Analytical Solution for a Fluid-Structure Interaction Problem in Comparison with Finite Element Solution

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Summery:
An analytical solution for a fluid-structure system regarding dam-water system is presented in this paper. The analytical response is used to evaluate the accuracy of the finite element method through a numeric example with especial consideration of high frequencies of excitation. In the example, the structure is modeled as a cantilever beam with rectangular cross-section including only shear deformation and the reservoir is assumed semi-infinite rectangular filled with compressible water. It is observed that finite element results deviate from the analytical responses in relatively higher frequencies which is the case for the near field earthquakes.

Keywords: Fluid-structure interaction, frequency domain, shear effect, analytical solution, finite element

1. INTRODUCTION

Most of physical problems in engineering can be expressed by a boundary value problem. Fluid-structure interaction is one of these problems which have been the subject of many researches for decades. Numerical methods as well as analytical approaches exist for the solution of the corresponding BVP. However, due to complexity of the BVP, limited numbers of analytical solutions only for especial cases are available (Westergaard, 1933, Chwang, 1978, Tsai, 1992). Also, mixed analytic-numeric approaches have been used in literature (Nath, 1971, Liam Finn and Varoglu, 1973, Avilés and Li, 1998). For complex geometries of dam and reservoir, numerical solution is inevitable and has been developed by many researches (Maity, 2005, Aznárez et al., 2006, Aftabi Sani and Lotfi, 2010).

A closed-form formulation was presented by Bouannani and Proulx (Bouaanani et al., 2003) for evaluation of hydrodynamic pressures on rigid dams in frequency domain. Their formulation included the effect of damping due to reservoir bottom together with infinite reservoir assumption. The effect of ice covering of reservoir was investigated by Bouannani and Paultre (Bouaanani and Paultre, 2005) using BEM. They also studied the effect of several reservoir far field boundary conditions including an analytical radiation boundary condition

The aim of this paper is to solve the BVP of fluid-structure system by providing an analytical solution. Moreover, analytical results are used to evaluate the accuracy of the finite element method especially for higher frequencies of excitation. Special attention is devoted to shear deformation of the structure. It should be noted that for a complicated differential equation with irregular boundaries, where closed-form solutions are not attainable, application of numerical methods such as FEM are inevitable.

The first section of this paper is concerned with definition of the governing differential equations of fluid-structure system and the corresponding boundary conditions. Subsequent sections deal with presentation of a closed-form and finite element solutions. Results of a numerical example are compared, to investigate the accuracy of each method as well as the effective parameters.

2. DEFINITION OF GOVERNING BVP

Any boundary value problem consists of one or more differential equations with related boundary conditions. In this study, the fluid-structure system consists of two differential equations regarding the fluid and the structure. Each domain includes independent boundary conditions as well as one dependent boundary condition usually referred to as interaction boundary condition. In the following, initially, the fluid differential equation accompanied with independent B.Cs. is introduced. Additionally, the structure differential equation and its independent B.Cs. will be presented. Finally, the interaction B.C. will be explained and incorporated to process of solution.

2.1. Fluid Governing Equations

Assuming irrotational flow, small amplitude of motion and inviscid fluid, the differential equation of the fluid can be derived from the well known Navier-Stokes equation. The resultant equation is referred to as Helmholtz equation (reduced wave equation) and is shown by Eqn. (2.1),

$$\frac{\partial^2 \mathrm{P}}{\partial x^2}+\frac{\partial^2 \mathrm{P}}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 \mathrm{P}}{\partial t^2} \tag{2.1}$$

where, c is the wave propagation speed and P is the unknown pressure. Apparently, in the above equation, pressure is a function of coordinates and time. Alternatively, one may prefer to write Eqn. (2.1) in frequency domain:

$$\frac{\partial^2 \mathrm{P}^*}{\partial \mathrm{x}^2}+\frac{\partial^2 \mathrm{P}^*}{\partial \mathrm{y}^2}+\frac{\omega^2}{\mathrm{c}^2}\mathrm{P}^*=0 \tag{2.2}$$

where, ω is frequency and P^* denotes pressure in frequency domain. For simplicity, P^* will be denoted by P hereafter. Likewise, boundary conditions are transferred to frequency domain by a similar transformation. These boundary conditions include both the specified values of pressure on one side and the flux of the pressure function (outward normal vector) on the other sides, as numbered in Figure1.

