A three-dimensional extension of the Ramberg-Osgood model. Comparison with other formulations

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SUMMARY:
The Ramberg-Osgood model, coupled with the extended Masing criterion, has been one of the most used constitutive relations in non-linear analyses of one-dimensional ground motion amplification. This model allows to appropriately account with stiffness reduction and hysteretic damping increase with increasing shear strain, and it permits, with few parameters, to better adjust these phenomena to laboratory data, when compared to other hysteretic models.
Having as goal the use of this model in finite-element codes, a three dimensional extension of the Ramberg-Osgood model coupled with the extended Masing criterion was presented by Chitas (2008). The proposed formulation describes the stiffness reduction as a function of the octahedral shear strain.
The formulation is applied to a case study. The results are compared with the ones obtained using other models well-known in geotechnical earthquake engineering, namely, the equivalent linear formulation.

Keywords: Ramberg-Osgood, Masing criterion, ground-motion amplification.

1. INTRODUCTION

For very small shear strains, most soils respond according to the linear elastic model. Considering the conservation equation and Hooke's law, it is possible to relate the elastic properties of soils and the wave propagation velocities.

However, even for small strains, soils exhibit energy dissipation and non-linear behaviour, which cannot be modelled by the linear elastic model. It is usual to define a threshold, in terms of shear strain, for which the linear elastic model is no longer an acceptable tool, named as the *linear cyclic threshold shear strain* (Vucetic, 1994; Santos, 1999).

A simple mechanical system that allows to consider energy dissipation of soils implies the parallel disposal of a spring and a dashpot. This system is known as the Kelvin-Voigt Model. Despite the fact that the rheological model contains a dashpot, experimental evidence contradicts the above statement in the range of frequencies of engineering interest. For cycles of constant amplitude (harmonic excitation), the soil response is characterised by a hysteresis loop. This implies that the damping is due to hysteresis (friction between soil particles), and that the damping coefficient is the parameter that characterizes energy dissipation, and not the viscosity. Hence, the Kelvin-Voigt Model is usually presented as a direct function of the damping coefficient. The Kelvin-Voigt model is normally presented within the framework of one-dimensional analysis.

The simple use of linear viscoelastic models don't take into account these effects, as the soil properties are strain-independent. In order to surpass this shortcoming, and having as purpose the use of linear viscoelastic models, with all the presented features and equations inherent to these models, the *Equivalent Linear Method* was developed.

Another approach used to model energy dissipation and stiffness reduction is to consider non-linear elastic models, coupled with the so-called *Masing criterion*. The use of these models is based on the following definition: Hysteretic behaviour of soils may be modelled by two curves, one relative to monotonic loading, known as the backbone or skeleton curve, and the other one relative to unloading/reloading conditions. Masing (1923) related both curves, under the following premises: *the secant shear modulus for the strain (or stress) reversals is equal to the initial shear modulus* and *the shape of the unloading/reloading curve is equal to the backbone curve, but it is scaled up by a factor equal to two*.Figure 1 translates the stress/strain curve associated to the Masing criterion for cyclic loading of constant amplitude.

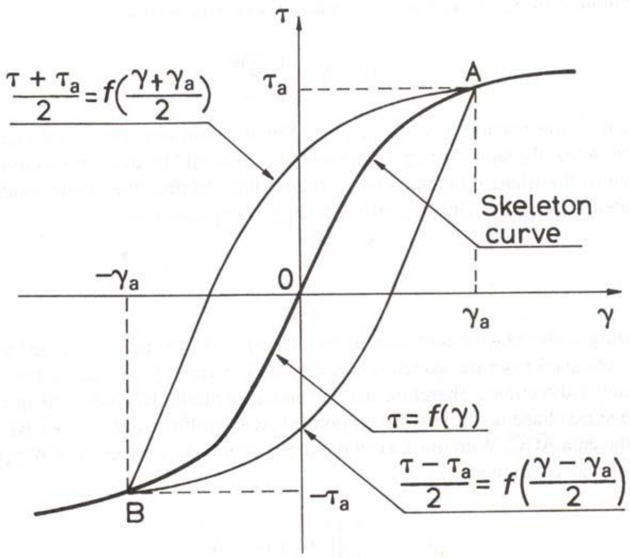

Figure 1. Stress/strain curve according to the Masing criterion for harmonic loading (Ishihara, 1996)

