Rotational ground motion effects from body waves decomposition at a site

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SUMMARY:
Ground rotations may derive from strong, near field seismic effects, in form of special, hypothetical rotational waves or wave reflections/refractions at a site (see e.g., special issue of BSSA – Lee et al., 2009). Recent developments in seismological instrumentation gave the rotational seismology a status of an emerging science. However until clear, strong motion rotational records are acquired in the epicentral areas of major earthquakes, the rotational seismic engineering will still be in its infancy. So far then, the rotational effects are analyzed indirectly from translational measurements. One of such the methods is presented in this paper, where the rotational ground motions (torsion about vertical axis and rocking about horizontal axis) are formulated from the wave decompositions of translational ground motions. Respective formulae for the torsional and rocking power spectral densities are formulated in terms of translational acceleration of seismic components. A brief parametric, numerical analysis is included.

Keywords: ground rotations, body wave reflections, spatial seismic effects, spectral density.

1. INTRODUCTION

The surface seismic rotations, can be an effect of direct surface or gravity wave propagation, hypothetical rotational waves appearing in the near field of strong earthquakes (Teisseyre et al, 2007) or they may appear as an effect of upcoming body waves (Trifunac, 1982). The purpose of this paper is to present, in a concise way, the derivations leading to rocking ground motion (about horizontal axis) and torsional motion (about vertical axis) in terms of three, translational ground motions and respective wave propagation parameters (angles and velocities of propagation) formulated in form of a representation of a stochastic field of the ground motions.

2. FORMULATION OF THE PROBLEM

Consider system of the so called "principal axes" on the ground surface, in which one of the axes is directed towards epicentre and the other one is vertical (see Fig. 1). Penzien and Watabe (1975) have

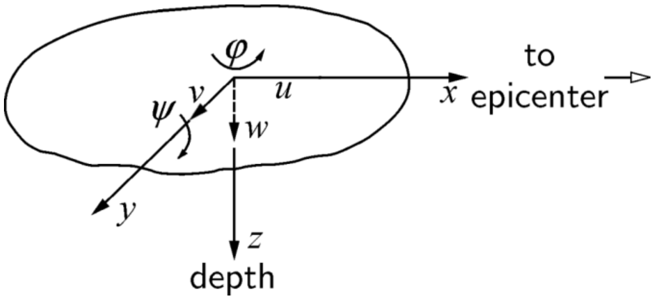

Figure 1. System of principal axes at a site and two rotations φ and ψ

shown that ground motions $u(t)$, $v(t)$ and $w(t)$ appear as uncorrelated. What is more (Zembaty 1997), when spatial seismic effects at two distinct surface points *A* and *B* are analyzed, the respective coherence matrix transforms as a tensor when the system of coordinates is changed. For the system of principal axes the problem of wave propagation at building site can be simplified to plane wabe propagation. In this case, from body waves reflections one can derive rocking $\psi(t)$ about horizontal *y* axis and torsion $\varphi(t)$ about *z* axis. Applying familiar, solid mechanics formulae (see e.g. Castellani & Boffi 1989) one can obtain the two rotations as:

$$\psi(t) = \frac{\partial w(t,x,y)}{\partial x} \tag{2.1}$$

$$\varphi(t) = \frac{1}{2}\left(\frac{\partial v(t,x,y)}{\partial x} - \frac{\partial u(t,x,y)}{\partial y}\right) \tag{2.2}$$

3. ROCKING (ABOUT HORIZONTAL AXIS)

Consider *P* waves incident at an angle Θ_P to the free surface (Fig. 2a). Each incident *P* wave generates a reflected, down going *P* wave, under the same angle $\Theta_{PP} = \Theta_P$ and a reflected *SV* wave at an angle Θ_{PS} with the angles and propagation velocities c_P (for *P* waves) and c_S (for shear waves - *S*) fulfilling the familiar geometrical optics formula.

