A Comparison of Closed-form and Finite-Element Solutions for the Free Vibration Analysis of a Sloping-frame

Davoud Nezamolmolki
Civil Engineering Department, Ferdowsi University, Mashhad, Iran.

Ahmad Aftabi Sani
Civil Engineering Department, Mashhad Branch, Islamic Azad University, Mashhad, Iran.

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SUMMARY:
This article deals with the free vibration analysis and determination of the seismic parameters of a sloping-frame. The members of the frame are assumed to be governed by the transverse vibration theory of Euler-Bernoulli beam. To solve this classical problem, a closed-form solution is firstly proposed and then, a numerical analysis is performed for some verification purposes. The closed-form solution is developed by solving the frame equations of motion, directly. For this reason, some mathematical techniques are utilized, such as Fourier transform and the well-known complementary solutions. In this way, some differential equations must be solved, and several boundary conditions should be satisfied. Moreover, these results are obtained by the use of the finite element method. In this comparison process, some differences are observed between the closed-form and numerical results. This fact motivated us to propose some modifications in the characteristic matrices of the finite element model of the frame.

Keywords: Free vibration analysis, sloping-frame, boundary condition, Euler-Bernoulli theory.

1. INTRODUCTION

The free vibration analysis of beams and frames is an important problem in the structural engineering. In view of that, many researchers have devoted themselves to the study of this field, with more concentration on the beams [Kim 2001, Albarracin *et al.* 2004, Li 2001, Li 2000, Firouz-Abadi *et al.* 2007, Failla and Santini 2008] in contrast with the frames [Albarracin and Grossi, 2005]. Moreover, the study of the closed-form solution of vibrating frame structures along with the numerical solution and the comparison of these two approaches, in order to examine the accuracy and the precision of the numerical ones, such as Finite Element Method (FEM), Boundary Element Methods (BEM), etc., is hardly considered in the literature. Also, in contrast to the body of available information, the new sufficient data was not found for sloping-frames with variable slopes.

This article deals with the free vibration analysis and determination of the seismic parameters of a sloping-frame which consists of three members; a horizontal, a vertical, and an inclined member. The both ends of the frame are clamped, and the members are rigidly connected at joint points. The individual members of the frame are assumed to be governed by the transverse vibration theory of an Euler-Bernoulli beam. To solve this classical problem, a closed-form solution is firstly proposed and then, a numerical analysis is performed for some verification purposes. The closed-form solution is developed by solving the frame equations of motion, directly. For this reason, some mathematical techniques are utilized, such as Fourier transform and the well-known complementary solutions. In this way, some differential equations must be solved, and several boundary conditions should be satisfied. Herein, the more accurate derivation of the last boundary condition is the most important challenge of this work. This boundary condition is expressed as three distinctive versions, and the free vibration parameters of the frame are attained for these three versions. The additional descriptions could be found elsewhere in detail [Nezamolmolki and Aftabi Sani, 2012]. Moreover, the results are obtained by the use of the FEM. In this comparison process, some differences are observed between the closed-form and the numerical results. This fact motivated us to propose some modifications in the characteristic matrices of the finite element model of the frame, with focus on the mass matrix in this article. Finally, the natural frequencies are presented for a wide range of angles of the sloping member.

2. DEFINITION OF THE PROBLEM

Now consider a sloping-frame as shown in Fig. 2.1. The members at joint points are rigidly connected. The behavior of the individual members of the frame is assumed to be governed by the Euler-Bernoulli theory and the axial deformations effects are neglected. The geometrical and mechanical properties and the length of three uniform members are the same. The flexural rigidity of the member is denoted by EI. Also, ρ is the mass density and A is the cross-sectional area of the bending member.