For non-harmonic loading, the cycles no longer have constant amplitude and loading curves become far more complex. The two premises of the Masing criterion alone aren't enough to describe the soil behaviour. Kramer (1996) added two premises in order to have a better description of the soil model: *if the previous maximum shear strain (in absolute value) is overcome, the stress path follows the backbone curve* and *if the n^{th} stress/strain cycle is closed, the stress path may not surpass the path defined by the $(n\text{-}1)^{th}$ cycle*. Figure 2 contains the stress/ strain path for an arbitrary loading.

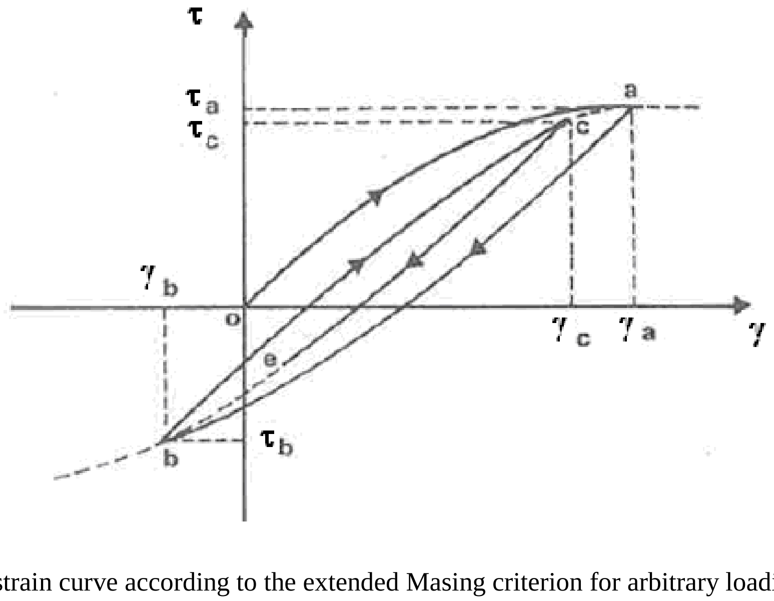

Figure 2. Stress/strain curve according to the extended Masing criterion for arbitrary loading (Pecker, 2006)

One of the main advantages of the use of these models is that, for general loading, it is possible to determine irreversible shear strains, *i.e.*, the use of the extended Masing criterion allows to consider plastic distortions *via* a non-linear elastic model. However, it is important to bear in mind that the use of this approach is only an approximation, as there isn't a complete mathematical description of soil behaviour, namely about volumetric behaviour. Despite these limitations, the use of constitutive models coupled to the Masing criteria have been widely used for the study of ground motion amplification for one-dimensional analyses. The most used models within the framework of Masing criterion are the *hyperbolic model* and the *Ramberg-Osgood model*.

Ramberg and Osgood (1943) first proposed a model with three parameters that would describe the stress/strain curves of aluminium-alloy and steel sheets. The first authors to use the Ramberg-Osgood model in soil modelling were Faccioli et al. (1973), in order to validate the shear modulus reduction curves first proposed by Seed and Idriss (1970) for sands. However, it was Idriss et al. (1978) who proposed the use of an adaptation of the Ramberg-Osgood model in order to obtain the shear modulus reduction. According to these authors, the stress/strain relation (backbone curve) follows Equation 1.1:

$$\frac{\gamma}{\gamma_y} = \frac{\tau}{\tau_y}\left[1+\alpha\left|\frac{\tau}{\tau_y}\right|^{r-1}\right] \tag{1.1}$$

where α and r are model (experimental) constants; and γ_g and τ_t are reference values for shear strain and shear stress. Normally, shear modulus reduction curves are strain-controlled; therefore, normally the backbone curve is defined as a function of the shear strain. Considering the definition of secant shear modulus, it is possible to rewrite the one-dimensional formulation of the Ramberg-Osgood model as in Equation 1.2 (Ishihara, 1996).