$$\sin(\Theta_P)/\sin(\Theta_{PS}) = c_P / c_S = S \tag{3.1}$$

An analogous formula can be written for the incident SV wave:

$$\sin(\Theta_S)/\sin(\Theta_{SP}) = c_S / c_P = 1/S \tag{3.2}$$

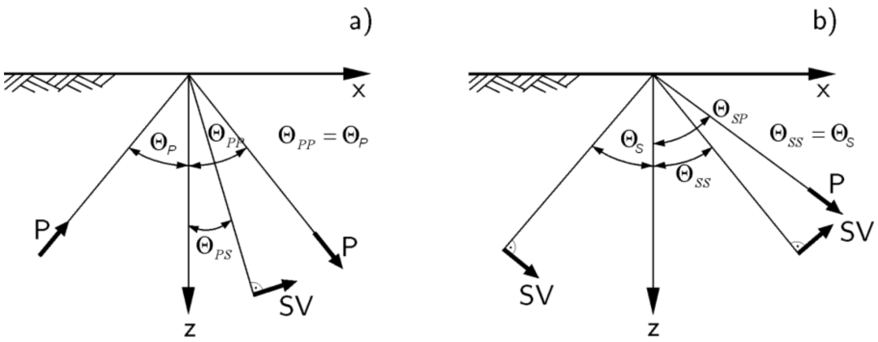

Figure 2. P and SV waves incident on the ground surface at two different angles Θ_P & Θ_S

Using figures 2a and 2b one can obtain the amplitudes of horizontal (A_X) and vertical (A_Z) surface ground motions in terms of respective amplitudes A_P and A_{SV} of the *P* and *SV* waves with particular circular frequency ω [rad/s]:

$$A_x = U_P A_P + U_S A_{SV} \tag{3.3}$$

$$A_z = W_P A_P + W_S A_{SV} \tag{3.4}$$

where coefficients U_P, U_S, W_P, W_S equal (compare figures 2a and 2b):

$$U_P = (1+P_P)\sin\Theta_P + P_P\cos\Theta_{PS} \tag{3.5}$$

$$W_P = (P_P - 1)\cos\Theta_P + P_S\sin\Theta_{PS} \tag{3.6}$$

$$U_S = (1+S_S)\cos(\Theta_S) + S_P\sin(\Theta_{SP}) \tag{3.7}$$

$$W_S = (1-S_S)\sin(\Theta_S) - S_P\cos(\Theta_{SP}) \tag{3.8}$$

The coefficients of the free surface reflections P_P, P_S, S_P, S_S can be found in the monographs on wave propagation (e.g. Aki, Richards, 1980):

$$P_p = \frac{-(S^2 - 2\sin^2\Theta_P)^2 + 4\sin^2\Theta_P\cos\Theta_P\cos\Theta_{PS}}{(S^2 - 2\sin^2\Theta_P)^2 + 4\sin^2\Theta_P\cos\Theta_P\cos\Theta_{PS}} \tag{3.9}$$

$$P_S = \frac{4S\sin\Theta_P\cos\Theta_P(S^2 - 2\sin^2\Theta_P)}{(S^2 - 2\sin^2\Theta_P)^2 + 4S\sin^2\Theta_P\cos^2\Theta_P\cos\Theta_{PS})} \tag{3.10}$$

$$S_p = \frac{4\sin\Theta_S\cos\Theta_S(1 - 2\sin^2\Theta_S)}{S(1 - 2\sin^2\Theta_S)^2 + 4\sin^2\Theta_S\cos\Theta_{SP}\cos\Theta_S} \tag{3.11}$$

$$S_S = \frac{S(1 - 2\sin^2\Theta_S)^2 - 4\sin^2\Theta_S\cos\Theta_{SP}\cos\Theta_S}{S(1 - 2\sin^2\Theta_S)^2 + 4\sin^2\Theta_S\cos\Theta_{SP}\cos\Theta_S} \tag{3.12}$$

It is important to note that the S_P and S_S coefficients are becoming complex when the incidence angle becomes critical $\Theta_S = \Theta_{cr}$ and instead of being reflected the *S* wave propagates as a surface wave (Aki, Richards, 1980).