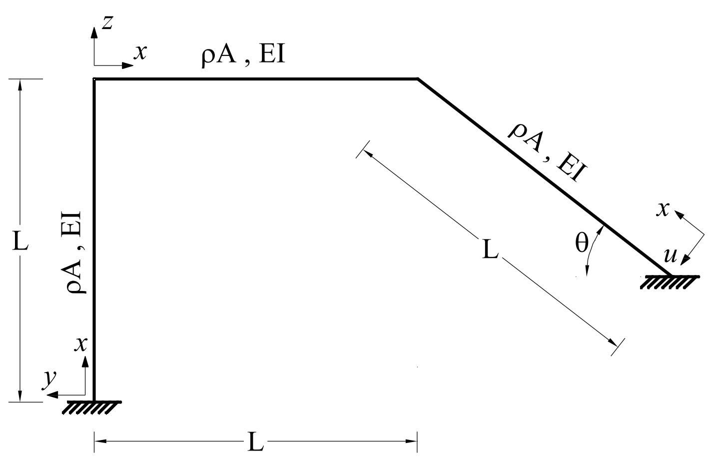

Figure 2.1. The semi-inclined frame under study

The angle among the inclined member and the horizontal direction is shown by θ which is assumed between 0° and 90°. Furthermore, the displacement functions for the vertical, horizontal and inclined members are y, z and u, respectively.

2.1. Differential Equation

Consider a uniform Euler- Bernoulli beam as an individual member of the frame shown in Fig. 1. The equation of motion for free flexural vibrations of a uniform elastic beam ignoring shear deformation and rotary inertia effects is:

$$\mathrm{EI}\frac{\partial^4 y(x,t)}{\partial x^4} + \rho\,\mathrm{A}\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{2.1}$$

where $y(x,t)$ is the lateral displacement at distance x along the length of the beam and time t, EI the flexural rigidity of the beam, ρ the mass density and A the cross-sectional area of the beam.

As it is clear, Eqn. 2.1 is expressed in the time domain which the Fourier transform can easily convert it into the frequency domain. Therefore:

$$Y^{IV} - \alpha^4 Y = 0 \qquad ; \qquad \left(\alpha^4 = \frac{\rho\,\mathrm{A}}{\mathrm{EI}}\omega^2\right) \tag{2.2}$$

where $Y(x,\omega)$ is the Fourier transform of $y(x,t)$ and ω the frequency. Obviously, Eqn. 2.2 is a homogenous ordinary differential equation with the following complementary solution:

$$Y(x,\omega) = c_1 \sin\alpha\, x + c_2 \cos\alpha\, x + c_3 \sinh\alpha\, x + c_4 \cosh\alpha\, x \tag{2.3}$$

Let us return to the frame shown in Fig. 1. Due to the fact that three members of the frame has the same $\rho\mathrm{A}$, EI and ω, then the general form of the above-mentioned solution is similar for all members. However, the coefficients are different and consequently, each element has an individual displacement function: $Y(x,\omega)$ for vertical member, $Z(x,\omega)$ for the horizontal and $U(x,\omega)$ for inclined one

$$Y(x,\omega) = c_1 \sin\alpha x + c_2 \cos\alpha x + c_3 \sinh\alpha x + c_4 \cosh\alpha x \tag{2.4}$$

$$Z(x,\omega) = c_5 \sin\alpha x + c_6 \cos\alpha x + c_7 \sinh\alpha x + c_8 \cosh\alpha x \tag{2.5}$$

$$U(x,\omega) = c_9 \sin\alpha x + c_{10} \cos\alpha x + c_{11} \sinh\alpha x + c_{12} \cosh\alpha x \tag{2.6}$$

After this, capital letters Y, Z and U are replaced by y, z and u for simplicity in notation.

2.2. Boundary Conditions (B.C.s)

As it is clear from Eqn. 2.4-2.6, the entire system has 12 unknown constants, which can be solved through the satisfaction of 12 boundary conditions. These boundary conditions are thoroughly illustrated elsewhere in detail. Herein, only the continuity conditions of displacements between each pair of the three bars meeting at the joints are presented for instance.

