Using Spring-Mass Models to Determine the Dynamic Response of Two-Story Buildings Subjected to Lateral Loads

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SUMMARY
This paper shows the use of spring-mass models to develop the equations of motion so the dynamic analysis of two-story buildings having linear stiffness on each floor can be carried out. If a two-story building is modelled as a shear-building, a scale-down physical spring-mass model can be constructed and tested using a shake table. On the other hand, if the model represents a moment resistant frame, a physical model using actual masses and springs becomes difficult to construct as one of the springs attached to the second floor may have a 'negative' stiffness. The main objective of this paper is to present an equivalent spring-mass model incorporating an equivalent structure, instead of using a spring with negative stiffness. The use of this equivalent spring-mass model will allow to obtain the dynamic response of two-story buildings subjected to horizontal loads, such as wind or ground acceleration.

Keywords: dynamic response, two-story building, spring-mass model, shear-building, shaking-table tests.

1. INTRODUCTION

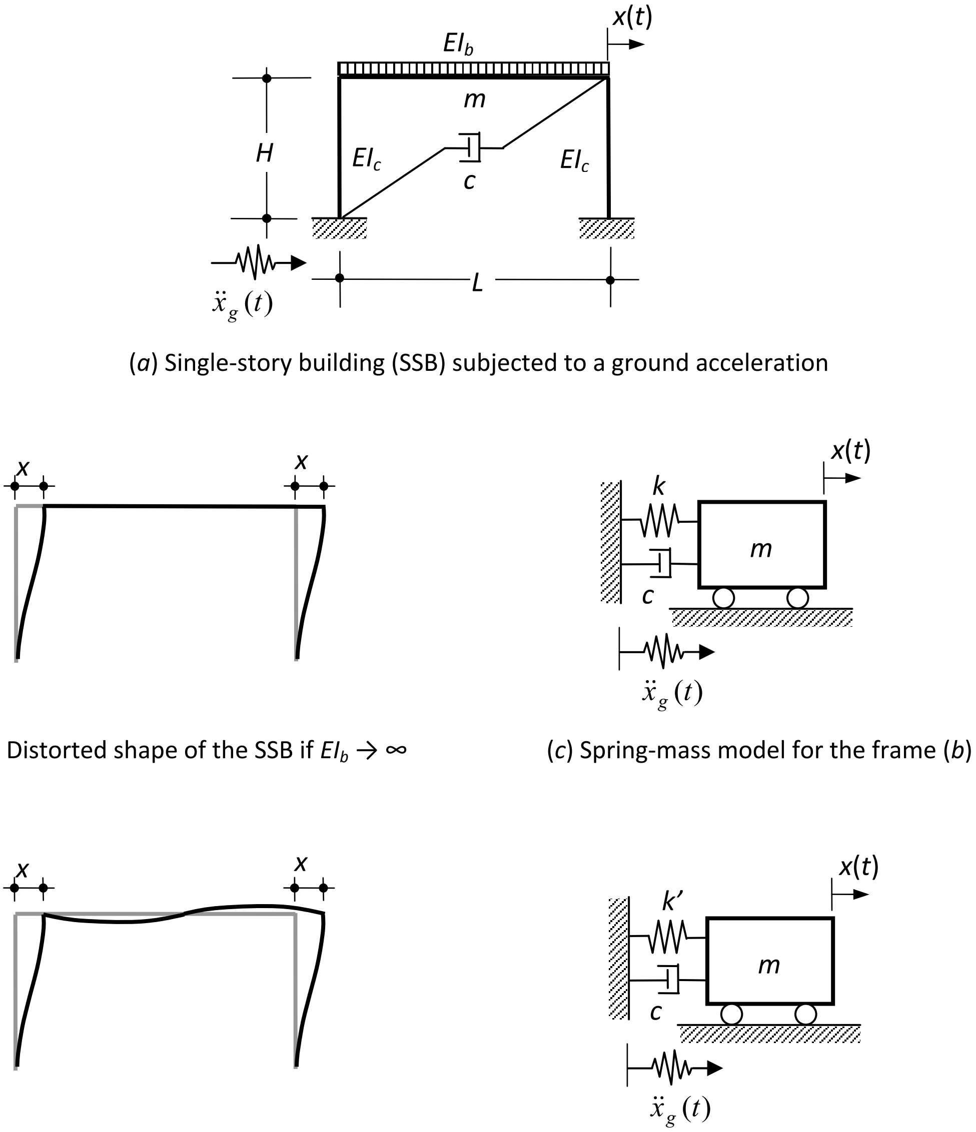
The mechanical models of structures can be represented by spring-mass systems (Brennan et al., 2008; Delhomme et al., 2007; Wu, 2004). When dealing with building structures subjected to earthquake loads, single-story buildings (SSBs) and multi-story buildings (MSBs) can also be represented by spring-mass models (SMMs) (De la Cruz and López-Almansa, 2006; Wilkinson and Thambiratnam, 2001). For example, in Fig. 1.1*a* a SSB is shown. The length of the girder is represented by L and the height of the frame is represented by H. If the girder flexural stiffness, represented by EI_b, is infinitely rigid, the distorted shape of the SSB when subjected to the ground acceleration $\ddot{x}_g(t)$ is shown in Fig. 1.1*b*, where the horizontal coordinate x stands for the single degree of freedom of the frame. The corresponding SMM is sketched in Fig. 1.1*c*. The quantities m, c and k represent, respectively, the mass of the girder, the viscous damping of the SSB and the overall stiffness of the SSB. In this case, this stiffness k is a function of the flexural stiffness, EI_c, of both columns, and the height H. On the other hand, if EI_b is no longer infinitely rigid, the assumed motion of the SSB is shown in Fig. 1.1*d*, and its SMM is shown in Fig. 1.1*e*. In this case, the actual stiffness k' is a function of both flexural stiffnesses, EI_b and EI_c, and the length L and height H.