$$\frac{G}{G_0} = \frac{1}{1+\alpha\left|\frac{G}{G_0}\cdot\frac{\gamma}{\gamma_r}\right|^{r-1}} \tag{1.2}$$

So, according to the Ramberg-Osgood model, it takes four parameters to define the secant shear modulus: the initial shear modulus, the reference shear strain, and the constants α and r.
The damping coefficient, according to this model, may be determined according to Equation 1.3.

$$\xi = \frac{2}{\pi}\cdot\frac{r-1}{r+1}\cdot\left(1-\frac{G}{G_0}\right) \tag{1.3}$$

The Ramberg-Osgood model implies an explicit relation between the secant shear modulus reduction and the damping coefficient increase. Santos (1999) refers that, for medium strains, it is reasonable to admit this assumption. Another feature that can be seen in Equation 1.6 is that is the parameter r that controls the energy dissipation inherent to a given soil. For increasing values of r , the greater is the dissipation capacity.

The parameters α and r allow a good fitting to experimental curves.

2. THREE-DIMENSIONAL EXTENSION OF THE RAMBERG-OSGOOD MODEL

2.1. Mathematical description

The mathematical description of the adopted formulation closely follows what is described in Chen and Mizuno (1990), concerning the isotropic non-linear incremental (*i.e.*, tangential) formulation based on the secant bulk modulus, K_S, and the secant shear modulus, G_S. In the developed model, the secant shear modulus was considered to be a function of the octahedral shear strain, γ_{oct}, according to the Ramberg-Osgood model. The octahedral shear strain is described by Equation 2.1:

$$\gamma_{oct} = \frac{2}{3}\sqrt{(\varepsilon_{11}-\varepsilon_{22})^2+(\varepsilon_{22}-\varepsilon_{33})^2+(\varepsilon_{33}-\varepsilon_{11})^2+6\cdot\varepsilon_{12}^2+6\cdot\varepsilon_{23}^2+6\cdot\varepsilon_{31}^2} \tag{2.1}$$

The octahedral shear strain is a scalar measure of the shear deformation on a given point (Maranha, 2005), and is closely related to the second invariant of the strain tensor (just as happens with the deviatoric stress, *q*, for the stress tensor).

Considering what has been mentioned, the secant shear modulus is given by Equation 2.2.

$$G_S = \frac{G_0}{1+\alpha\left|\frac{G_S}{G_0}\cdot\frac{\gamma_{oct}}{\gamma_r}\right|^{r-1}} \tag{2.2}$$

Considering isotropic linear elastic behaviour in terms of volumetric strain, the tangent bulk modulus, K_t, is equal to the secant bulk modulus, K_S. This means that, for the proposed formulation, the Poission coefficient may vary, as a function of the shear modulus decay. The secant constitutive relation is the following:

$$\sigma_{ij} = K_S\cdot\varepsilon_{kk}+2\cdot G_S\cdot\varepsilon_{ij} \tag{2.3}$$

In terms of numerical implementation within finite-element codes, it is more interesting to express the constitutive relation incrementally. The full mathematical description on obtaining the incremental formulation is presented in Chitas (2008). In index notation, the incremental formulation is given by Equation 2.4.

$$d\sigma_{ij} = 2\left[\left(\frac{K_t}{2}-\frac{G_S}{3}\right)\cdot\delta_{ij}\cdot\delta_{kl}+G_S\cdot\delta_{ik}\cdot\delta_{jl}+\eta\cdot e_{kl}\cdot e_{ij}\right]\cdot d\varepsilon_{kl} \tag{2.4}$$

The term η is equal to:

$$\eta = \frac{4}{3}\cdot\frac{G_t-G_S}{\gamma_{oct}^2} \tag{2.5}$$

The constitutive relation, in its matrix form, may be written as:

$$\{d\sigma\} = [C_t]\cdot\{d\varepsilon\} \tag{2.6}$$

where $[C_t]$ is *the tangential stiffness matrix*. This matrix may be expressed as the sum of two parts.