$$y(\mathrm{L}) = u(\mathrm{L}) \sin\theta \tag{2.7}$$

$$z(0) = 0 \tag{2.8}$$

$$y(\mathrm{L}) = -z(\mathrm{L}) \tan\theta \tag{2.9}$$

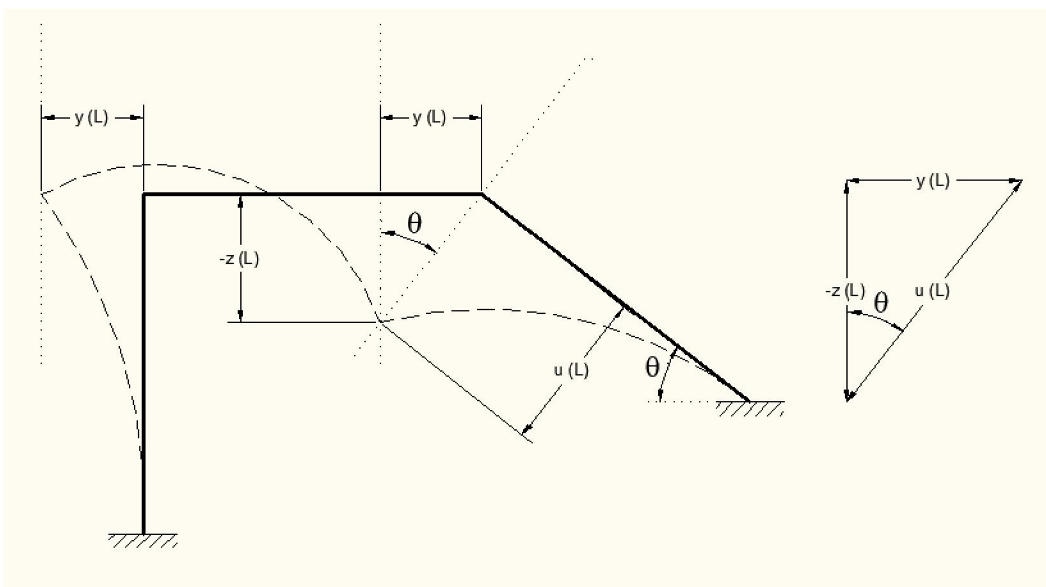

Figure 2.2. Bending deformation diagram based on neglecting of the axial deformation

The other boundary conditions, such as the natural boundary conditions of the fixed ends, the continuity conditions for slopes, moments at the joints, and the equation of equilibrium of forces and moments which could be governed and expressed in three versions, are omitted for brevity.

2.3. Determination of the closed-form solution

Substituting Eqn. 2.4-2.6 in boundary conditions, one obtains a set of 12 homogeneous equations in the constants $c_i, i = 1, 2, ..., 12$. Since the system is homogeneous for existence of a non-trivial solution the determinant of coefficients must be equal to zero. This procedure yields the frequency equation:

$$S(\mathrm{L}, \alpha, \theta) = 0 \tag{3.1}$$

It should be mentioned, the set of 12 homogeneous equations could be easily converted to the set of 7 homogeneous equations. The linear systems corresponded to each version of 12$^{\text{th}}$ boundary condition are only different in the last row.

3. FINITE ELEMENT METHOD OF NUMERICAL SOLUTION

In the finite element method, we divide the given frame (beam and columns) into several elements and assume a suitable solution within each of the elements. From this we formulate the necessary equations from which the approximate solution can be obtained easily.

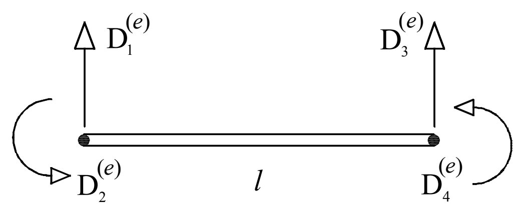

Figure 3.1. The bending element

the stiffness matrix $\mathbf{K}^{(e)}$ and mass matrix $\mathbf{M}^{(e)}$ as [Rao, 2004]:

$$\mathbf{K}^{(e)} = \frac{2\,\mathrm{EI}}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} \tag{3.1}$$

$$\mathbf{M}^{(e)} = \frac{\rho \mathrm{A} l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{3.2}$$

Obviously, to obtain the more accurate results, the frame should be divided into smaller elements. Herein, each member is subdivided into NE elements of equal length. Therefore, the frame contains $3 \times \mathrm{NE}$ elements as shown in Fig. 3.2.

