{2}\right) - 7\} \tag{4.8}$$

$$G''' = \frac{R(4Z+5)}{6\mu(Z+1)^2} \tag{4.9}$$

where $Z = \sqrt{1+\varepsilon^2 R^2}$ is selected for simplicity.

By substituting Eqns. 4.7, 4.8 and 4.9 into Eqns. 3.13 and 3.14, the final simplified relations of ψ_{kl}^{m} and η_{kl}^{m} may be obtained as followings:

$$\psi_{kl}^{m} = \frac{\delta_{kl}}{12\mu(\lambda+2\mu)\varepsilon^{2}(Z+1)}\{4(2\lambda+5\mu)Z^{2}-(\lambda+\mu)Z-6(\lambda+3\mu)(Z+1)\ln\left(\frac{Z+1}{2}\right)- (7\lambda+19\mu)\}-\frac{(\lambda+\mu)R_{,k}R_{,l}}{6\mu(\lambda+2\mu)\varepsilon^{2}(Z+1)}\{2Z^{2}-Z-1\} \tag{4.10}$$

and

$$\eta_{kl}^{m} = \frac{n_l R_{,k}+\delta_{kl}R_{,n}}{3(\lambda+2\mu)\varepsilon^{2}R(Z+1)}\{(\lambda+4\mu)Z^{2}+(\lambda+\mu)Z-(2\lambda+5\mu)\}+ \frac{n_k R_{,l}}{3(\lambda+2\mu)\varepsilon^{2}R(Z+1)}\{(\lambda-2\mu)Z^{2}+(\lambda+\mu)Z-(2\lambda-\mu)\}+ \frac{2(\lambda+\mu)R_{,k}R_{,l}R_{,n}}{3(\lambda+2\mu)\varepsilon^{2}R(Z+1)}\{Z^{2}-2Z+1\} \tag{4.11}$$

It should be noted that after appropriate selection C_i in complementary solution, the expressions of ψ_{kl}^{m} and η_{kl}^{m} kernels are nonsingular and their limiting values of coincidence of collocation point and boundary node, as $R\rightarrow 0$, can be evaluated as the following forms:

$$\lim_{R\rightarrow 0}\psi_{kl}^{m} = \lim_{R\rightarrow 0}\eta_{kl}^{m} = 0 \tag{4.12}$$

5. NUMERICAL DISCRETIZATION

By substituting Eqn. 2.3 into Eqn. 2.2 and using Eqns. 3.4 and 3.5, the following expression may be obtained:

$$c_{lk}(\xi)u_k(\xi)+\int_{\Gamma} p_{lk}^{*}u_k \mathrm{d}\Gamma = \int_{\Gamma} u_{lk}^{*}p_k \mathrm{d}\Gamma + \sum_{m=1}^{M}[c_{lk}(\xi)\psi_{kj}^{m}+\int_{\Gamma} p_{lk}^{*}\psi_{kj}^{m}\mathrm{d}\Gamma-\int_{\Gamma} u_{lk}^{*}\eta_{kj}^{m}\mathrm{d}\Gamma]\alpha_{j}^{m} \tag{5.1}$$

This equation can be employed as the basic equation for a boundary element approach with no domain integration. Quadratic shape functions are used for any element to represent its displacements and tractions. Finally, the results of discretizing the boundary into elements can be written in compact form as follows:

$$\mathbf{M\ddot{u}}+\mathbf{Hu}=\mathbf{Gp} \tag{5.2}$$

where $\mathbf{M}$ is mass matrix, $\mathbf{\ddot{u}}$ is vector of acceleration, and $\mathbf{H}$, $\mathbf{u}$, $\mathbf{G}$, and $\mathbf{p}$ are the usual boundary element vectors and matrices (for more details, see Dominguez, 1993)

Eqn. 5.2 shows a system of dynamic equilibrium equations that can be solved with a time-marching scheme. Given certain boundary conditions and initial values, these equations are solved according to the Houbolt finite difference scheme (Zienkiewicz and Taylor, 2000).