Assume now, that the incident *P* and *S* waves are random processes with Stieltjes-Fourier representations allowing their classic, spectral decompostion. Then, the horizontal acceleration signals $\ddot{u}(t)$, $\ddot{v}(t)$ and the vertical one $\ddot{w}(t)$ can be written as:

$$\ddot{u}(t) = \int_{-\infty}^{\infty} e^{i\omega t} d\hat{\ddot{u}}(\omega); \quad \ddot{v}(t) = \int_{-\infty}^{\infty} e^{i\omega t} d\hat{\ddot{v}}(\omega); \quad \ddot{w}(t) = \int_{-\infty}^{\infty} e^{i\omega t} d\hat{\ddot{w}}(\omega) \tag{3.13}$$

where dashed symbols are random processes in the frequency domain with orthogonal increments:

$$\left\langle d\hat{\ddot{v}}(\omega_1) d\hat{\ddot{v}}(\omega_2)^* \right\rangle = \begin{cases} \left\langle \left| d\hat{\ddot{v}}(\omega) \right|^2 \right\rangle = S_{\ddot{v}}(\omega)\mathrm{d}\omega & \text{for} \quad \omega_1 = \omega_2 = \omega \\ 0 \quad \text{for} \quad \omega_1 \neq \omega_2 \end{cases} \tag{3.14}$$

In formula 3.14, *v* stands either for *v* itself or *u* and *w* as they appear in equations 3.13, $S_{\ddot{v}}(\omega)$ is the power spectral density .of the acceleration of *v*(t). An analogous formula holds for respective cross spectral densities of all the three translation processes e.g. for processes $\ddot{u}(t)$ and $\ddot{w}(t)$:

$$\left\langle \hat{\ddot{u}}(\omega_1) d\hat{\ddot{w}}^*(\omega_2) \right\rangle = \begin{cases} \left\langle \left| d\hat{\ddot{u}}(\omega) d\hat{\ddot{w}}^*(\omega) \right| \right\rangle = S_{\ddot{u}\ddot{w}}(\omega)\mathrm{d}\omega & \text{for} \quad \omega_1 = \omega_2 = \omega \\ 0 \quad \text{for} \quad \omega_1 \neq \omega_2 \end{cases} \tag{3.15}$$

where $S_{\ddot{u}\ddot{w}}(\omega)$ it is respective cross spectral density. Writing the horizontal and vertical accelerations along *x* and *z* axes (Fig. 2a and 2b) as infinitesimal, harmonic contributions of two stochastic processes in the frequency band *(ω, ω+dω)*:

$$e^{i\omega\tau} d\hat{\ddot{u}}(\omega), \qquad e^{i\omega\tau} d\hat{\ddot{w}}(\omega) \tag{3.16}$$

leads to following wave contributions from *P* and *SV* waves respectively:

$$\begin{cases} d\ddot{u} = d\ddot{u}_p + d\ddot{u}_s \\ d\ddot{w} = d\ddot{w}_p + d\ddot{w}_s \end{cases} \tag{3.17}$$

Substituting the spectral representations (3.16) for $\ddot{u}$ and $\ddot{w}$ into (3.17) one obtains:

$$\begin{cases} e^{i\omega\tau} d\hat{\ddot{u}}(\omega) = U_p e^{i\omega\tau} d\hat{\ddot{\Phi}}_p(\omega) \\ \qquad + U_s e^{i\omega\tau} d\hat{\ddot{\Phi}}_s(\omega) \\ e^{i\omega\tau} d\hat{\ddot{w}}(\omega) = W_p e^{i\omega\tau} d\hat{\ddot{\Phi}}_p(\omega) \\ \qquad + W_s e^{i\omega\tau} d\hat{\ddot{\Phi}}_s(\omega) \end{cases} \tag{3.18}$$