Once the stiffness and mass matrices of the complete frame are available, we formulate the eigenvalue problem as

$$\mathbf{K\,D} = \lambda \mathbf{M\,D} \tag{3.3}$$

where $\mathbf{D}$ is the eigenvector, and λ is the eigenvalue. The solution of Eqn. 3.3 gives us the natural frequencies ($\lambda = \omega^2$) and the corresponding mode shapes of the frame.

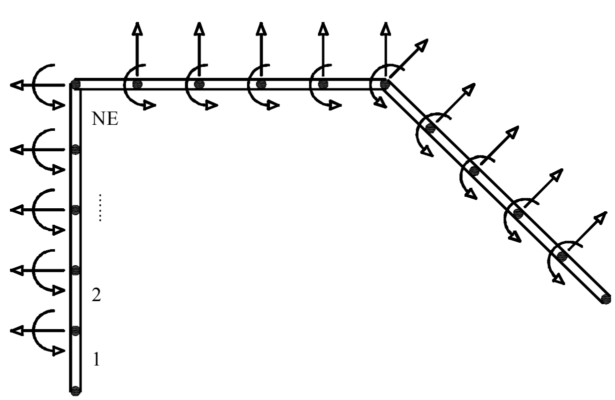

Figure 3.2. Each member divided into NE element

4. RESULTS

As explained in the previous section, the finite element method was utilized as a numerical analysis approach in the present study such as many other studies which are carried out recently in this field. Therefore, a special purpose computer program was developed based on the theory explained in the previous section. The program is utilizing the both original and modified mass matrices. The sloping-frame members are discretized by bending element introduced in Fig.3.1. There are several options available in this analysis tool to add several type of added mass. Also, a simple code was produced for the closed-form solution to obtain the natural frequencies and mode shapes of the frame. Utilizing the above-mentioned programs, the free vibration responses of sloping-frame are obtained for several values of θ, as a discrete spectrum ($\theta = 0, 10, \ldots, 90^{o}$). The first value represents L-shape frame, while $\theta = 90^{o}$ represents the well-known portal frame. It should be mentioned that response quantities which will be presented in this section are the values of dimensionless eigen frequency λ. These values are obtained at the first ten modes for various amounts of angle θ. The results are presented in Tables 4.1-4.2 for several models and $\theta = 0, 10, \ldots, 90^{o}$. As it is clear from the tables, these values are the same for the modified finite element method with added mass and the first version of closed-form solution.

Table 4.1. Frequency Parameters obtained by Finite Element Analysis without Added Mass

	Mode Number									
	1	2	3	4	5	6	7	8	9	10
θ (degree)										
0	2.1878	3.6245	4.3818	5.3538	6.7702	7.5121	8.4941	9.9118	10.6544	11.6361
10	2.2315	3.6309	4.4176	5.3105	6.7592	7.4845	8.5343	9.9183	10.6898	11.5926
20	2.2720	3.6315	4.4609	5.2656	6.7454	7.4642	8.5694	9.9192	10.7327	11.5478
30	2.3064	3.6262	4.5106	5.2214	6.7310	7.4500	8.5976	9.9144	10.7819	11.5038
40	2.3323	3.6156	4.5641	5.1803	6.7183	7.4406	8.6172	9.9046	10.8348	11.4633
50	2.3474	3.6014	4.6175	5.1452	6.7098	7.4348	8.6264	9.8912	10.8873	11.4293
60	2.3491	3.5856	4.6656	5.1198	6.7079	7.4317	8.6233	9.8764	10.9342	11.4055
70	2.3352	3.5710	4.7024	5.1075	6.7152	7.4302	8.6063	9.8627	10.9697	11.3950
80	2.3035	3.5604	4.7238	5.1095	6.7343	7.4297	8.5738	9.8528	10.9900	11.3984
90	2.2525	3.5564	4.7300	5.1216	6.7677	7.4296	8.5254	9.8490	10.9960	11.4114