$$[C_t] = [A] + [B] \tag{2.7}$$

Matrix $[A]$ results from the sum of the first two parts of Equation 2.4, and it has a similar shape to the isotropic linear elastic stiffness matrix, but with G_S and K_t replacing G and K. The tangential stiffness matrix is strain-dependent, as the secant shear modulus is a function of the actual strain state (*via* γ_{oct}). Matrix $[B]$ is given by Equation 2.8.

$$[B] = 2 \cdot \eta \cdot \{e\} \cdot \{e\}^T \tag{2.8}$$

where $\{e\}$ is the deviatoric part of the strain tensor, expressed as a vector. This part of the tangential stiffness matrix is clearly dependent on the actual strain state and presents two main features. In one hand, this matrix is clearly unsymmetrical, as the deviatoric part of the strain state may have all of its components different from 0. On the other hand, matrix $[B]$ depends of the second-order term $(\gamma_{oct})^2$ (*via* η) When dealing with a shear modulus decay, η assumes a negative value.

An important issue that is inherent to the adopted formulation is the independence between volumetric and distortional behaviours, as the formulation is a direct adaptation of the isotropic linear elastic formulation. For medium to large strains, the mentioned behaviours are clearly intertwined, making the present formulation inadequate for this strain range.

2.2. Definition of the extended Masing criterion

For the present formulation, the mentioned premises were fulfilled only when distortional behaviour was concerned. Despite having a three-dimensional meaning, Equation 2.2 is formally similar to Equation 1.5, which falls in the definition of a backbone curve, developed from a direct relation between the octahedral shear stress, τ_{oct}, and the octahedral shear strain, γ_{oct}. Therefore, the application of the extended Masing criterion was made exactly as for the one-dimensional case, but relating the octahedral values of stress and strain.

In order to be able to apply the extended Masing criterion in terms of octahedral shear strain to the adopted incremental formulation, several issues had to be tackled, mainly concerning the detection of octahedral shear strain reversals, unloading and reloading situations, and maintenance of the stress path envelope.

First, to verify the reversals of the octahedral shear strain in the backbone curve, comparisons, between the current and the new value, was made at each calculation step. If is verified a decrease in terms of octahedral shear strain, means that there was a strain reversal. The detection of strain reversals, when the strain path is in the backbone curve, implied storing the previous maximum octahedral shear strain in the strain history for each step. This maximum is usually known as backstress, for stress-controlled descriptions. Recalling Figure 1, the backstress would correspond to the stress at point *a*.

The scale factor in unloading/reloading situations was made adapting Equation 2.2. Assuming a maximum octahedral shear strain, γ_a, according to the second premise of the Masing criterion, the scale factor in the unloading/reloading curve is equal to 2. This means that the octahedral shear strain used to determined the secant shear modulus in the unloading/reloading curve is half of the real value. As seen in Equation 2.9, this last statement may be replaced by the following equivalent statement: the reference shear strain used to determine shear modulus in the unloading/reloading curve is twice the real value.

$$G_S\left(\frac{\gamma_a-\gamma_{oct}}{2}\right)=\frac{G_0}{1+\alpha\left|\frac{G_S}{G_0}\cdot\frac{\gamma_a-\gamma_{oct}}{2\cdot\gamma_r}\right|^{r-1}} \tag{2.9}$$

This result proved to be important in the proposed incremental formulation. As matrix $[B]$ depends on the actual deviatoric strain tensor quadratically, it would be difficult to implement the scale factor to the stiffness matrix via the octahedral shear strain. Using this result, both matrices $[A]$ and $[B]$ are easily determined, without any interference with the strain tensor.

The third and fourth premises of the extended Masing criterion were the most difficult to implement. Both premises imply that, in order not to surpass the previously defined hysteretic loops, the octahedral shear strain where the strain reversal occurs must be stored, whether it concerns the backbone curve or the previous cycles. As the adopted formulation is incremental, and, therefore, depends on the deviatoric part of the strain tensor, $\{e\}$, *via* matrix $[B]$, it is also needed to store the independent components of the strain tensor. These issues are conceptually easy to understand; the soil needs to have a kind of "memory" in order to retake the previous stress/strain path. In order to fulfil the fourth premise, one must be able to store large amounts of information concerning the stress/strain reversal.