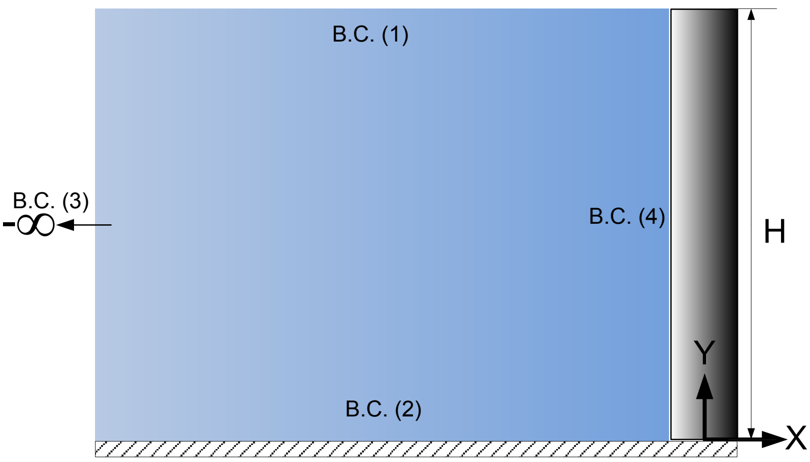

Figure 1. Fluid-structure system and fluid boundary conditions

The mentioned boundary conditions are expressed mathematically in section 2.3.

2.2. Structure Governing Equations

In this study, the structure is considered as a vertical shear beam with constant rectangular cross-section. Also, mass and shear stiffness are constantly distributed along the beam height. For the case of a gravity dam with considerable transverse thickness, shear deformations are significant. Therefore, despite the conventional assumption of flexural deformation for the beam, only shear deformations are considered in this paper. Moreover, ignoring rotational inertia and considering small amplitude of motion, one can derive the beam equation of motion in the form of:

$$\mathrm{GA}\frac{\partial^2 \mathrm{u}}{\partial y^2} = \rho\frac{\partial^2 \mathrm{u_T}}{\partial t^2} + \mathrm{P} \tag{2.3}$$

where, GA, ρ and u are constant shear rigidity, density and relative displacement of the beam, respectively. Also, $\mathrm{u_T}$ is the total displacements of the beam (sum of relative and absolute) and P denoted the pressure of water on beam. This equation can also be written in frequency domain as

$$GA\frac{\partial^2 \mathrm{u}}{\partial y^2} = -\rho\,\omega^2(\mathrm{u}+1) + \mathrm{P}|_{(0,y,\omega)} \tag{2.4}$$

2.3. Boundary Conditions

The related boundary conditions should be expressed in frequency domain and are presented briefly in Table 2.1.

Table 2.1. Fluid-Structure boundary value problem and its boundary conditions

	Fluid	Structure
Unknown Function	$\mathrm{P}(x,y,\omega)$	$\mathrm{u}(y,\omega)$
Governing differential equation	$\frac{\partial^2 \mathrm{P}}{\partial x^2} + \frac{\partial^2 \mathrm{P}}{\partial y^2} + \frac{\omega^2}{c^2}\mathrm{P} = 0$	$\mathrm{G}A\frac{\partial^2 \mathrm{u}}{\partial y^2} + \rho\,\omega^2 \mathrm{u} = -\rho\,\omega^2 + \mathrm{P}|_{(0,y,\omega)}$
Boundary conditions	(1) $\mathrm{P}(x,\mathrm{H},\omega) = 0$ (2) $\partial \mathrm{P}/\partial y|_{(x,0,\omega)} = 0$ (3) $\lim_{x\to-\infty} \mathrm{P}(x,\mathrm{y},t) = 0$ (4) $\partial \mathrm{P}/\partial x|_{(0,y,\omega)} = \rho_\mathrm{F}\,\omega^2(\mathrm{u}+1)$	$\mathrm{u}(0,\omega) = 0$ $\partial \mathrm{u}/\partial y|_{(\mathrm{H},\omega)} = 0$

3. ANALYTICAL SOLUTION OF FLUID-STRUCTURE SYSTEM

To solve the mentioned boundary value problem, one may apply some mathematical techniques such as separation of variables, using the orthogonality properties of some special functions and etc. Consequently, the solution is an infinite series with unknown coefficients. These coefficients are obtained from a system of linear equations. Although this system of equations comprise infinite terms, it can be shown that considering limited number of terms provides accurate results. Alternatively, one may use more terms of series to achieve desired accuracy. The comprehensive details will be shown elsewhere thoroughly (Keivani, 2012).