Table 4.2. Frequency Parameters obtained by Finite Element Analysis with Added Mass (which is exactly similar to the First Version of the Closed-form Solution)

	Mode Number									
	1	2	3	4	5	6	7	8	9	10
θ (degree)										
0	2.1878	3.6245	4.3818	5.3538	6.7702	7.5121	8.4941	9.9118	10.6544	11.6361
10	2.2127	3.6286	4.4151	5.2774	6.7563	7.4827	8.4811	9.9128	10.6843	11.5229
20	2.2014	3.6231	4.4487	5.1539	6.7382	7.4601	8.3927	9.9016	10.7068	11.3396
30	2.1628	3.6112	4.4794	5.0240	6.7231	7.4455	8.2888	9.8869	10.7188	11.1879
40	2.1077	3.5969	4.5050	4.9150	6.7133	7.4371	8.2009	9.8740	10.7213	11.0936
50	2.0447	3.5833	4.5243	4.8347	6.7086	7.4327	8.1339	9.8643	10.7175	11.0420
60	1.9793	3.5720	4.5369	4.7811	6.7077	7.4307	8.0837	9.8573	10.7112	11.0155
70	1.9145	3.5635	4.5429	4.7493	6.7100	7.4299	8.0455	9.8527	10.7050	11.0027
80	1.8513	3.5583	4.5437	4.7341	6.7152	7.4296	8.0157	9.8499	10.7002	10.9973
90	1.7901	3.5564	4.5419	4.7300	6.7233	7.4296	7.9919	9.8490	10.6975	10.9960

For the first and second modes, the amount of eigenvalues λ is plotted versus the values of angle θ for a significant range as shown in Fig. 4.1-4.2. Furthermore, some comparisons are illustrated in each figure for the analytical and numerical approaches. This would help to capture a feeling of the accuracy obtained in the closed-form solution versus the finite element technique. However, prior to this presentation, it is worthwhile to have a glance at the comparison for natural frequencies. It is observed that the response of the first version of the closed-form solution in each mode matches very well with the modified finite element response, such that it is hardly distinguishable from the corresponding exact curve. It should be mentioned that the comprehensive explanations about these three versions of the closed-form solution could be found elsewhere in detail [Nezamolmolki and Aftabi Sani, 2012].