3. CASE STUDY

3.1. Introduction

The presented formulation has been implemented in PLAXIS as a user-defined model. The detailed description of its implementation, as well as its source-code, is presented in Chitas (2008).

In order to analyse the formulation and its behaviour, a direct comparison was made between a two-dimensional plane-strain model in PLAXIS with a width much larger than its height and lateral absorbent boundaries (henceforth, FEM-RO, standing for "Finite-Element Method - Ramberg Osgood"), and a soil column modelled in SHAKE2000 (Ordoñez, 2006). In the central area of the model, for an imposed acceleration at the base of the model, the obtained result should be similar to the obtained in a one-dimensional analysis, as lateral effects are not significant. The model has a height of 10m and a width of 250m. A single material was considered, having the properties shown in Table 3.1.

Table 3.1. Soil properties used in the case study

G_0 [kPa]	K_t [kPa]	γ [kN/m^3]	α []	r []	γ_{ref}- []
20,0x10^3	20,0x10^4	20,0	50	2,5	1,0x10^{-2}

Figure 3 shows the finite-element model used in the case study.

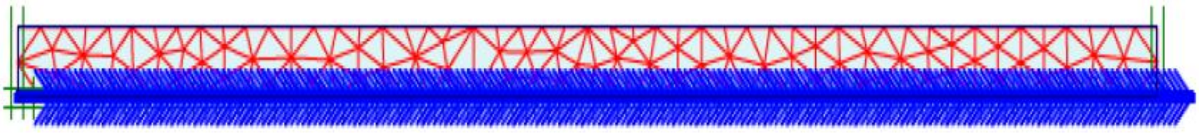

Figure 3. FEM-RO model used to compare results with one-dimensional analysis (Chitas, 2008)

The standard numerical integration procedure in FEM-RO analysis admits a Newmark-beta algorithm with numerical damping. Here the calculation with an average-acceleration version was followed, with a Rayleigh-type damping equal to 2% adjusted to the first and third vibration modes.

Prescribing a given self-weight meant that FEM-RO model would perform a full dynamic calculation.

The displacement set at the base followed a well-known acceleration time series, obtained at Gilroy #1 array, for the 1989 Loma Prieta earthquake.

For the adopted model, the stiffness reduction was such that it, for the maximum shear strain, the shear modulus, G_S, would be around half of maximum shear modulus, G_0. For these values, the Poisson coefficient varies from 0,45 to 0,475, corresponding to a near-saturation situation.

The one-dimensional analysis was done using the commercial program SHAKE2000 (from now on, ELM, standing for "Equivalent Linear Method"), with an equivalent shear strain ratio equal to 0,65. A soil column of 10m was considered, with underlying bedrock with a shear stiffness 100 times greater. The stiffness reduction and material damping curves were compliant with the adopted Ramberg-Osgood parameters, bearing in mind that the octahedral shear strain, in plane-strain for a two-dimensional pure-shear analysis, doesn't assume the same value as the applied shear strain, as shown in Equation 3.1.

$$\gamma_{oct} = \frac{2}{3}\sqrt{6\cdot\left(\frac{\gamma}{2}\right)^2} \approx 0{,}816\cdot\gamma \tag{3.1}$$

The scale factor between the octahedral and the plane shear strains, as demonstrated with Equation 3.1, was accounted, dividing the shear strain that served as input by 0,816. The mentioned curved are shown in Figure 4 and Figure 5.