4. FINITE ELEMENT METHOD

According to widespread implementation of the finite element method, the methodology and corresponding numerical model are briefly presented in this study.

In the finite element method, a discretization is made over the continuous unknown function of the governing differential equations by dividing a region into sub-regions referred to as elements. This results in a system of linear equations in which the responses are values of the unknown function in finite number of points referred to as nodes. Also, in order to evaluate the function value of arbitrary point in the region, shape functions are employed as means of interpolation.

For this study, the types of elements used are:
1- Shear-beam element with one translational degree of freedom at each end.
2- Four-node iso-parametric fluid element with one pressure degree of freedom per node.
3- Fluid hyperelement with two nodes in each sub-layer.

Accordingly, shear-beam elements are used to model a cantilever beam. The stiffness and mass matrices of the beam are:

$$\mathbf{K}=\frac{\mathrm{GA}}{\mathrm{H}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix},\quad \mathbf{M}=\rho\mathrm{A}\begin{bmatrix}\frac{1}{3} & \frac{1}{6}\\ \frac{1}{6} & \frac{1}{3}\end{bmatrix} \tag{4.1}$$

The fluid domain adjacent to the beam and the infinite fluid domain are modeled by four-node and hyperelements, respectively. A typical finite element model is shown in Figure 2. In addition, since the accuracy of response relies on the number of elements, different meshes are investigated in this study. However, a 10×10 mesh for the near field reservoir and ten elements for the beam and the hyper-element provide sufficient accuracy.

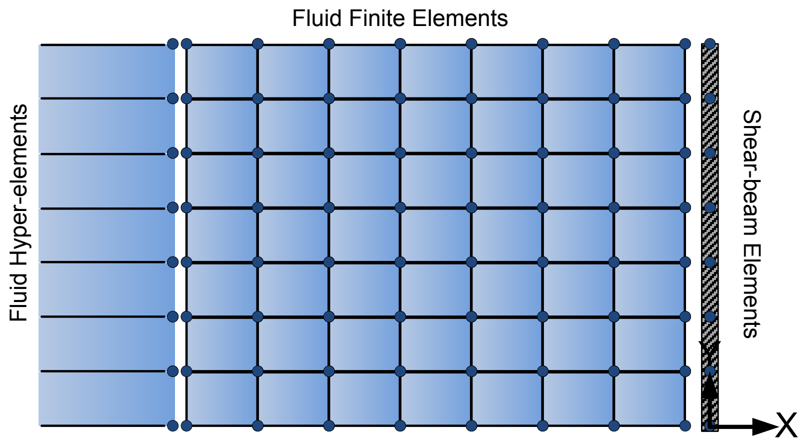

Figure 2. A schematic finite element mesh for the beam and reservoir

The corresponding finite element matrix equation of fluid-structure system for horizontal excitation may be defined by the following equation in frequency domain:

$$\begin{bmatrix}\mathbf{K}-\omega^2\mathbf{M} & -\mathbf{B}^{\mathrm{T}}\\ -\rho_{\mathrm{F}}\,\omega^2\mathbf{B} & \mathbf{H}+\mathbf{H}_{\mathrm{h}}-(\omega^2/c^2)\mathbf{G}\end{bmatrix}\begin{pmatrix}\mathbf{U}\\ \mathbf{P}\end{pmatrix}=\begin{pmatrix}-\mathbf{M}\,\mathbf{J}\,\mathbf{a}_{\mathrm{g}}\\ -\rho_{\mathrm{F}}\,\mathbf{B}\,\mathbf{J}\,\mathbf{a}_{\mathrm{g}}\end{pmatrix} \tag{4.2}$$

Where, $\mathbf{K},\mathbf{M}$ are stiffness and mass matrices of the structure and H,G represent corresponding matrices of the fluid domain. $\mathbf{H}_h$ is hyper-element contribution part in the far field and $\mathbf{B}$ is the interaction matrix.