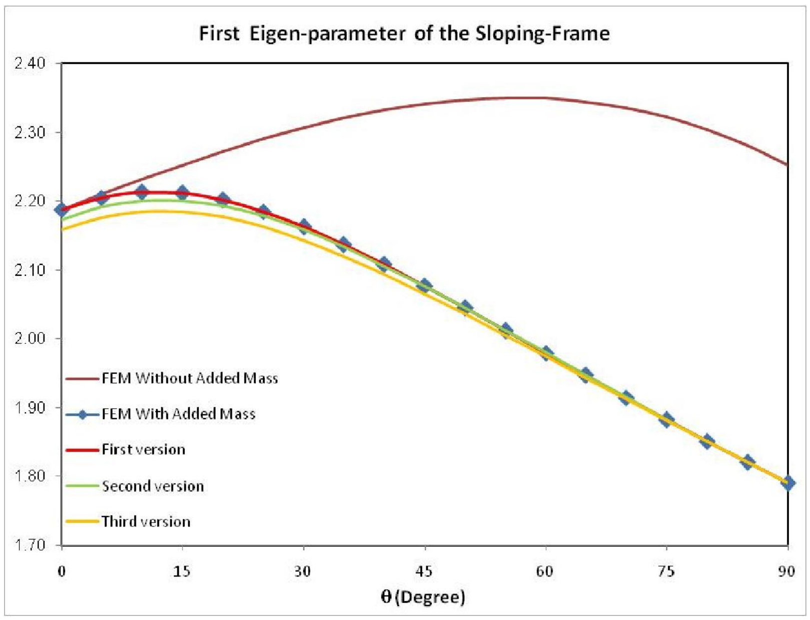

Figure 4.1. The first mode

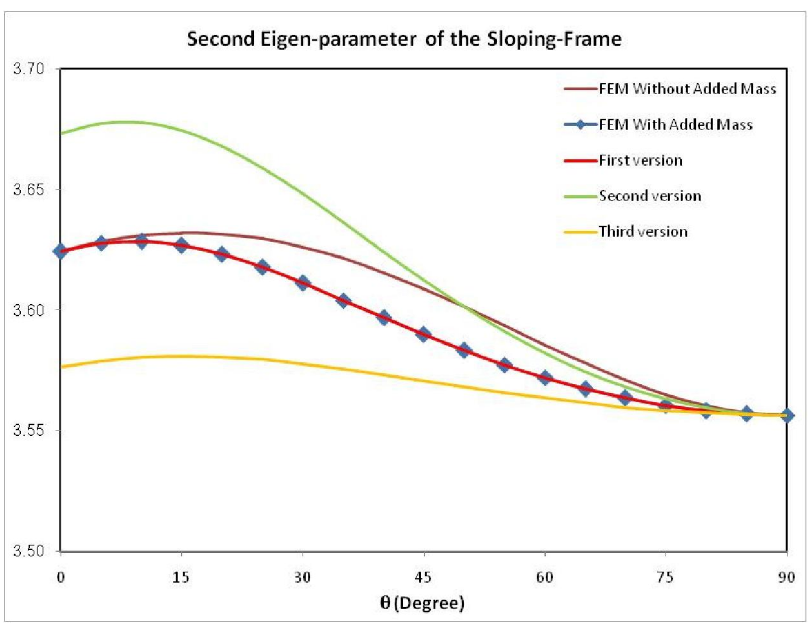

Figure 4.2. The second mode

Also, Fig. 4.3-4.4 illustrates the effect of the number of bending elements (which is denoted by abbreviation NE) on the accuracy of the modified finite element response. This investigation is carried out for the first 10 modes and $\theta = 30, 60^{\circ}$ by using several NE.

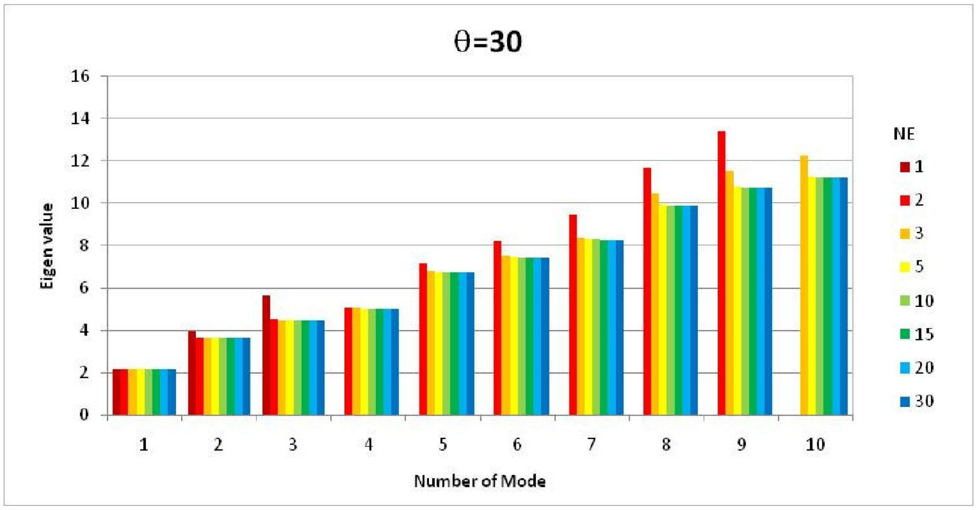

Figure 4.3. Effect of the number of elements (NE) on the accuracy of the modified finite element response $\theta = 30^{\circ}$