6. NUMERICAL EXAMPLES

The above-mentioned formulation has been implemented in a 2D BEM code in which, a minimum required number of internal points is considered. In order to verify the nature and general behavior of the method, two numerical examples are taken into account. To evaluate the accuracy and stability of the proposed method, numerical results are compared with those obtained by exact analytical solutions and/or by other boundary element results available in the literature.

6.1. Rectangular domain under step function loading

In the first example, verification of the algorithm especially in comparison with exact analytical solutions is the main aim. A rectangular domain in plane strain condition (see Fig. 6.1), whose height is twice its width, with tangential-traction free boundary condition sides is considered. The top side of the rectangle domain is uniformly subjected to a Heaviside step function $P(t) = H(t - 0)$ representing a suddenly unit applied load as shown in Fig. 6.1. The material constants are ρ = 1.0 kg/m^3, $\mu = 4 \times 10^4$ Pa, and $\lambda = 4 \times 10^4$ Pa. Six boundary elements with no internal points are employed to discretize the rectangular domain, as depicted in Fig. 6.1. The numerical results of the proposed RBF, for cases in which ε = 0.25, 0.5, 0.75, are compared with that of the analytical solution (Dominguez, 1993). Moreover, the results of two other RBFs (i.e., CSRBF (Rashed, 2002b) and MQRBF (Samaan and Rashed, 2007)) are considered for more comparison. It should be noted that the results of Refs. (Samaan and Rashed, 2007, and Rashed, 2002b) are obtained using 12 boundary elements as well as a few internal points, while the results of the proposed IMQ RBF correspond to a mesh of 6 boundary elements with no internal point. Figs. 6.2 and 6.3 illustrate the vertical displacement of point A and normal traction histories at point B, respectively. From these figures, one may easily observe that the results of the present RBF show a good agreement with the analytical solution.

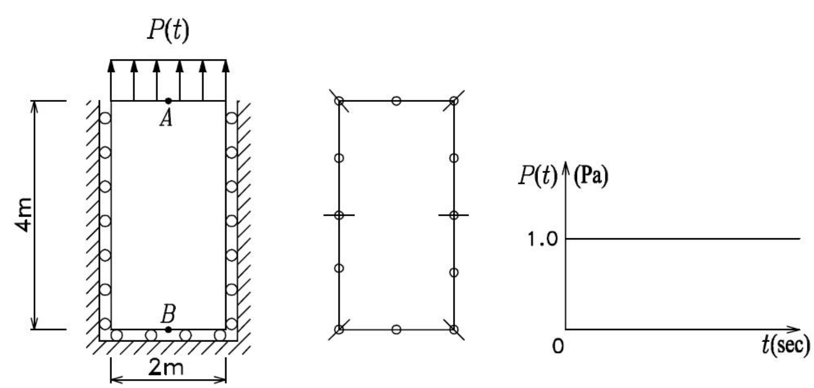

Figure 6.1. The geometry of the rectangular plane strain domain, modeled by 6 quadratic boundary elements with no internal points, and the time history of the normal traction