5. NUMERICAL EXAMPLE

Analytical solution and finite element method for the fluid-structure system were explained in the previous sections. Hereafter, a numerical example is presented and the results of the mentioned methods are compared. The geometric and mechanical properties of the model are:

$$\text{Structure}: \begin{cases} E = 2\times10^{10} & N/m^2 \\ \upsilon = 0.2 & \\ G = 8.33\times10^{9} & N/m^2 \\ A = 1\times 20 & m^2 \\ \rho = 2500\times 20\times 1 & Kg/m \end{cases} \quad , \quad \text{Fluid}: \begin{cases} \rho_F = 1000\times 1 & Kg/m^3 \\ c = 1400 & m/sec. \\ H = 200 & m \\ L/H = 1 & \end{cases}$$

The model consists of a cantilever concrete beam with 200 meters height and a 20x1 rectangular cross section which pure shear deformations are considered. It is assumed that the fluid is compressible water which fills a semi-infinite reservoir of 200 meter height. Moreover, the length of the near field fluid mesh, containing 4-node fluid elements, is taken equal to two times of its height in the finite element model.

As for the first response, the frequency response function (FRF) for the acceleration of the beam crest is plotted in Figure 3. The figure represents the analytical results, considering 5, 10 and 20 terms of the series for unit horizontal ground acceleration.

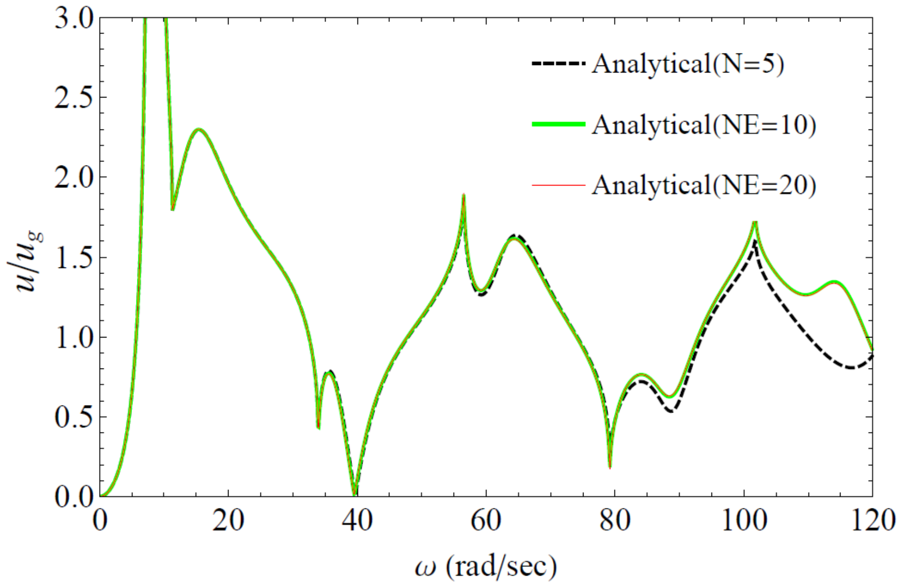

Figure 3. Analytical FRF of the beam crest for different number of series terms considered

It is observed that the three curves are in good agreement within the natural frequencies of 0 and 80 rad/sec. However, the curve with (N=5) gradually diverges after this frequency due to insufficient number of terms considered in the series. The curves with N=10 and N=20 are in significant agreement within the shown rage of frequencies. This indicates that taking only ten terms of the series leads to accurate results and the solution can be referred as exact.

Since the analytical response for the desired range of frequencies is at hand, accuracy of finite element method can be readily verified at this stage. As a result, the acceleration of the beam crest with a mesh of ten elements in height is shown in Figure 4, in contrast to the analytical solution.