where $\hat{\ddot{\Phi}}_P(\omega)$, $\hat{\ddot{\Phi}}_S(\omega)$ are the random functions with orthogonal increments (eqs. 3.14-3.15), and U_P, U_S, W_P, W_S are the coefficients given by the formulae (3.5-3.8). Solving the system of equations (3.18) for the *'P'* and *'SV'* waves contributions one obtains the inverse of eqns. 3.18:

$$\begin{cases} e^{i\omega t} d\hat{\ddot{\Phi}}_p(\omega) = \dfrac{W_s}{D} e^{i\omega t} d\hat{\ddot{u}}(\omega) \\ \qquad - \dfrac{Us}{D} e^{i\omega t} d\hat{\ddot{w}}(\omega) \\ e^{i\omega t} d\hat{\ddot{\Phi}}_s(\omega) = \dfrac{U_p}{D} e^{i\omega t} d\hat{\ddot{w}}(\omega) \\ \qquad - \dfrac{W_s}{D} e^{i\omega t} d\hat{\ddot{u}}(\omega) \end{cases} \tag{3.19}$$

where $D=U_P W_S - W_P U_S$. The vertical motion can then be presented as a sum of two wave terms propagating in the *x* direction with different velocities as follows:

$$d\ddot{w}(t,\omega,x) = W_p \exp\left[i\omega\left(t - \frac{x\sin(\Theta_P)}{c_p} \right)\right] d\hat{\ddot{\Phi}}_p(\omega) + W_s \exp\left[i\omega\left(t - \frac{x\sin(\Theta_S)}{c_s} \right)\right] d\hat{\ddot{\Phi}}_s(\omega) \tag{3.20}$$

The infinitesimal rocking acceleration equals then:

$$d\ddot{\psi}(t,\omega,x) = \frac{\partial}{\partial x} d\ddot{w}(t,\omega,x)\Big|_{x=0} \tag{3.21}$$

$$\begin{aligned} d\ddot{\psi}(t,\omega,x) &= W_p\left(-i\omega \frac{\sin(\Theta_P)}{c_p} \right) \\ &\times \exp\left[i\omega\left(t - \frac{x\sin(\Theta_P)}{c_p} \right)\right] d\hat{\ddot{\Phi}}_p(\omega) + W_s\left(-i\omega \frac{\sin(\Theta_S)}{c_s} \right) \\ &\times \exp\left[i\omega\left(t - \frac{x\sin(\Theta_S)}{c_s} \right)\right] d\hat{\ddot{\Phi}}_s(\omega) \end{aligned} \tag{3.22}$$

Substituting *x*=0 and taking into account eq. 3.19 one obtains

$$d\ddot{\psi}(t,\omega) = W_p\left(-i\omega\frac{\sin(\Theta_P)}{c_p}\right)e^{i\omega t}\frac{W_s}{D}d\hat{u}(\omega)$$
$$-W_p\left(-i\omega\frac{\sin(\Theta_P)}{c_p}\right)e^{i\omega t}\frac{U_s}{D}d\hat{w}(\omega) + W_s\left(-i\omega\frac{\sin(\Theta_S)}{c_S}\right)e^{i\omega t}\frac{U_p}{D}d\hat{w}(\omega) \tag{3.23}$$
$$-W_s\left(-i\omega\frac{\sin(\Theta_S)}{c_s}\right)e^{i\omega t}\frac{W_p}{D}d\hat{u}(\omega)$$

Introducing new coefficients

$$W_x = \frac{1}{c_S}\frac{W_pW_s}{D}\left[\frac{\sin(\Theta_P)}{S} - \sin(\Theta_S)\right] \tag{3.24}$$
$$W_z = \frac{1}{c_S}\left[\frac{W_pU_s}{D}\frac{\sin(\Theta_P)}{S} - \frac{W_SU_P}{D}\sin(\theta_S)\right] \tag{3.25}$$

where S=c_P/c_S and extending the analysis into the whole frequency domain, allows one to formulate the rocking acceleration stochastic process in terms of its horizontal and translational components:

$$\ddot{\psi}(t) = \int_{-\infty}^{\infty}(-i\omega)W_xe^{i\omega t}d\hat{u}(\omega) + \int_{-\infty}^{\infty}(-i\omega)W_ze^{i\omega t}d\hat{w}(\omega) \tag{3.26}$$