The equation of motion of the model shown in Fig. 1.1*c* is

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{x}_g(t) \tag{1.1}$$

where the upper dots represent the time derivatives of x. The equation of motion of the model shown in Fig. 1.1*e* is the same Eqn. (1.1) provided that the actual stiffness k' will replace k.

Now, a two-story building (TSB) is shown in Fig. 1.2*a*; the length of the girders is represented by L and the heights of the frame floors are represented by H_1 and H_2; the degrees of freedom of the TSB are the horizontal displacements x_1 and x_2. Again, assuming that the girders are infinitely rigid (i.e., $EI_{b1} \rightarrow \infty$ and $EI_{b2} \rightarrow \infty$), the distorted shape of the TSB when subjected to a ground acceleration $\ddot{x}_g(t)$ is shown in Fig. 1.2*b*. The corresponding SMM is sketched in Fig. 1.2*c*. In contrast, if the flexural rigidity of the girders, EI_{b1} and EI_{b2} is finite, the actual distorted shape will be as shown in Fig. 1.2*d* and the corresponding SMM is depicted in Fig. 1.2*e*. Note that an additional spring k_3 has to be 'connected' from the rigid wall —representing the ground— to the mass m_2 —representing the second floor— since the stiffness of the overall system has to be modified to include both the lateral and the rotational degrees of freedom (Cheng, 2001).



(*a*) Single-story building (SSB) subjected to a ground acceleration

(*b*) Distorted shape of the SSB if $EI_b \rightarrow \infty$

(*c*) Spring-mass model for the frame (*b*)

(*d*) Distorted shape of the SSB if $EI_b \neq \infty$

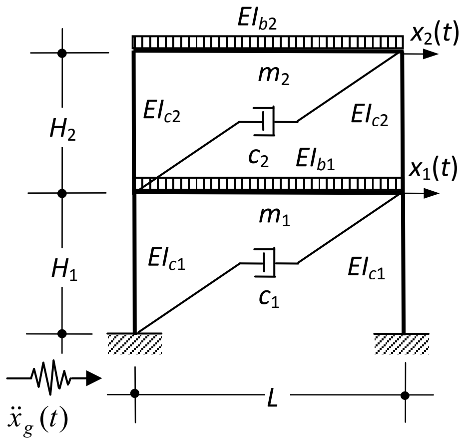
(*e*) Spring-mass model for the frame (*d*)