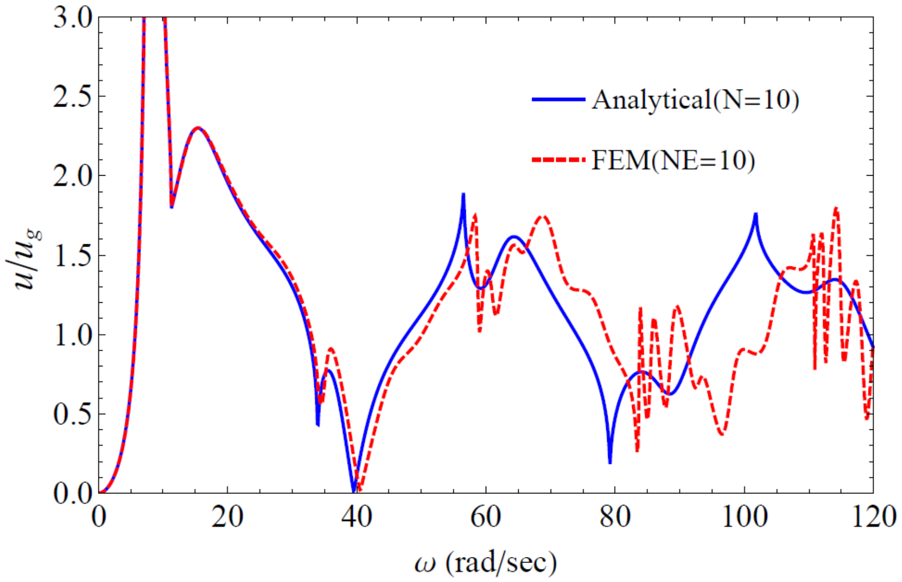

Figure 4. FEM vs. Analytical displacement of the beam crest

It is observed that the finite element response begins to deviate from the analytical response at natural frequencies of 25 rad/sec. This deviation considerably develops along with frequency increase. To increase accuracy, one may consider that mesh refinement in the finite element. This is investigated by doubling the number of beam and the reservoir elements along the height. It should be noted that since fluid hyperelements are implemented vertical mesh refinement does not affect the results. Figure 7 provides a comparison between the refined mesh (NE=20) and the results of the analytical solution.

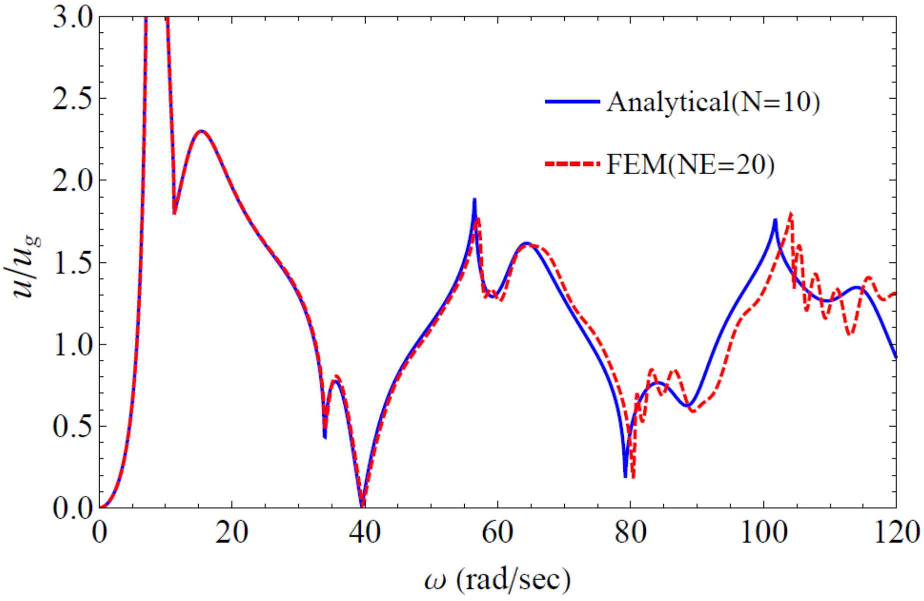

Figure 5. FEM vs. Analytical displacement of the beam crest

The above figure indicates that even with doubling the number of elements, finite element does not provide sufficient accuracy for higher range of frequencies.

Likewise, the frequency response function of the pressure on the beam base can be determined. This is carried out using finite element (with 10 elements) and the analytical solution (with10 terms) as displayed in Figure 6.