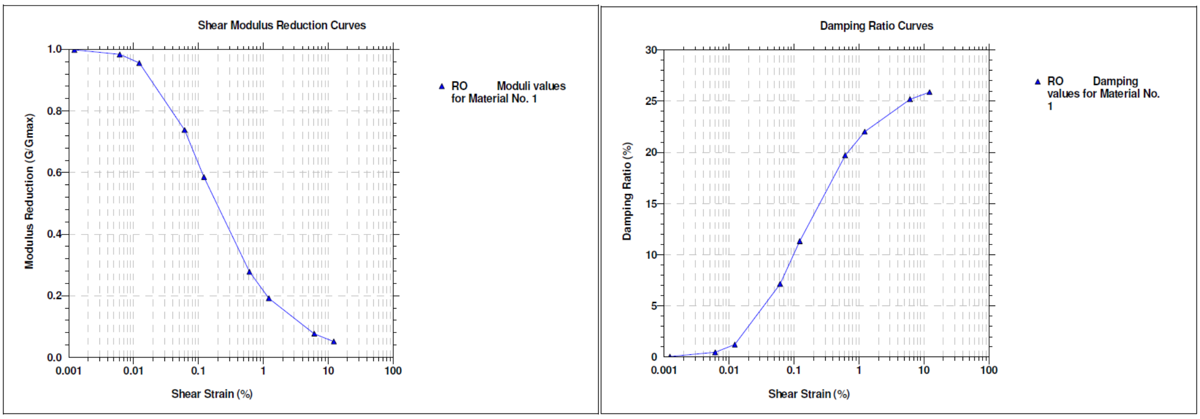

Figure 4. Stiffness reduction and damping curves used in ELM analysis

3.2 Results analysis

In order to compare the results obtained by both models, the acceleration and the shear stress time series at different depths was retrieved. These depths were 0 m (at surface), 1 m and 7 m. In terms of acceleration time series, the results are shown in Figure 5, Figure 6, and Figure 7.