Figure 1.1. Single-story building modelled as a single-degree-of-freedom system

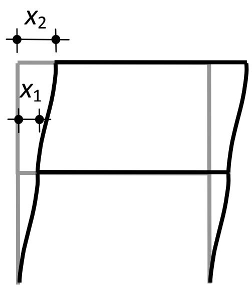
The equation of motion of the model shown in Fig. 1.2*c* is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = -\mathbf{M}\mathbf{r}\ddot{x}_g(t) \tag{1.2}$$

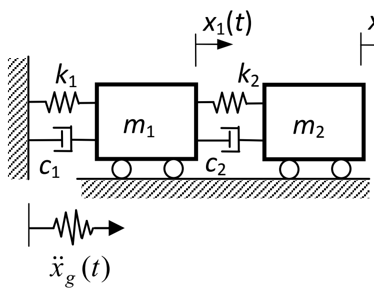
where **M, C** and **K** represent the mass, damping and stiffness matrices, respectively; **x** represents the displacement vector and **r** is a unit vector.



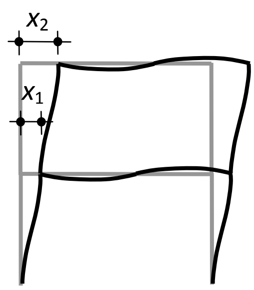
(a) Two-story building (TSB) subjected to a ground acceleration



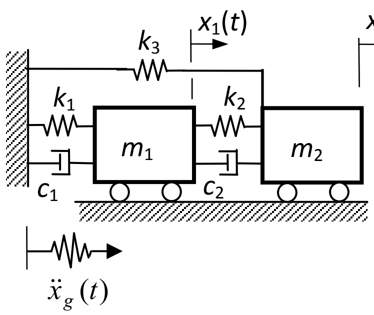
(b) Distorted shape of the TSB if $EI_{bi} \to \infty$



(c) Spring-mass model for the frame (b)



(d) Distorted shape of the TSB if $EI_{bi} \neq \infty$



(e) Spring-mass model for the frame (d)

Figure 1.2. Two-story building modelled as a multi-degree-of-freedom system

Expanding Eqn. (1.2), we obtain

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=-\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\ddot{x}_g(t) \tag{1.3}$$

where subscripts 1 and 2 refer to floor 1 and floor 2, respectively.

On the other hand, the equation of motion of the model shown in Fig. 1.2*e* is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}^*\mathbf{x} = -\mathbf{M}\mathbf{r}\ddot{x}_g(t) \tag{1.4}$$

Expanding Eqn. (1.4), we obtain

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\ddot{x}_g(t) \tag{1.5}$$

The new stiffness matrix $\mathbf{K}^* = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$ is generally obtained after making a static matrix condensation (Cheng, 2001).Note that Eqn. (1.2) is not longer valid for the model shown in Fig. 1.2*e* as the overall stiffness matrix **K*** in Eqn. (1.4) is obtained by adding a new stiffness coefficient k_3 to the former matrix **K.**

As a final remark of this introduction, we know that for the SSB both conditions ('shear building', SB, and 'moment resistant frame', MRF) can be represented for the same equation of motion —Eqn. (1.1)— since the stiffness of the overall system, *k*, can be modified to include both the lateral and the rotational degrees of freedom of the nodes (De la Cruz and López-Almansa, 2006). For the TSB, however, a similar modification cannot be carried out since we are dealing with a stiffness matrix, instead of a single coefficient, which is the case of the SSB.

This paper deals with a procedure to get the dynamic response of TSBs (either SBs or MRFs) represented by SMMs when subjected to lateral forces (i.e., wind, earthquake).

2. DEVELOPMENT

2.1. The Two-Story Building (TSB)

The TSB shown in Fig. 1.2*a* is going to be considered for this analysis. As already seen, when the rigidity of the girders, $(EI_b)_i$, approaches to infinity, the assumed distorted shape of the frame is shown in Fig. 1.2*b*, its SMM is depicted in Fig. 1.2*c* and the equation of motion is equal to Eqn. (1.2). Now, considering the flexural rigidity of both girders and columns, the distorted shape of the frame is depicted in Fig. 1.2*d*, its SMM is shown in Fig. 1.2*e* and the equation of motion is equal to Eqn. (1.4).