Taking into account the orthogonality conditions (3.14, 3.15) one obtains the equation for spectral density of the rotational process

$$S_{\ddot{\psi}}(\omega) = |W_X|^2\,\omega^2 S_{\ddot{u}}(\omega) + 2W_XW_Z^*S_{\ddot{u}\ddot{w}}(\omega) + |W_Z|^2\,\omega^2 S_{\ddot{w}}(\omega) \tag{3.27}$$

which, in turn, when integrated gives mean square rocking:

$$\sigma_{\ddot{\psi}}^2 = \int_{-\infty}^{\infty}|W_x|^2\omega^2 S_{\ddot{u}}(\omega)d\omega + 2\int_{-\infty}^{\infty}W_xW_z^*\omega^2 S_{\ddot{u}\ddot{w}}(\omega)d\omega + \int_{-\infty}^{\infty}|W_z|^2\omega^2 S_{\ddot{w}}(\omega)d\omega \tag{3.28}$$

It can be seen from formulae 3.27-3.28, that the rotational spectral density is a function of the first derivatives of the vertical and horizontal accelerations (the ω^2 multiplier) i.e., the function of the third derivative of respective translational displacements because $S_{\ddot{u}}(\omega) = \omega^6 S_u(\omega)$. Closer examination of the final formula (3.27) reveals that the rocking ground motion driven by body wave reflections depends on horizontal and vertical ground motions $\ddot{u}$ and $\ddot{w}$ as well as on the wave propagation velocities c_S, c_P through S=c_P/c_S and Poisson coefficient ν since S = √(2-2ν)/√(1-2ν).

In Fig. 3 the modulae of both coefficients W_X and W_Z, which drive the actual translation-rocking transition, are shown normalized with respect to c_S velocity, as functions of equal incidence angles Θ_P=Θ_S, the same for both waves and for the S ratio equal to 1.73, (ν=0.25). We may observe, that, as the incidence angle goes down to zero, the rocking component also goes to zero. With both incidence angle increasing, the vertical ground motion component has much larger contribution to overall rocking than the horizontal one. For overcritical angles, the horizontal translations are more pronounced, however the vertical ones are increased too.

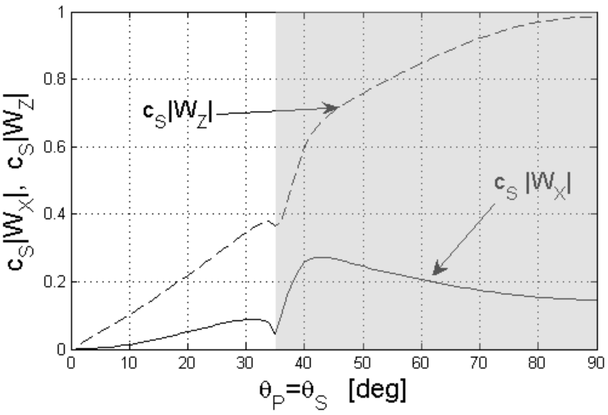

Figure 3. Plots of modulae of coefficients W_X and W_Z (normalized with respect to shear wave velocity) representing horizontal and vertical ground motion contributions to the rocking component. The $|W_X|$ and $|W_Z|$ are shown for equal incidence angles $\Theta_P = \Theta_S$.