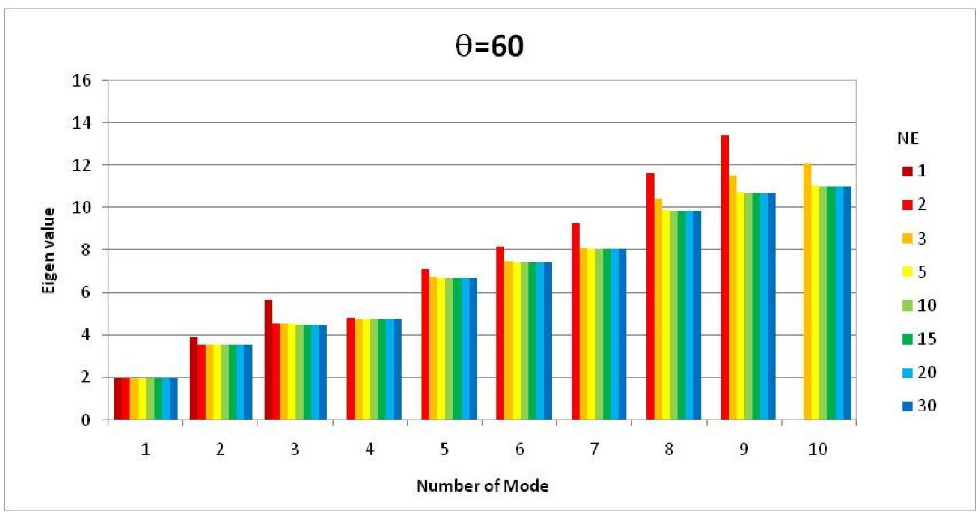

Figure 4.4. Effect of the number of elements (NE) on the accuracy of the modified finite element response $\theta = 60^{\circ}$

As the last result, some mode shapes are illustrated for some values of $\theta = 30, 60^{\circ}$ which are shown in Fig. 4.5-4.6.

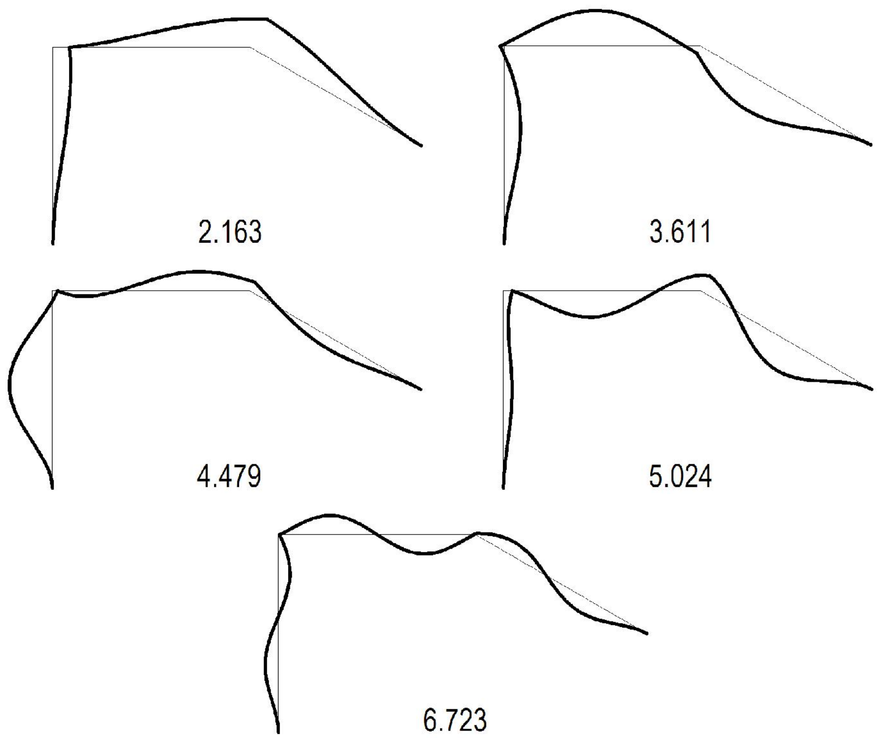

Figure 4.5. The first 5 modes of the sloping-frame with $\theta = 30^o$

Figure 4.6. The first 5 modes of the sloping-frame with $\theta = 60^o$

8. CONCLUSION

The free vibration analysis of a sloping-frame was studied and the seismic parameters of the system were determined. For this reason, a closed-form solution was developed which was based on the satisfaction of the both differential equations and boundary conditions, simultaneously. Also, the finite element method was utilized and the results were obtained to compare with the closed-form solutions. The natural frequencies and mode shapes were presented for different values of the angle θ in inclined member. Overall, the main conclusions obtained by the present study can be listed as follows:

* It is observed that natural frequencies of the original finite element method are generally smaller than the natural frequencies obtained by the analytical approach. However, it is noted that results corresponding to the modified finite element model with change in the mass matrix are getting closer to and coincided with the closed-form solution. Moreover, the comparison between the analytical and numerical techniques in related with the variation of the angle θ in inclined member for several modes are thoroughly discussed, as follow:

* In the first mode, the difference between the natural frequency of the original finite element method and the results of the analytical approach increases gradually as the angle θ grows. The similar trend is observed for the mode number three and four. On the contrary, the mentioned difference increases for the median values of the angle θ in the second mode.

* In general, the parameter θ has a significant effect on the frequency parameters of the sloping-frame. This fact could be clearly proved by the comprehensive sensitivity analysis that was carried out in the result section. Also, this is true for all three versions of the analytical solution considered.

* Almost in all modes, a single special value of θ could be introduced as an optimum value which lead to the maximum/minimum amount of the natural frequency. For example, in the first mode, this optimum value is about ten degrees (lead to max.) and in the fifth mode, it is about sixty degrees (lead to min.).

* As it is obvious from the presented results for all investigated modes, the finite element solutions converges into the first version of closed-form solution, of course by imposing the proposed modification on the mass matrix. It should be mentioned that the best modification which coincided appropriately with the closed-form results was to add the single translational mass which excited only by the horizontal acceleration of the beam. Moreover, the amount of the mentioned mass should be equal to the beam mass.

* Although, the analytical approach is not in general as simple and programmable as the numerical techniques, it is a reliable approach and more accurate. Furthermore, it is much easier than the experimental efforts.

* It is worthwhile to mention that the accuracy of the finite element results decrease with the increase in the mode number. Whereas, the closed-form solution is frequency-independent, and its accuracy is appropriate for both high and low frequencies. This fact could be considered as a valuable object especially for the analysis in near-fault earthquakes which are approximately observed with the high dominant frequencies. In fact, this type of earthquakes excite the high modes of the structure and hence, these modes should be evaluated more accurate which is almost impossible in finite element method.

REFERENCES

Albarracın, C.M., Zannier, L. and Grossi, R.O. (2004). Some observations in the dynamic of beams with intermediate supports. *Journal of Sound and Vibration.* **271**, 475-80.

Albarracin, C.M. and Grossi, R.O. (2005).Vibration of elastically restrained frames. *Journal of Sound and Vibration*. **285**, 467-76.

Kim, H.K. and Kim, M.S. (2001). Vibration of beam with generally restrained boundary conditions using Fourier series. *Journal of Sound and Vibration.* **245**, 771-84.

Failla, G. and Santini, A. (2008). A solution method for Euler–Bernoulli vibrating discontinuous beams. *Mechanics Research Communications*. **35**,517-29.

Firouz-Abadi, R.D., Haddadpour, H. and Novinzadeh, A.B. (2007). An asymptotic solution to transverse free vibrations of variable-section beams. *Journal of Sound and Vibration.* **304**, 530-40.

Li, W.L. (2000). Free vibration of beam with general boundary conditions. *Journal of Sound and Vibration*. **237**, 709-25.

Li, W.L. (2001).Dynamic analysis of beams with arbitrary elastic supports at both ends. *Journal of Sound and Vibration.* **246,** 751-56.

Li, W.L. (2002). Comparison of Fourier sine and cosine series expansions for beam with arbitrary boundary conditions. *Journal of Sound and Vibration.* **255**, 185-94.

Nezamolmolki, D. and Aftabi Sani, A. (2012). Free Vibration Analysis of a Sloping-frame: Closed-form Solution versus Finite Element Solution and Modification of the Characteristic Matrices. To Be Submitted.

Rao, S.S. (2004). The Finite Element Method in Engineering. Elsevier Science & Technology Books.