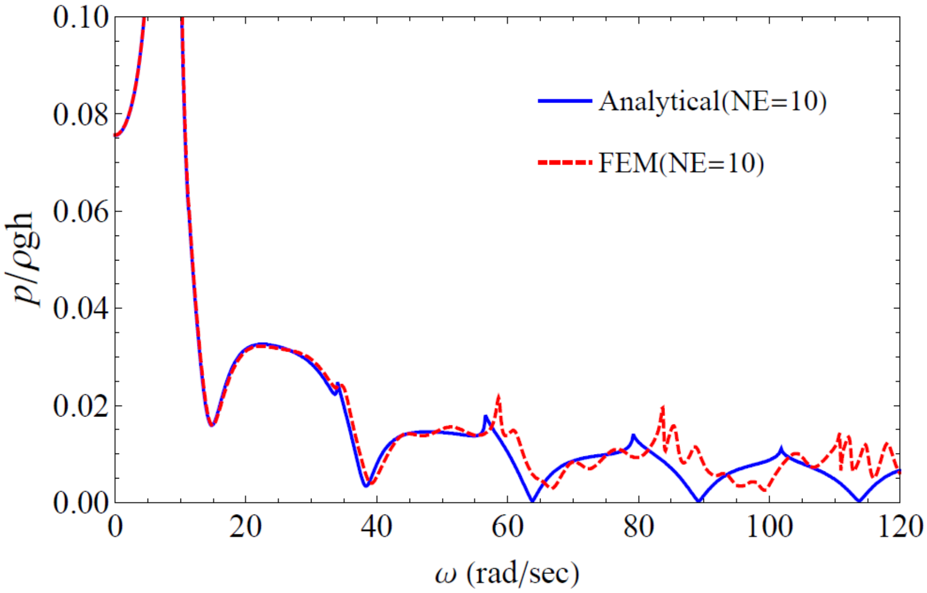

Figure 6. Pressure frequency response function at base level

The loss of accuracy in FE results has also been observed in Figure 6 for higher frequencies. This is mostly due to linear approximation in solid region which is in interaction with fluid. The overall deviations of the curves and their location of occurrence are similar to that of the displacement responses. Nevertheless, for the pressures, the curves are smoother and the picks display relatively blunt changes.

Moreover, it is convenient to study several special cases. As for the first case, a rigid model of the beam is compared with the flexible beam. In practice, rigidity is achieved by assuming GA to be infinity (a large numeric value). The effect of structure rigidity on pressure frequency response function is shown in Fig. 8.

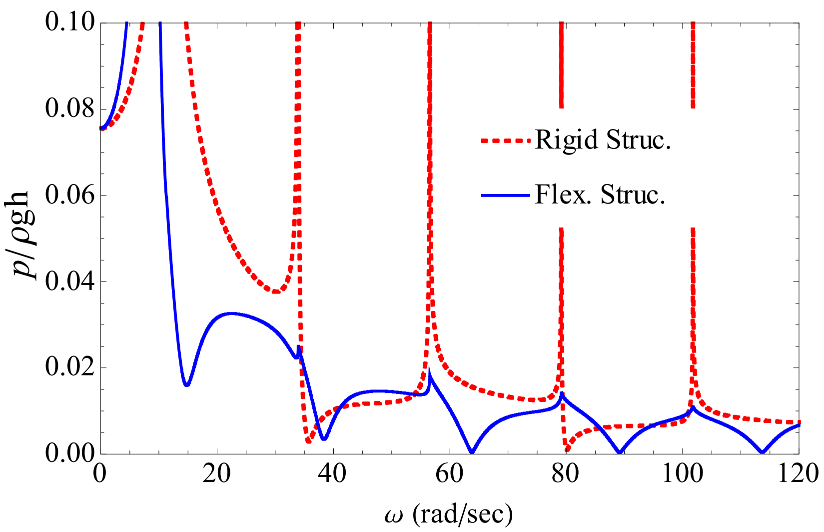

Figure 7. FRF of pressure at base level for rigid and flexible cases of Structure

Interestingly, it is observed that only the first resonance frequency of the system has been reduced when flexibility is brought into account. In contrast, other resonant frequencies have indicated only a very slight reduction. However, the values of pressures at resonance frequencies of the rigid structure are considerably higher from the values of flexible structure.

Another important case to investigate is an empty reservoir against a full reservoir. This can be achieved by assigning a small value to fluid density. The effect of damping induced by the reservoir can be simply investigated in Figure 8.