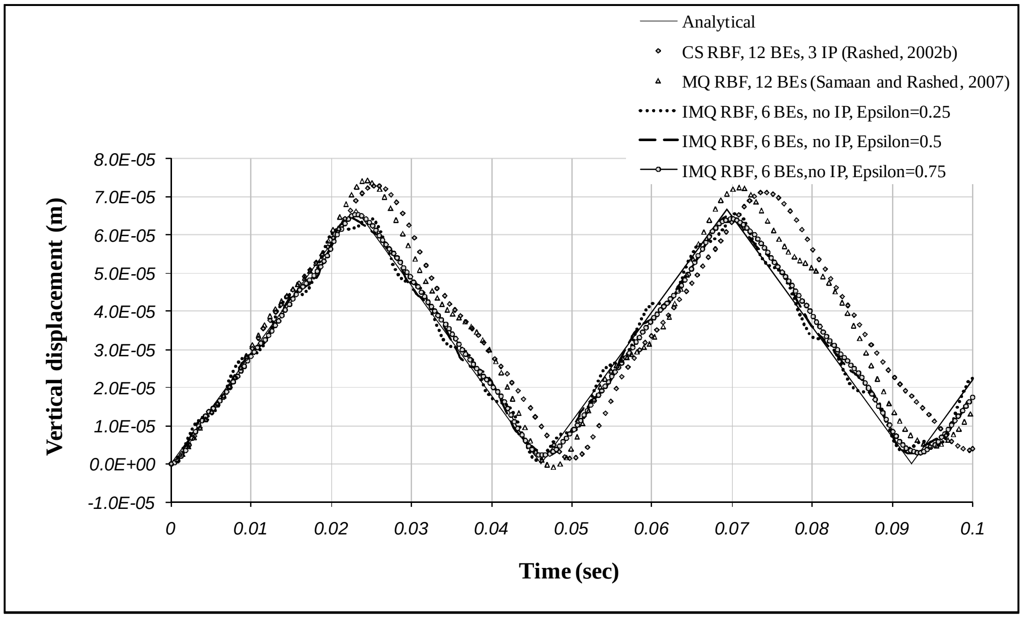

Figure 6.2. Dynamic response of vertical displacement at point A for the first example

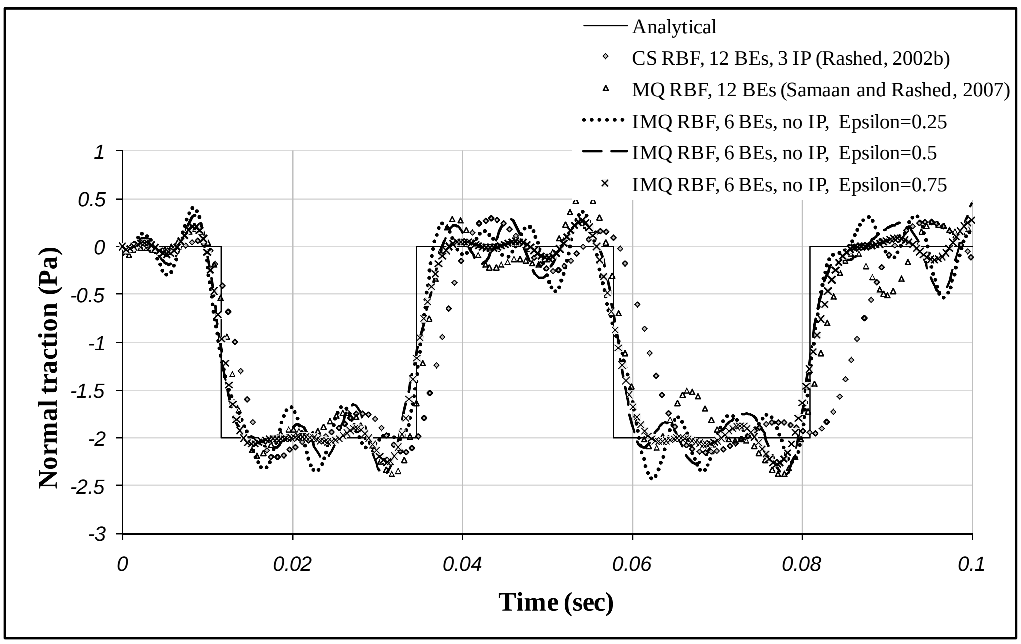

Figure 6.3. Dynamic response of normal traction at point B for the first example

6.2. Perforated strip subjected to tension stress

The second example illustrates a rectangular perforated plate in axial tension under plane stress conditions (Fig. 6.4). The left and right edges of the plate are subjected to Heaviside step function loading $P(t) = 7500H(t - 0)$. The material constants and geometrical properties of this example are as follows: $\rho = 0.00785$ kg/m^3,$\mu = 8.08\times10^6$ N/cm^2, $\lambda = 1.21 \times 107$ N/cm^2, length is 36 cm, and breadth is 20 cm. Due to the symmetry condition (Fig. 6.4), only one quarter of the plate is assumed in BEM analyses. In addition, the results of two other RBFs (i.e., $(1 + R)$ RBF (Agnantiaris et al., 1996), and CSRBF (Rashed, 2002b)) are illustrated for more comparison. According to three shape parameters (ε = 0.25, 0.5, 0.75) from the proposed RBF, the displacement time history of point A (Fig. 6.4) is shown in Fig. 6.5 on which the results of two other RBFs (Agnantiaris et al., 1996, and Rashed, 2002b) are also drawn.