Taking into account that the vertical and horizontal ground motions are uncorrelated, as reasonably can be assumed (e.g. Penzien and Watabe, 1975), than equations 3.24-25 and 3.27 can further be simplified.

$$S_{\ddot{\psi}}(\omega) = |W_X|^2\ \omega^2 S_{\ddot{u}}(\omega) + |W_Z|^2\ \omega^2 S_{\ddot{w}}(\omega) \tag{3.29}$$

$$S_{\ddot{\psi}}(\omega) = \frac{1}{c_S^2}\left\{\left|\frac{W_p W_s}{D}\left[\frac{\sin(\Theta_P)}{S} - \sin(\Theta_{SV})\right]\right|^2 \omega^2 S_{\ddot{u}}(\omega) + \left|\left[\frac{W_p U_s}{D}\frac{\sin(\Theta_P)}{S} - \frac{W_S U_P}{D}\sin(\theta_{SV})\right]\right|^2 \omega^2 S_{\ddot{w}}(\omega)\right\} \tag{3.30}$$

As one can see from equations 3.29 and 3.30, the rocking ground motion is inversely proportional to the shear wave velocity at the site (the actual spectral density is inversely proportional to the square of shear wave velocity). This means that one can expect the rocking ground motion to increase for the softer sites. It can also be seen, that the rocking signal should be frequency shifted compare to the both translational ones. In figures 4, 5 and 6 the modulae of both coefficients W_X and W_Z are shown as functions of the two, independent, incidence angles Θ_P and Θ_S.

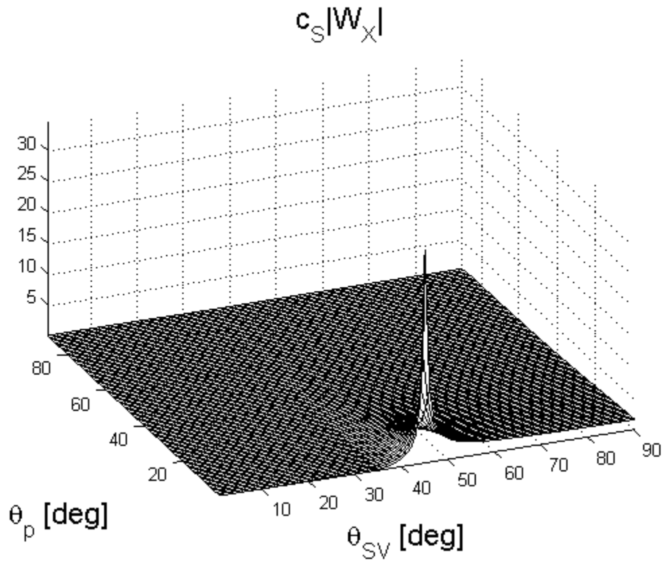

Figure 4. Plot of modulus of coefficient W_X (normalized with respect to shear wave velocity) shown as a function of two, independent incidence angles Θ_P and Θ_S..

The cross-sections of these 3D plots, going diagonally for $\Theta_P=\Theta_S$, display the results analyzed in Fig. 3. The plot of $|W_X|$ shown in Fig. 4 as a 3D plot is rescaled compare to figures 4 and 5 due substantial increase of $|W_X|$ in the area where the S wave comes at close to critical angle while the P wave comes at a very small angle. Besides the $|W_X|$ coefficient stays rather small compare to $|W_Z|$.

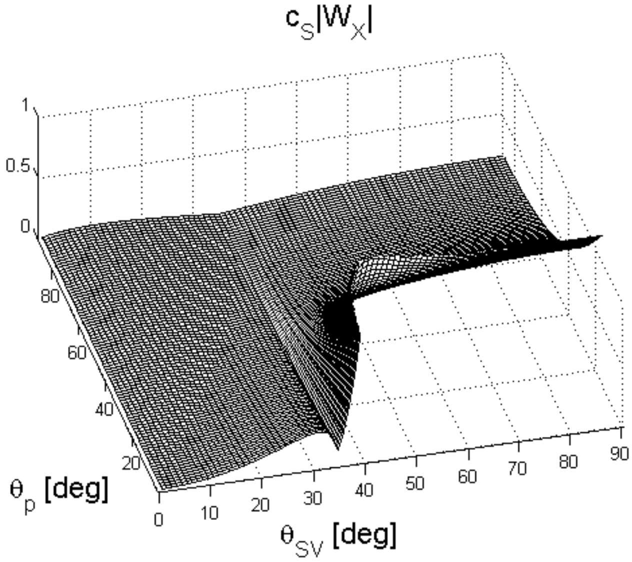

Figure 5. Plot of modulus of coefficient W_X (normalized with respect to shear wave velocity) shown as a function of two, independent incidence angles Θ_P and Θ_S. and in scale with the $|W_Z|$ plot from Fig. 6