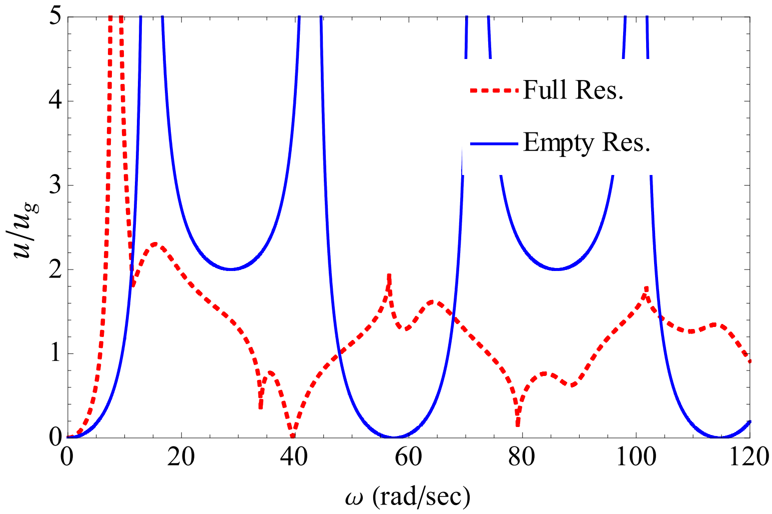

Figure 8. FRF of beam crest for full vs. empty reservoir cases

Fig. 8 shows that interaction of reservoir decreases the first fundamental frequency of the system. Drift of the subsequent resonance frequencies are significant and the coupled system is considerably damped. Moreover, additional resonant frequencies are inserted to the system.

It is interesting to investigate the effect of shear stiffness of the structure on the behavior of the system. Thus, the FRFs of the beam crest for different widths of the cross sections are plotted in Figure 9. In all cases the depth of the beam is assumed unity. It is observed that the position of resonance frequencies has slightly been affected by variation of the structure width. However, frequencies in all cases are shifted to a greater value.

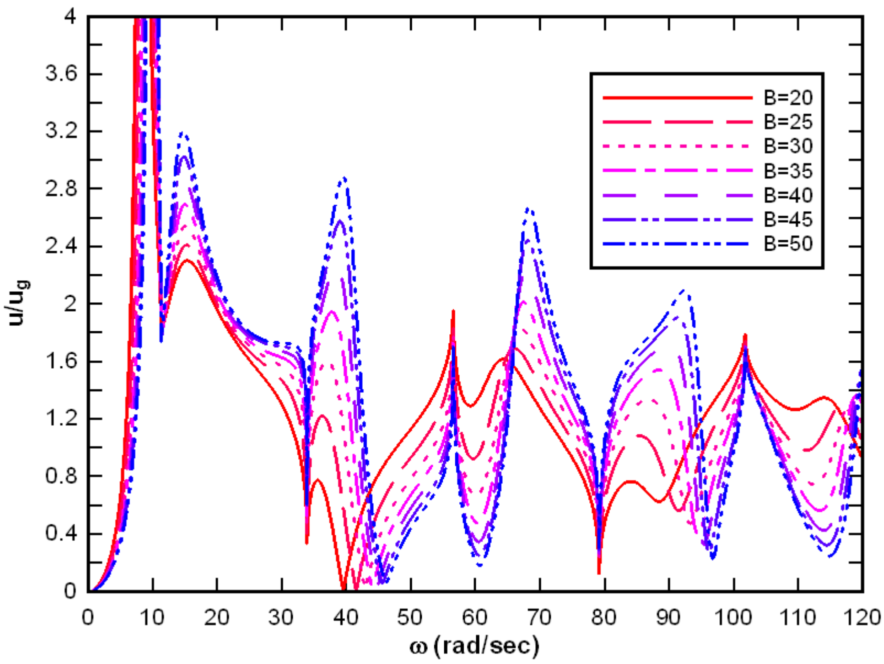

Figure 9. FRF of displacements at beam crest for various widths of beam

Although frequency response functions represent the behavior of the system, they are restricted to a single point of the system. In contrast, graphical representation of the pressure contours provides the ability of investigation of the whole system at a certain frequency. This is an effective way to evaluate the accuracy of two analyses since the results contain the whole concerning region and are not restricted to a single point. Accordingly, pressure contours for analytical and finite element are shown in Figures 10 for several selected frequencies. The pressure values of the following figures are normalized to hydrostatic pressure in all cases.