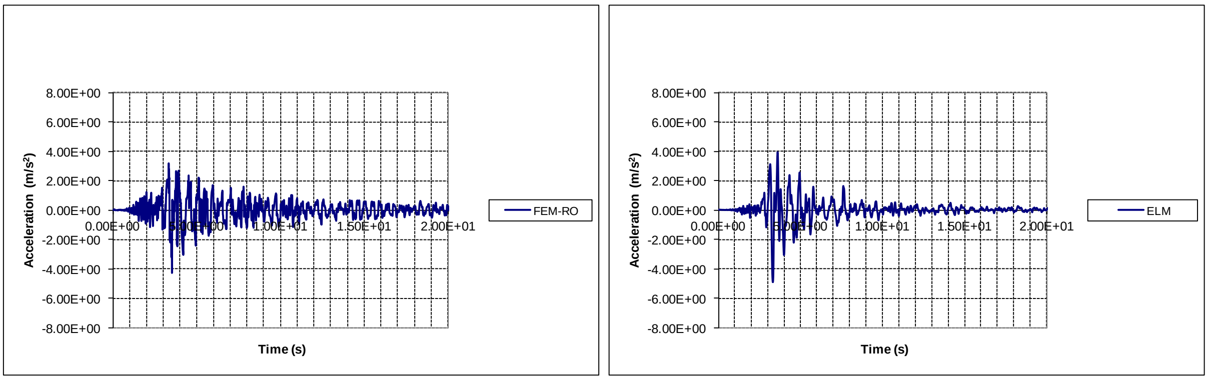

Figure 5. Acceleration time series by both models at the surface

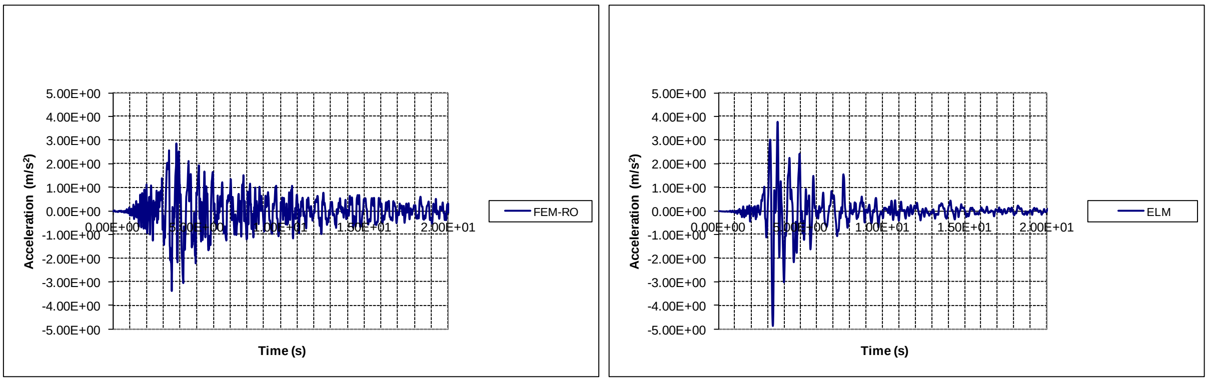

Figure 6. Acceleration time series by both models for a depth equal to 1m

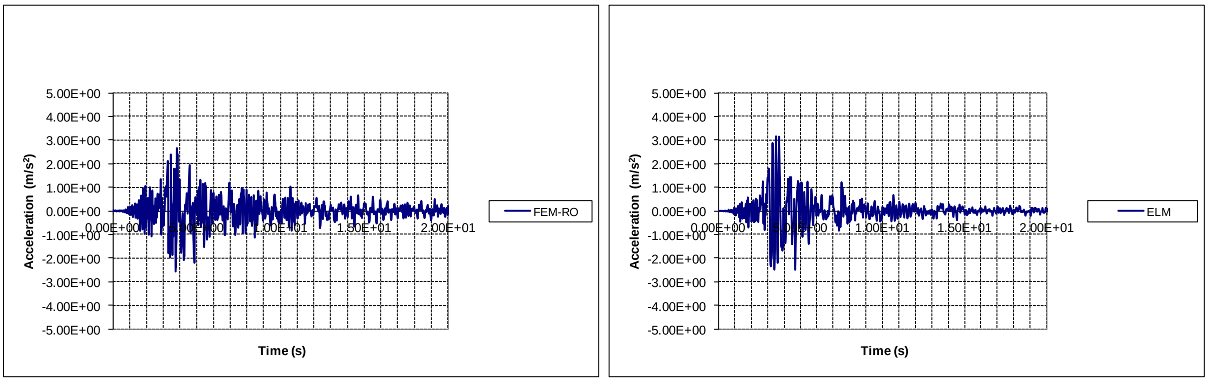

Figure 7. Acceleration time series by both models for a depth equal to 7m

There are differences between the two models. The acceleration time series obtained using the FEM-RO formulation exhibit less damping at the end of the time series for the most superficial points of control, when comparing with the ELM analysis. This may be due to the fact that, in FEM-RO model, at the end of the time series, it is likely to be constantly in unloading/reloading conditions. For these conditions, there is almost no hysteretic behaviour, which leads to greater stiffness and lesser damping behaviour than the one obtained using the ELM model.

Concerning PGA, the FEM-RO model leads consistently to lower values than the ELM analysis, which indicates that, for the peak strain, the proposed formulation leads to greater damping than the ELM formulation.

Equivalent linear analyses may lead to oversoftened response. But in somehow the computed ground response obtained from this case study seems to be more realistic compared with the shape of real records. This apparent contradiction is due to the lack of radiation damping in the numerical simulations. If radiation damping is accounted in the model, equivalent linear analyses usually lead to an overdamped system when the peak strain is much larger than the remainder of the shear strains.

These comments are coherent with the obtained shear strain time series, which are shown in Figure 8 and Figure 9.