The stiffness coefficient $k_2 + k_3$ of the stiffness matrix **K*** is what makes difficult to build a physical model of the TSB as the term k_3 can be either positive or negative (De la Cruz and López-Almansa, 2006). The latter case, i.e., $k_3 < 0$, means to attach a spring with negative stiffness to mass m_2. In this work, a procedure is developed to deal with either possibility —a positive or negative k_3.

2.2. Construction of an Equivalent Spring-Mass Model

It is possible to construct a SB 'equivalent' to a MRF using a spring-mass model, provided that the angular frequencies ω_{n1} and ω_{n2} of the new model are equal to those of the original model. Fig. 2.1 illustrates this affirmation.

In order to convert model of Fig. 2.1*b* whose new matrices are **M', C'** and **K'** into model of Fig. 2.1*a*, whose original matrices are **M, C** and **K*,** it is necessary to determine parameters γ and ε as follows:

$$\gamma = \frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{(k_1 + k_2)k_2 - k_2^2} \tag{2.1}$$

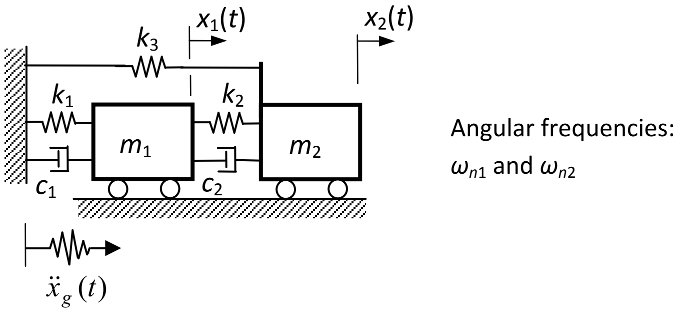
$$\varepsilon = \frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{\sqrt{\gamma}\left[(k_1 + k_2)\beta m_2 + k_2 \alpha m_1\right]} \tag{2.2}$$

where parameters *α, β* and *ε* in Eqn. (2.2) must accomplish the following:

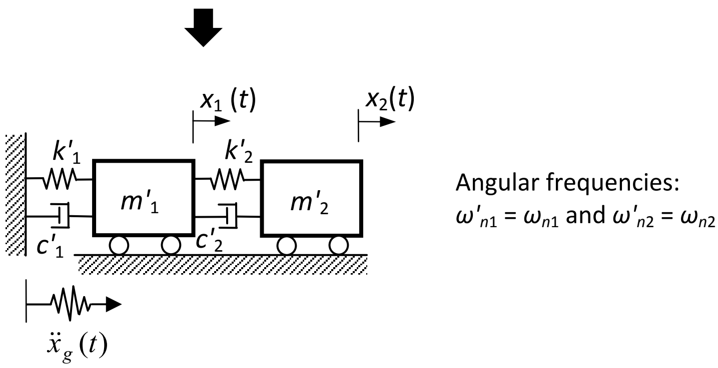
$$\alpha\beta\varepsilon^2 = 1 \tag{2.3}$$

The new mass and stiffness matrices will be

$$\begin{aligned} \mathbf{M}' &= \begin{bmatrix} \alpha\varepsilon m_1 & 0 \\ 0 & \beta\varepsilon m_2 \end{bmatrix} \\ \mathbf{K}' &= \begin{bmatrix} \sqrt{\gamma}(k_1 + k_2) & -\sqrt{\gamma}k_2 \\ -\sqrt{\gamma}k_2 & \sqrt{\gamma}k_2 \end{bmatrix} \end{aligned} \tag{2.4}$$



(*a*) Spring-mass model of a two-story MRF subjected to a ground acceleration



(*b*) Spring-mass of a shear-building (SB) 'equivalent' to model (*a*)

Figure 2.1. Spring-mass models for a MRF and its 'equivalent' SB

The damping matrix **C'** can be the same **C** as before.