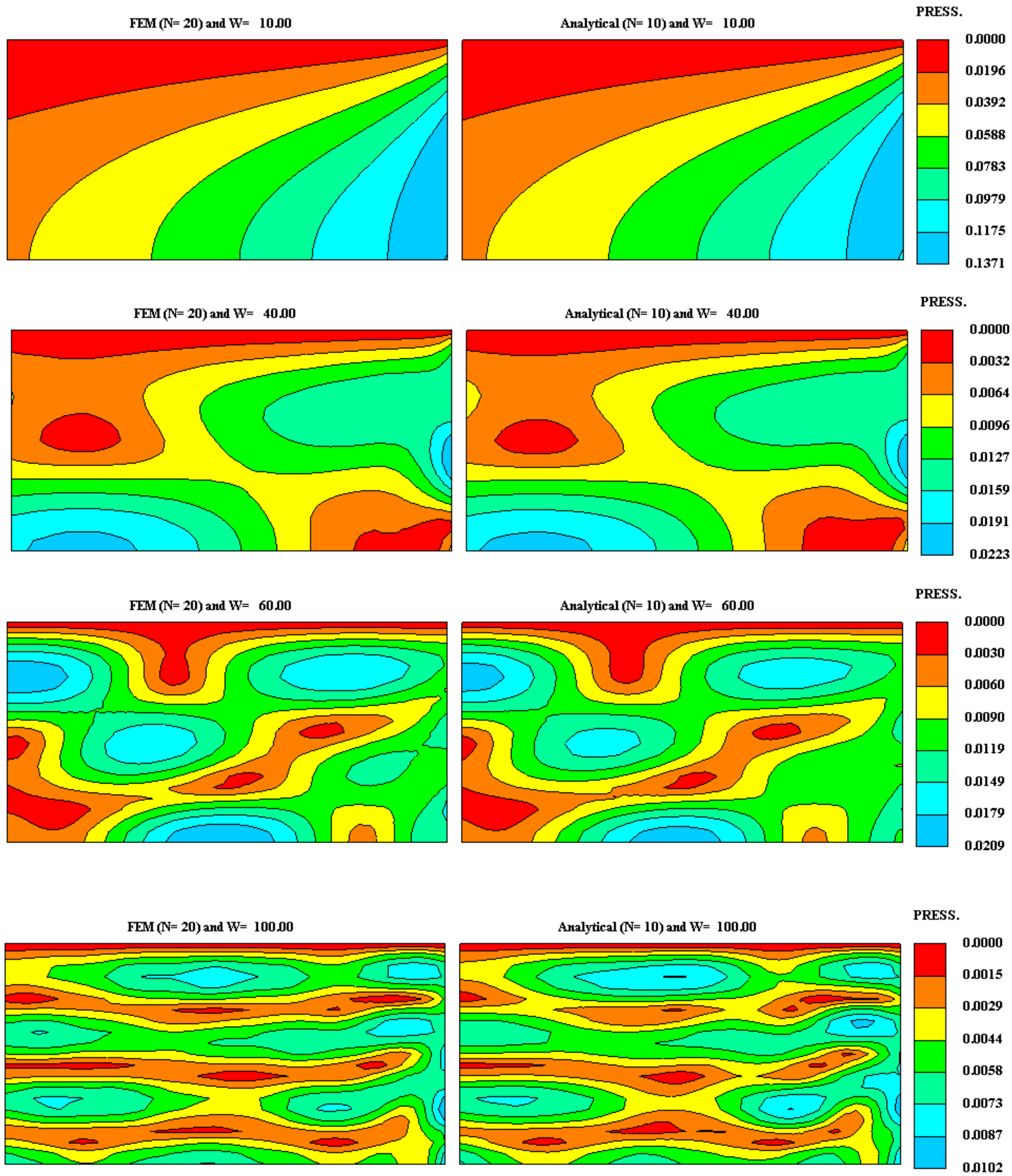

Figure 10. Pressure distribution of FEM vs. Analytical results for various ω

6. CONCLUSION

An analytical solution of the fluid-shear beam differential equation was presented in frequency domain under the semi-infinite compressible and irrotational assumptions for the fluid region and horizontal ground excitation. The closed-form solution was used to investigate the accuracy of the conventional finite element method through a numeric example with approximate dimensions of a gravity dam. The displacement of the beam and pressures as comparing parameters and special cases such as rigid beam and empty reservoir were investigated in the analysis. Also, sensitivity analysis was carried out on several effective parameters. The following conclusions were drawn:

- For low frequencies both methods are in good agreement, while for higher frequencies the finite element responses are not accurate and considerably deviate from analytical responses. Moreover, mesh refinement slightly increases the accuracy whereas errors cannot be ignored. One possible reason for this observation may be the implementation of frequency-independent shape functions in the FE method.
- The computation time for the analytical solution is significantly less than finite element method. Although one may benefit from efficient storage techniques in finite element method to increase time efficiency, a quite fine mesh is required to achieve the same accuracy.
- For the case of near field earthquakes, where excitation frequencies are high, the analytical solution provides accurate results.
- In many finite element codes, modal analysis is implemented due to efficiency. In such cases, number of participating modes needs to be increased to cover higher frequencies of excitation. Therefore, the efficiency of the method reduces. In contrast, such difficulty is not encountered for analytical solution.

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