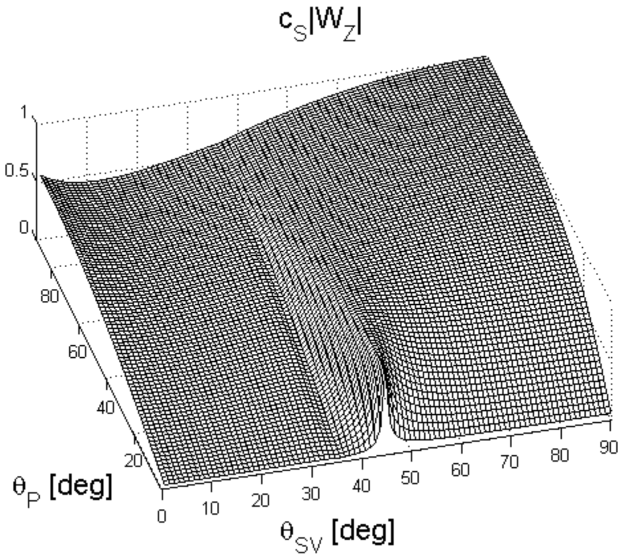

Figure 6. Plot of modulus of coefficient W_Z (normalized with respect to shear wave velocity) shown as a function of two, independent incidence angles Θ_P and Θ_S

4. TORSION FROM SH WAVES REFLECTIONS

Consider now an *SH* wave incident on the free surface at an angle Θ_S (Fig. 7). This wave is reflected at the same angle Θ_S, and the same amplitude is kept for the reflected wave (Aki, Richards, 1980). Thus, the amplitudes of the ground motions *u(t)* and *w(t),* along axes *x* and *z* respectively, are equal to zero, while the amplitude of the ground motion *v(t)* along axis *y* does not depend on the angle of incidence and equals $A_V=2A_{SH}$. From formula (2.2) it is evident that the torsional component (ground rotations around vertical axis) will be build by the derivatives of two horizontal motions *u(t)* and *v(t).* For plane waves and the principal coordinate system from Fig.1, the *SH* component *v*(t) along *y* axis depends only on coordinate *x*, so:

$$\varphi = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\frac{\partial v}{\partial x} \tag{3.31}$$

Now, assuming that the process $\ddot{v}(t)$ can be presented in form of the Stieltjes-Fourier representations

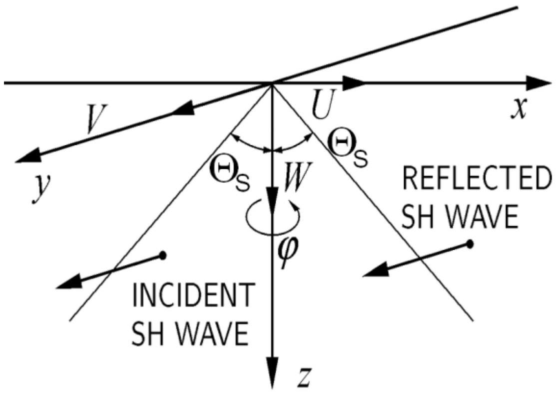

Figure 7. SH wave incident on the ground surface.