It is important to notice that the new matrices **M'** and **K'** will yield equal angular frequencies to those obtained with the original matrices **M** and **K.**

3. APPLICATIONS

3.1. Numerical Simulation

The following **M, C** and $\mathbf{K}^*$ matrices belong to a scale-down TSB which was actually built to be tested (De la Cruz et al., 2007; López-Almansa et al., 2011):

$$\mathbf{M} = \begin{bmatrix} 533.5 & 0 \\ 0 & 552.5 \end{bmatrix} \text{ kg}$$

$$\mathbf{C} = \begin{bmatrix} 72.692 & -4.065 \\ -4.065 & 68.688 \end{bmatrix} \text{ N}\cdot\text{s/m}$$

$$\mathbf{K}^* = \begin{bmatrix} 808.375 & -351.467 \\ -351.467 & 267.115 \end{bmatrix} \text{ kN/m}$$

For this structure, the angular frequencies are: ω_{n1} = 13.0982 rad/s and ω_{n2} = 42.7447 rad/s.

The ground acceleration shown in Fig. 2.2 will be used as the external driving force for the numerical evaluation of the response.

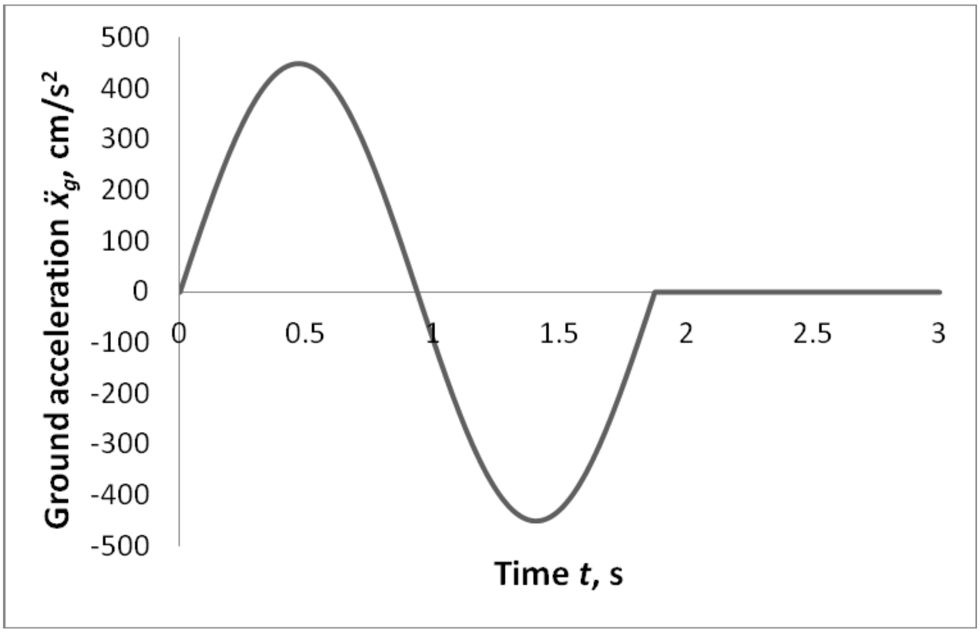


Figure 2.2. Ground acceleration for the TSB of the example

3.1.1. Conversion of the MFR to a SB

The values of α and β were set equal to 0.55 and 1.20, respectively. With these values, and using the coefficients of matrices **M** and **C,** the following values for γ and ε were obtained: γ = 0.57539 and ε = 1.21528. The matrices **M', K'** and **C'** were found to be

$$\mathbf{M}' = \begin{bmatrix} 356.6 & 0 \\ 0 & 805.7 \end{bmatrix} \text{ kg}$$

$$\mathbf{C}' = \begin{bmatrix} 72.692 & -4.065 \\ -4.065 & 68.688 \end{bmatrix} \text{N} \cdot \text{s/m}$$

$$\mathbf{K}' = \begin{bmatrix} 613.185 & -266.602 \\ -266.602 & 266.602 \end{bmatrix} \text{ kN/m}$$

For this 'converted' structure, the angular frequencies are: ω'_{n1} = 13.0811 rad/s and ω'_{n2} = 43.3512 rad/s.