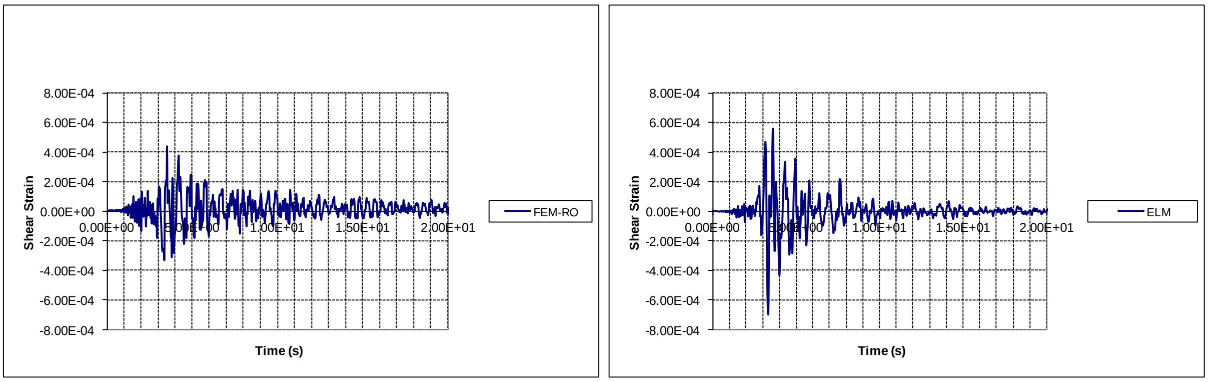

Figure 8. Shear strain time series by both models for a depth equal to 1m

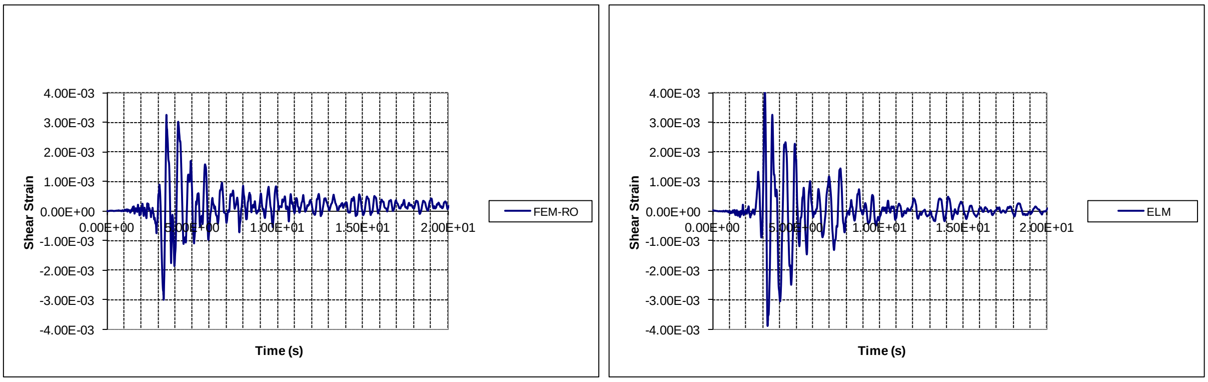

Figure 9. Shear strain time series by both models for a depth equal to 7m

There is an interesting feature concerning the shear strain time series obtained using the FEM-RO model. There is evidence that at the end of the calculation process, the value of the shear strain isn't null, which is in agreement with the Masing criterion formulation. In order to conclude the comparison, the transfer function relating the acceleration at the surface and at the bedrock was calculated for both programs. Figure 10 contains the transfer function.

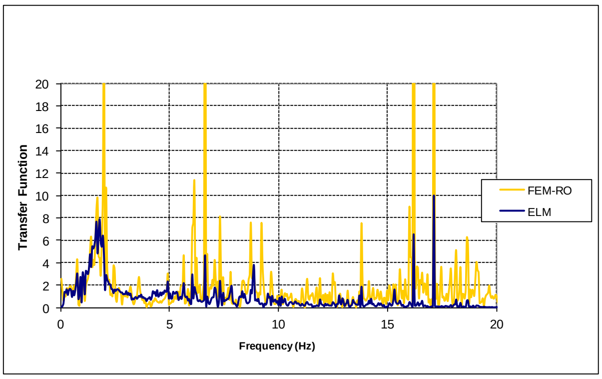

Figure 10. Transfer function for both programs

The models exhibit the same vibration modes. Comparing the level of amplification, the FEM-RO model exhibits higher amplification values for the vibration modes. This falls in line with the previous results, which showed that FEM-RO leads to less damping. Another remark that may be made is that the FEM-RO model amplifies much more than the ELM model for high frequencies.

4. CONCLUSIONS

A full three-dimensional constitutive model based on the Ramberg-Osgood model was exposed, showing all the details. Important remarks concerning eventual numerical implementation were done at this point. Issues concerning the extended Masing criterion were also shown, focusing not only on theoretical aspects, but mainly on the consequences in terms of the implementation of all its premises.

A case study was shown, comparing a two-dimensional finite-element model using the presented constitutive relation (FEM-RO) and a one-dimensional soil column model using the equivalent linear formulation (ELM). Considering the results, the proposed formulation leads to greater damping for the peak response when comparing with the equivalent linear formulation, whereas, for the lower strains associated to the end of the time series, the opposite is true.

ACKNOWLEDGEMENT

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