In this example, the boundary is modeled into 15 boundary elements, and 17 internal nodes are employed to improve the accuracy of the obtained results (see Fig. 6.4). As may be observed from Fig. 6.5, the results of the present study in the all cases are in good agreement with the results of Refs. (Agnantiaris et al., 1996, and Rashed, 2002b) which have been obtained using much more degrees of freedom (37 boundary elements with 136 internal points).

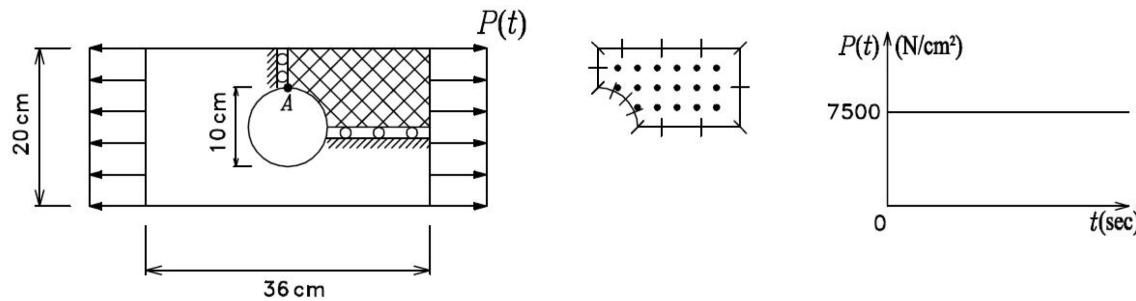

Figure 6.4. The geometry of the perforated plate subjected to edge tensile traction, modeled by 15 quadratic boundary elements and 17 internal points, and the time history of the traction

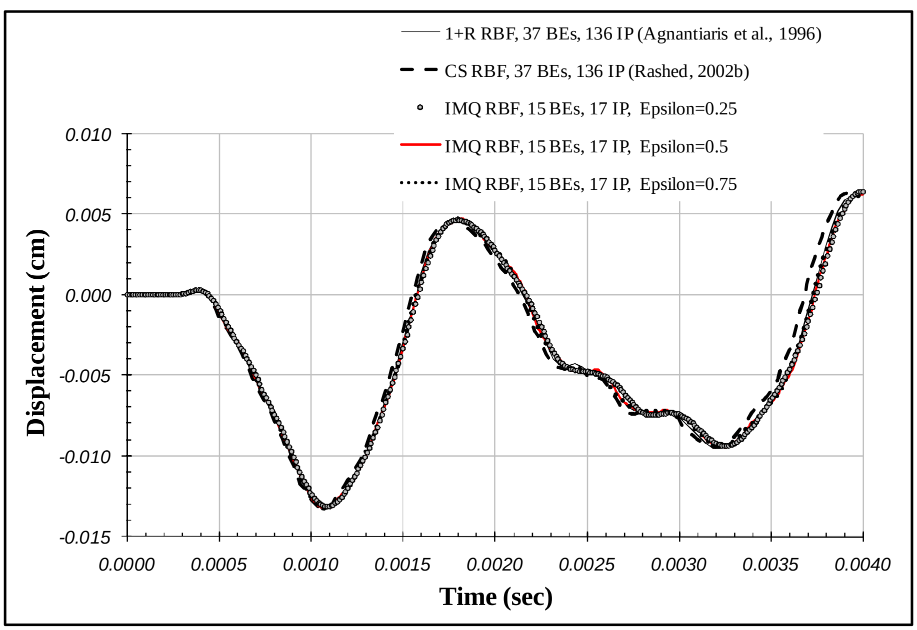

Figure 6.5. Displacement time history at point A for the second example

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