3.1.2. Displacement response using step-by-step algorithms

Using a numerical tool, such as ALMA (De la Cruz, 2004) or MS Excel®, it is possible to find the dynamic response of the TSB. In this case, the latter was used applying the well known linear acceleration method (Clough and Penzien, 2003). The time-history of the second-floor displacement is shown in Fig. 2.3 for the first 3 seconds.

In Fig. 2.3, the dotted line corresponds to the displacement response of the second floor of the original structure, with matrices **M, C** and **K*,** while the solid line corresponds to the displacement response of the second floor of the 'converted' structure, with matrices **M', C'** and **K'.** In this case, $\alpha\beta\varepsilon^2$ is found to be equal to 0.975. As the value of $\alpha\beta\varepsilon^2$ approaches to 1, there should be no difference between both responses.

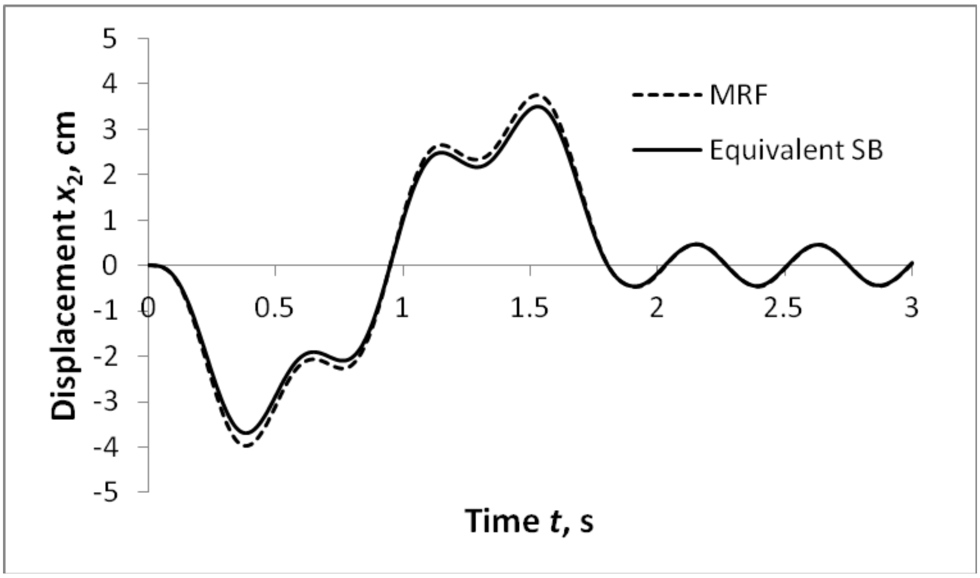


Figure 2.3. Second floor responses for the MRF and SB models, subjected to the ground acceleration of Fig. 2.2

3.1.2. Displacement response using commercial software

The above data can be implemented rapidly into commercial software such as ADINA (Bathe, 1996). In ADINA, the viscous dampers and the springs (stiffness coefficients) can be entered as linear elements, making the modelling very simple (De la Cruz et al., 2009). Besides, the time-history response obtained with ADINA is equal to that obtained in subsection 3.1.2.

3.2. Shaking-table Tests

The SMM shown in Fig. 1.2*c* can be built so a physical model of an actual TSB (either SB or MRF) can be tested. Moreover, the nonlinear behaviour (e.g., bi-linear stiffness) can be simulated by using friction devices attached to the springs (De la Cruz et al., 2010).

4. CONCLUSIONS AND FUTURE RESEARCH

This paper presents a procedure to make spring-mass models as an option to analyze the dynamic behaviour of two-floor building structures in which the flexural rigidities of the beams, in addition to those of the columns, are considered. The procedure presented here consists in transforming the mass and stiffness matrices of the original structure. The main advantage of using this procedure is that any bi-dimensional two-story structure can be modelled as a spring-mass system; therefore, its dynamical analysis is easy to carry out using commercial software. Finally, it is important to notice that a nonlinear dynamic analysis can be made using the 'equivalent' spring-mass models, as described in this paper. For example, when dealing with nonlinear stiffness, this can be incorporated into the mass-spring models by adding the necessary nonlinear spring elements on each mass. Finally, it is important to point out that currently the authors are working on the physical testing of the spring-mass models.

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