(3.13), the acceleration $\ddot{v}(t)$ in the frequency band *(ω, ω+dω)*) can be written as a wave, propagating along *x* direction:

$$d\ddot{v} = 2\exp\left[i\omega\left(t - \frac{x\sin(\Theta_S)}{c_S}\right)\right]d\hat{\ddot{\Phi}}_{SH}(\omega) \tag{3.32}$$

From this equation, taking into the account orthogonality condition (3.14), one can derive the spectral density of the torsional component in an analogous way as it was done for the rocking component:

$$S_{\ddot{\varphi}}(\omega) = \frac{1}{(2c_S)^2}\sin^2(\Theta_S)\omega^2 S_{\ddot{v}}(\omega) \tag{3.33}$$

It can be seen from formula (3.33), that the resulting torsional spectrum will be frequency shifted in a similar way like the rocking component (ω^2 factor). In Fig. 8 the coefficient $\sin^2(\Theta_S)/(2c_S)^2$ from equation (3.33) is plotted vs. angle of incidence of shear waves Θ_S. It can be seen from this figure and eq. 3.33 that the torsional component increases with increasing angle of incidence θ_S and that it is inversely proportional to the shear wave velocity, which means that it will be more pronounced for softer soils, the same way as it was the case of the rocking ground motion.

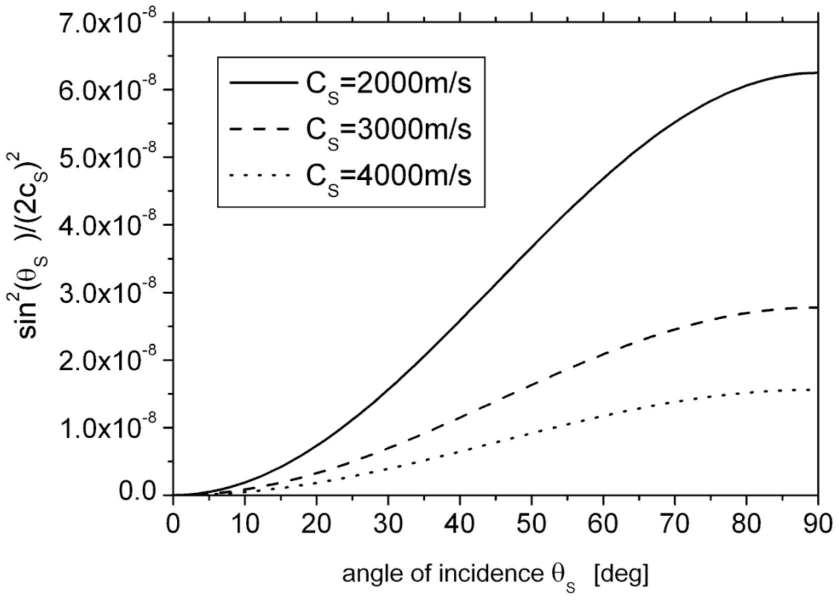

Figure 8. Coefficient $[\sin(\Theta_S)/2c_S]^2$ of equation (3.33) versus angle of incidence Θ_S.

5. CONCLUSIONS

A numerical analysis of the effect body waves reflections from the free surface, on the respective torsional (rotation about vertical axis) and rocking ground motion (rotation about horizontal axis) was presented. The rocking from the P and SV waves reflections was obtained as a function of the horizontal and vertical translational components, while the torsional ground motion was obtained from *SH* wave reflection data and the horizontal component of ground motion. Respective engineering formulas were derived (3.30 and 3.33). The ω^2 coefficient, present in both formulae, results in a phase shift of the rocking and torsional power spectral densities, compare to the translational ones. This means that the rocking and torsional accelerations are functions of the third time derivative of translational displacements. Both rocking and torsional ground motions are inversely proportional to shear wave velocity at the site, thus they will be more pronounced for compliant sites than for the hard ones. If *P* and *S* waves propagate vertically, they neither produce surface torsion nor rocking. The rocking component increases as the angles of incidence of both waves uniformly increase while the torsional component increases with increasing angle of incidence of shear waves. However it is the vertical ground motion component which dominates in the resulting rocking ground motion. It should finally be noted, that the actual torsion and rocking ground motions will depend not only on the body waves reflections but also on the surface, Love and Rayleigh wave contributions. However prediction of the actual contribution of surface waves in the overall ground motion at a particular site is not unique and constitutes one of the difficult questions of engineering seismology.

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